Abelian varieties and Fourier–Mukai transforms

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Originally developed more as a technical tool, derived categories have recently found themselves recast as intricate geometric invariants. Either is good motivation to study them, and the main point of this overview is to convince you of two things:

- Derived categories are the natural setting for homological algebra[1];
- Derived categories have a powerful and interesting bearing on geometry, often in surprising ways.

1. Derived categories as homological algebra

Homological algebra takes place in the context of abelian categories, which generalize the properties of modules over a ring:

Definition 1.1. An abelian category $\mathcal{A}$ is a category $\mathcal{A}$ satisfying:

(1) $\mathcal{A}$ is enriched in abelian groups, meaning that the Hom set between any two objects is endowed with the structure of an abelian group in such a way that the composition maps are group homomorphisms;
(2) $\mathcal{A}$ has all small limits and colimits. This essentially means that
  (a) $\mathcal{A}$ has a zero object $0 \in \mathrm{Ob}(\mathcal{A})$;
  (b) any two objects $M, N \in \mathrm{Ob}(\mathcal{A})$ have a direct sum $M \oplus N$ (categorically this is both the coproduct and product)
  (c) any morphism $f : M \to N$ has a kernel and cokernel.
(3) The second isomorphism theorem: for any map $f : M \to N$ the natural map on the right in the below diagram is an isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & \operatorname{coker} f & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longleftarrow & \operatorname{coim} f & \longleftarrow & N & \longleftarrow & \operatorname{im} f & \longleftarrow & 0 \end{array}$$

Example 1.2. (1) The category Ab of abelian groups is abelian.
(2) For any ring A with unit (not necessarily commutative), the category $\mathrm{Mod}(A)$ of A-modules is an abelian category.
(3) For any scheme X the category $\mathbf{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules is an abelian category.

Recall that an additive functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories is a functor for whom the morphisms on Hom sets are homomorphisms. We say that F is left exact if for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

the functor F preserves exactness in the first two places:

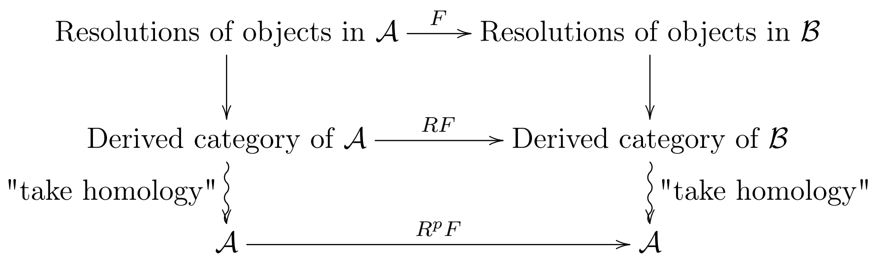
$$0 \to F(M') \to F(M) \to F(M'') \tag{1}$$

[1] One of course could (reasonably) argue that higher categorical structures provide an even more natural setting.

So for example, for X, Y schemes and $f : X \to Y$ a morphism, the push-forward functor $f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ is left exact.

In this situation (with some additional technical assumptions) one can associate derived functors to $F : \mathcal{A} \to \mathcal{B}$: this is a collection of additive functors $R^pF : \mathcal{A} \to \mathcal{B}$ for $p \geq 0$ with $R^0F = F$. For example, in the above example we get the higher direct image functors R^pf_*. The main point of derived functors is that they repair the failure of exactness in (1) by embedding it in a long exact sequence, and their applications are ubiquitous.

If you recall, the derived functors $R^pF(M)$ are computed by taking a particularly nice resolution of M, applying F, and then taking homology; afterwards we forget the resolution. The main defect of the derived functors picture is that there is additional information contained in the resolution which, on the one hand, doesn't depend on the resolution (and is therefore a homological invariant of M), but on the other hand cannot be reconstructed from the objects $R^pF(M)$ alone. Keeping track of resolutions is also the wrong thing to do, because there is a lot about the resolution we *do* want to suppress. The derived category is an intermediate which forgets "just enough": it remembers all the homological information while remembering as little about the resolutions as possible.



