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Poincaré bundle and Fourier-Mukai transforms

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0 Reminders

We are going to recall some facts of the previous two talks which we will use.

0.1 The Poincaré bundle

Let A be an abelian variety over an algebraically closed field k of dimension g and let $\mathcal{L} \in \operatorname{Pic}(A)$ be ample. Consider the *Mumford line bundle*

$$\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \in \operatorname{Pic}(A \times A),$$

where $p_i\colon A \times A \to A$, for $i = 1, 2$, are the natural projections. Let $A^\vee \simeq A/K(L)$ be the dual abelian variety and denote by $\pi\colon A \to A^\vee$ the canonical projection. Using descent data on $\Lambda(\mathcal{L})$ for the flat morphism

$$(\operatorname{id} \times \pi)\colon A \times A \to A \times A^\vee,$$

we constructed a line bundle $\mathcal{P}$ on $A \times A^\vee$, called the *Poincaré bundle*, such that $(\operatorname{id} \times \pi)^*\mathcal{P} = \Lambda(\mathcal{L})$, and satisfying the following properties:

1. $\mathcal{P}|_{e_A \times A^\vee} \simeq \mathcal{O}_{A^\vee}$.
2. For all $\xi \in A^\vee$, $\mathcal{P}_\xi := \mathcal{P}|_{A\times\xi} \in \operatorname{Pic}^0(A)$.
3. The map $\xi \to \mathcal{P}_\xi$ is a group isomorphism $\lambda\colon A^\vee \xrightarrow{\sim} \operatorname{Pic}^0$.
4. $\mathcal{P}$ satisfies a universal property.

Last time, property 4 was stated but not proved. We will prove it here. (See Section 3 for the precise statment and proof of the universal property of the Poncaré bundle.)

0.2 Some exact functors between derived categories

Let X be a smooth algebraic variety over an algebraically closed field k. We denote by $\mathrm{D}(X)$ the bounded derived category of coherent sheaves on X.
Any morphism $f\colon X \to Y$ between varieties as above induces two exact functors between their bounded derived categories:
The direct image functor

$$Rf_*\colon \mathrm{D}(X) \to \mathrm{D}(Y)$$

and the inverse image functor

$$Lf^*\colon \mathrm{D}(Y) \to \mathrm{D}(X)$$

which is left adjoint to Rf_*.
Also, an object $\mathcal{E} \in \mathrm{D}(Y)$ defines both the derived tensor product

$$\otimes^{\mathbb{L}}\mathcal{E}\colon \mathrm{D}(Y) \to \mathrm{D}(Y)$$

and the derived Hom, which is its right adjoint

$$RHom_f\colon \mathrm{D}(X) \to \mathrm{D}(Y).$$

We will use these standard derived functors to introduce a new large class of exact functors between $\mathrm{D}(X)$ and $\mathrm{D}(Y)$, namely the so called *Fourier-Mukai transforms*.

1 Fourier-Mukai transforms

We will assume that all our varieties are defined over an algebraically closed field k of characteristic 0.

Definition 1.1. Let X and Y be smooth and complete algebraic varieties and let $K \in \mathrm{D}(X \times Y)$. We define the *Fourier-Mukai transform with kernel* K to be the functor

$$\phi_K \colon \mathrm{D}(X) \to \mathrm{D}(Y)$$

given by

$$\mathcal{F} \mapsto Rp_{2_*}(Lp_1^* \mathcal{F} \otimes^{\mathbb{L}} K)$$

where $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ are the natural projections.

We do a couple of remarks:

Remark 1.2. Note that the Fourier-Mukai transform ϕ_K is a composition of exact functors and hence it is exact. (Recall that being an exact functor between triangulated categories means that it takes triangels to triangles.)

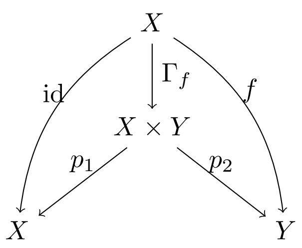
Remark 1.3. Note the analogy with classical integral transforms with kernel: These are given by

$$f(x) \to g(y) = \int_x f(x)k(x,y)\mathrm{d}x.$$

Here, the function of two variables $k(x,y)$ is called the kernel of the integral transform.
Indeed, taking the derived push forwrad Rp_{2_*} correponds to integrating, the derived pullback Lp_1^* corresponds to viewing the function $f(x)$ as a function of two variables x, y, and tensoring corresponds to taking the product.
If we look at the classical Fourier transform, which is the integral transform with kernel $k(x,y) = e^{2\pi i x y}$, one of its most important properties is that it is invertible. This can be seen analogous to the fact that the Fourier Mukai transform with kernel the Poincaré bundle gives us an equivalence between the derived category of coherent sheaves of an abelian variety and that of its dual. (See Section 2.)