Slightly more precisely, to any abelian category $\mathcal{A}$ we associate a category $D(\mathcal{A})$ which contains $\mathcal{A}$ as a full subcategory and such that any left exact functor $F : \mathcal{A} \to \mathcal{B}$ yields a *single* functor $RF : D(A) \to D(B)$. This functor is exact in a particular way, and all of the derived functors R^pF can be recovered from it by "taking homology." Likewise, a right exact functor $G : \mathcal{A} \to \mathcal{B}$ yields a functor $LG : D(A) \to D(B)$.

Why is this useful? There are at least 3 reasons:

- Historically the derived category was introduced by Grothendieck and Verdier to formulate Serre duality for singular spaces: for X an arbitrary variety of dimension n, we want there to be a sheaf $\omega_X \in \mathbf{QCoh}(X)$ so that $H^i(X, F) \cong \mathrm{Ext}_X^{n-i}(F, \omega_X)^\vee$. This is only true for X Cohen–Macaulay, but on the level of derived categories is true in absurd generality. Likewise, there is a relative version for a map $f : X \to Y$ but once again is best understood on the derived category.
- In retrospect, it will be clear for a variety of reasons that derived categories "fix" many aspects of homological algebra. For example, given a morphism $f : X \to Y$ of varieties, the functors f_* and f^* are naturally adjoint:

  $$\mathrm{Hom}_Y(E, f_*F) \cong \mathrm{Hom}_X(f^*E, F)$$

  The relationship between their derived functors is however given by a complicated spectral sequence, and it certainly isn't true that L^pf^* and R^pf_* are adjoint in any sense. But it *is* true in the derived category:

  $$\mathrm{Hom}_{D(Y)}(E, Rf_*F) \cong \mathrm{Hom}_{D(X)}(Lf^*E, F)$$

  Somehow the morphisms in the derived category secretly encode the spectral. Here we denote $D(X) = D(\mathbf{QCoh}(X))$, and likewise for Y.

For another example, for any two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we of course have $(g \circ f)_* = g_* \circ f_*$, and there is a Leray spectral sequence between the higher direct images $R^p g_*(R^q f_*(-)) \Rightarrow R^{p+q}(g \circ f)_*$, and on the level of derived categories this simply means formation of Rf_* commutes with composition:

$$Rg_* \circ Rf_* = R(g \circ f)_*$$

- Derived categories actually naturally come up. This happens abstractly in many contexts, but the simplest case I can give you is the Riemann–Hilbert correspondence, which is an equivalence between vector bundles with flat connection on a variety X and local systems (*i.e.* locally constant $\mathbb{Z}_X$-modules) given by taking flat sections. To extend this equivalence to more general sheaves of solutions to differential equations, the image really needs to be in the derived category of $\mathbb{Z}_X$-modules.

2. Derived categories as geometry

Let X be a variety. A common way to understand the geometry of X is by understanding linear objects on it. By the following classical theorem, the category $\mathbf{QCoh}(X)$ of all such linear objects completely describes X:

Theorem 1 (Gabriel, Rosenberg)**.** *Let X, Y be quasi-separated schemes.*

1. *The categories $\mathbf{QCoh}(X)$ and $\mathbf{QCoh}(Y)$ of quasi-coherent sheaves on X and Y are equivalent as abelian categories if and only if $X \cong Y$.*
2. *Moreover, every equivalence $F : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ is of the form $F = f^*(-) \otimes L$ for $f : X \to Y$ an isomorphism and L a line bundle on X.*

Remark 2. By an equivalence of abelian categories A, A' we just mean is an additive equivalence, *i.e.* an equivalence $F : A \to A'$ such that the maps on Homs are homomorphisms. This implies that F respects kernels and cokernels.

Remark 3. There is an analogous statement in the relative setting too: For $S = \operatorname{Spec} A$ an affine scheme, and $X, Y/S$ quasi-separated schemes over S, X and Y are isomorphic over S if and only if $\mathbf{QCoh}(X)$ and $\mathbf{QCoh}(Y)$ are equivalent as A-linear categories.

Corollary 4. *The group of autoequivalences of $\mathbf{QCoh}(X)$ is precisely*

$$\operatorname{Aut}(X) \rtimes \operatorname{Pic}(X).$$

At the same time, Theorem 1 is discouraging because it means that $\mathbf{QCoh}(X)$ is uninteresting as an invariant, in the sense that $\mathbf{QCoh}(X)$ is not particularly amenable to classification and there is no hope of varieties having equivalent categories of quasi-coherent sheaves exhibiting interesting geometry.

In this context, it is natural to ask the same about the derived category:

Question 2.1. *To what extent does $D(X) := D(\mathbf{QCoh}(X))$ determine X?*

This question has been actively investigated ever since Bondal and Orlov proved the following celebrated theorem:

Theorem 5 (Bondal–Orlov)**.** *Suppose X is a smooth variety with either ω_X or $\omega_X^\vee$ ample. Then*

1. *X is determined by $D(X)$;*
2. *Any equivalence $D(Y) \xrightarrow{\cong} D(X)$ is of the form $f^*(-) \otimes L$ (modulo shifts) for an isomorphism $f : X \to Y$ and a line bundle L on X;*
3. $\operatorname{Aut}_0(D(X)) = \operatorname{Aut}(X) \rtimes \operatorname{Pic}(X)$ *(modulo shifts).*

Here the shift is an extra autoequivalence any derived category automatically has.

Thus, to find classes of varieties with potentially interesting derived equivalences we must specialize to those for which ω_X is neither ample nor anti-ample. One might guess that those with *trivial* canonical bundle might be the most likely to exhibit interesting behavior, and this is indeed the case.

In a seminal series of papers in the 80s, Mukai first raised the above question and studied it for K3 surfaces and abelian varieties. Recall that a K3 surface is a smooth simply connected projective surface X with $\omega_X \cong \mathcal{O}_X$. We then have, by work of a long list of people (among them at least Mukai, Huybrechts, Yoshioka, Toda, Bridgeland, Bayer, in no particular order),

Theorem 6. *Let X be a K3 surface.*

(1) *Any variety Y with $D(X) \cong D(Y)$ is also a K3 surface;*
(2) *There are only finitely such Y for a fixed X;*
(3) *All such Y are moduli spaces of sheaves on X;*
(4) *For Y another K3 surface, $D(X) \cong D(Y)$ if and only if $H^*(X,\mathbb{Z}) \cong H^*(Y,\mathbb{Z})$ in a particular way—you can think of this as a derived Torelli theorem;*
(5) *For X general (*i.e. $\mathrm{Pic}(X) \cong \mathbb{Z}$*),* $\mathrm{Aut}(D(X))$ *is the semi-direct product of* $\mathrm{Aut}(X)$ *be a large group of autoequivalences associated to special vector bundles on X, including* $\mathrm{Pic}(X)$.

At the moment, a description of the group of autoequivalences in (5) for the higher Picard rank is conjectured but not known.

3. Abelian varieties

Most of the results in Theorem 6 require a lot of hard work; an easier class of varieties with $\omega_X \cong \mathcal{O}_X$ is provided by abelian varieties, which will be the focus of this course. This is *not* to say that abelian varieties are less interesting. On the contrary, abelian varieties lie at the intersection of a huge number of fields of mathematics, and can therefore be studied from many different angles. Conversely, understanding sheaves on them has powerful implications in those fields.

Definition 3.1. An abelian variety A is a projective variety A with an algebraic group structure.