Many of the usual functors between derived categories are of Fourier-Mukai type, as can be seen in the following examples.

Example 1.4. The derived direct image Rf_* of a morphism $f \colon X \to Y$ is naturally isomorphic to $\phi_{\mathcal{O}_{\Gamma_f}}$, the Fourier-Mukai transform with kernel the structure sheaf of the graph $\Gamma_f \subset X \times Y$ of f. Indeed, look at the following commuting diagram:



We have

$$\begin{aligned}
\phi_{\mathcal{O}_{\Gamma_f}}(\cdot) &= Rp_{2_*}(Lp_1^*(\cdot) \otimes^{\mathbb{L}} \mathcal{O}_{\Gamma_f}) \\
&= Rp_{2_*}(Lp_1^*(\cdot) \otimes^{\mathbb{L}} R\Gamma_{f\,*}(\mathcal{O}_X)) \\
&\overset{\text{P.F}}{=} Rp_{2_*} R\Gamma_{f\,*}(\mathcal{O}_X \otimes^{\mathbb{L}} L\Gamma_f^* Lp_1^*(\cdot)) \\
&= Rf_*(\cdot)
\end{aligned}$$

where we have used the derived version of the Projection Formula P.F (see the talk on exact functors).

Example 1.5. Let K be a sheaf on $X \times Y$ flat over X (e.g. K a line bundle on $X \times Y$). We can think of K as a family of sheaves parametrized by X. In this case, we can compute the image of the structure sheaf at a closed point $x \in X$ as $\phi_K(k(x)) = K_{x \times Y}$.
Indeed, we have

$$\begin{aligned} \phi_K(k(x)) &= Rp_{2_*}(Lp_1^* k(x) \otimes^{\mathbb{L}} K) \\ &= Rp_{2_*}(\mathcal{O}_{x \times Y} \otimes^{\mathbb{L}} K) \\ &= K_{x \times Y}. \end{aligned}$$

Note that p^* for smooth p is exact, hence p^* of a skyscraper sheaf in the derived and regular senses are identical. Also, flatness of K over X ensures that the tensor product is exact, so its higher derived functors vanish. Hence, the above string of equalities holds. We see that $k(x)$ thus behaves like a delta function at x.

Example 1.6. Let $\mathcal{E} \in \mathrm{D}(X)$. The functor

$$\mathcal{E} \otimes^{\mathbb{L}} (\cdot) \colon \mathrm{D}(X) \to \mathrm{D}(X)$$

is of Fourier-Mukai type. Its Kernel is $\Delta_* \mathcal{E}$ where $\Delta \colon X \to X \times X$ denotes the diagonal embedding morphism.
Indeed, we have

$$\phi_{\Delta_* \mathcal{E}}(\cdot) = Rp_{2_*}(Lp_1^*(\cdot) \otimes^{\mathbb{L}} \Delta_* \mathcal{E}) \stackrel{\text{P.F}}{=} Rp_{2_*} \Delta_* \mathcal{E} \otimes^{\mathbb{L}} (\cdot) = \mathcal{E} \otimes^{\mathbb{L}} (\cdot).$$

In particular, the Serre functor, which is the exact equivalence

$$S_X(\cdot) = (\cdot) \otimes^{\mathbb{L}} \omega_X[\dim X],$$

where ω_X denotes the canonical line bundle of X, is of Fourier-Mukai type.

The next proposition shows that being of Fourier-Mukai type is closed under compositions.

Proposition 1.7. Let X, Y and Z varieties as above. Let $K \in \mathrm{D}(X \times Y)$ and $L \in \mathrm{D}(Y \times Z)$. Then we have an isomorphism of functors

$$\phi_L \circ \phi_K \simeq \phi_{K * L} \colon \mathrm{D}(X) \to \mathrm{D}(Z)$$

where

$$K * L = Rp_{13*}(Lp_{23}^* L \otimes^{\mathbb{L}} Lp_{12}^* K).$$

The projections p_{ij} are the natural ones appearing in the following commuting diagram:

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{p_{12}} & X \times Y \\ {\scriptstyle p_{23}} \downarrow & & \downarrow \\ Y \times Z & \longrightarrow & Y \end{array}$$

Proof. See [Huy06][Prop. 5.10]. □

Definition 1.8. The operation $K * L$ is called the *convolution* of the kernels K and L.