Example 3.2. In dimension 1, abelian varieties are elliptic curves, which can also be thought of as

(1) plane cubics $y^2 = x^3 + ax + b$ (in characteristic different from 2 or 3);
(2) genus 1 curves with a marked point;
(3) over $\mathbb{C}$, $\mathbb{C}/\Lambda$ for $\Lambda \subset \mathbb{C}$ a lattice.

Example 3.3. In dimension $g > 1$ there isn't as simple a description, but we at least have a few examples:

(1) over $\mathbb{C}$, $\mathbb{C}^g/\Lambda$ is an abelian variety for certain lattices $\Lambda \subset \mathbb{C}^g$;
(2) for a curve C, its Jacobian $\mathrm{Jac}(C)$ is an abelian variety;
(3) for A an abelian variety, the identity component of $\mathrm{Pic}(A)$ is an abelian variety called the dual abelian variety $A^\vee$.

There is a wealth of reasons why abelian varieties are worth studying. To name a few:

- They have interesting geometry in their own right.
- They come up in the geometry of other varieties. The embedding of a curve C in it's Jacobian $\mathrm{Jac}(C)$ is extremely useful, and more generally any variety

X admits a canonical map to an abelian variety $\mathrm{Alb}(x)$ called the Albanese of X. More generally still, one can often associate other abelian varieties called intermediate Jacobians to X. For instance, Clemens and Griffiths prove the generic cubic threefold is irrational by studying its intermediate Jacobian.

- For abelian varieties A/K over number fields K, the group $A(K)$ of rational points is arithmetically interesting. By the Mordell–Weil theorem this group is finitely generated, and the rational points of more general varieties X/K can be controlled via the map $X \to \mathrm{Alb}(X)$. This is, for instance, how Faltings' theorem is proven.
- They have a well-behaved and geometrically interesting moduli space.

Last but not least,

- Their derived categories exhibit interesting and understandable equivalences.

Specifically, we'll be proving essentially all of the analogs of Theorem 6 for abelian varieties. First, a result of Mukai shows that every abelian variety is derived equivalent to its dual in a beautiful way:

Theorem 7 (Mukai). *For any abelian variety A, there is a natural equivalence $D(A) \cong D(A^\vee)$.*

This proof of this theorem is the first instance of a "Fourier–Mukai" functor, which can be thought of as a categorification of the classical Fourier transform. Next we'll classify when abelian varieties are derived equivalent:

Theorem 8 (Orlov). *Let A be an abelian variety. Any variety A' with $D(A) \cong D(A')$ is an abelian variety, and moreover there are only finitely many such A' and they are precisely described.*

Finally, we'll exactly compute the group of derived autoequivalences of an abelian variety:

Theorem 9 (Orlov, Polishchuk). *Let A be an abelian variety. The group $\mathrm{Aut}(D^b(A))$ is explicitly describable.*

Let me end with a list of some further directions:

- As we'll see, the Fourier–Mukai transform is a powerful tool in the study of sheaves on abelian varieties. In particular, it is very useful in understanding the geometry of moduli spaces of sheaves on abelian varieties.
- More generally, Birdgeland stability conditions are stability structures defined on the derived category, and are extraordinarily useful in understanding the birational geometry of moduli spaces of sheaves on many varieties (so far at the very least K3 surfaces, abelian varieties, and $\mathbb{P}^2$) via wall-crossing. They're also a useful tool for understanding the group of derived autoequivalences.
- Derived categories of coherent sheaves turn out to be an interesting birational invariant. For example, it is conjectured that any two birational varieties with $\omega_X = \mathcal{O}_X$ are derived equivalent.
- Derived categories of coherent sheaves are one side of Kontsevich's homological version of mirror symmetry; the other side is another triangulated category associated to the symplectic geometry of the mirror.
- One way of making sense of the idea of "noncommutative geometry" is using derived categories of coherent sheaves.
- Derived categories naturally come up (as we've seen) in the theory of D-modules.