Remark 1.9. More generally, an object $K \in \mathrm{D}(X \times Y)$ defines a family of functors

$$\phi_K \colon \mathrm{D}(X \times S) \to \mathrm{D}(Y \times S)$$

where S is any scheme. If we let

$$
\begin{array}{ccc}
 & X \times Y \times S & \\
 {}^{p_{s1}}\swarrow & & \searrow^{p_{s2}} \\
X \times S & & Y \times S
\end{array}
$$

be the natural projections, then

$$\phi_K(\mathcal{F}) := Rp_{s2*}(Lp_{s2}^*(\mathcal{F}) \otimes^{\mathbb{L}} K).$$

We still have a natural isomorphism of functors

$$\phi_L \circ \phi_K \simeq \phi_{K*L} \colon \mathrm{D}(X \times S) \to \mathrm{D}(Z \times S).$$

Naturality here means that they commute with derived pullback functors associated to morphisms $g \colon S \to S'$.

2 Equivalence of categories $\mathrm{D}(A) \simeq \mathrm{D}(A^\vee)$

Let A be an abelian variety of dimension g, $A^\vee$ its dual abelian variety and $\mathcal{P}$ the Poincaré bundle over $A \times A^\vee$. We start with an adjunction formula:

Lemma 2.1. For every scheme S, the functor

$$\phi_{\mathcal{P}^{-1}[g]} \colon A \times S \to A^\vee \times S$$

is left adjoint to

$$\phi_{\mathcal{P}} \colon A \times S \to A^\vee \times S.$$

Proof. Let $\mathcal{F} \in \mathrm{D}(A^\vee \times S)$ and $\mathcal{G} \in \mathrm{D}(A \times S)$. We have

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(A^\vee \times S)}(\mathcal{F}, \phi_{\mathcal{P}}) &= \mathrm{Hom}_{\mathrm{D}(A^\vee \times S)}\left(\mathcal{F}, Rp_{s2*}(Lp_{s1}^*\mathcal{G} \otimes^{\mathbb{L}} \mathcal{P})\right) \\
&\overset{Lp^* \vdash Rp_*}{\simeq} \mathrm{Hom}_{\mathrm{D}(A \times A^\vee \times S)}\left(Lp_{s2}^*\mathcal{F}, Lp_{s1}^*\mathcal{G} \otimes^{\mathbb{L}} \mathcal{P}\right) \\
&\overset{\otimes^{\mathbb{L}} \vdash Rhom}{\simeq} \mathrm{Hom}_{\mathrm{D}(A \times A^\vee \times S)}\left(Lp_{s2}^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}, Lp_{s1}^*\mathcal{G}\right) \\
&\overset{\text{S.D}}{\simeq} \mathrm{Hom}_{\mathrm{D}(A \times A^\vee \times S)}\left(Lp_{s1}^*\mathcal{G}, Lp_{s2}^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}[2g]\right)^\vee \\
&\overset{Lp^* \vdash Rp_*}{\simeq} \mathrm{Hom}_{\mathrm{D}(A \times S)}\left(\mathcal{G}, Rp_{s1*}(Lp_{s2}^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}[2g])\right)^\vee \\
&\overset{\text{S.D}}{\simeq} \mathrm{Hom}_{\mathrm{D}(A \times S)}\left(Rp_{s1*}(Lp_{s2}^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}[2g]), \mathcal{G}[g]\right) \\
&\simeq \mathrm{Hom}_{\mathrm{D}(A \times S)}\left(Rp_{s1*}(Lp_{s2}^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}[g]), \mathcal{G}\right).
\end{aligned}
$$

Here, S.D stands for Serre duality. □

Thw next proposition describes the cohomology of the Poincaré bundle. The equivalence of categories will be a direct consequence of this computation. Hence we do it very explicitely.

Proposition 2.2. Let $A, A^\vee$ and $\mathcal{P}$ as before and write $p_2 \colon A \times A^\vee \to A^\vee$ for the second projection. Then we have

$$R^n p_{2*}\mathcal{P} = \begin{cases} 0 & \text{if } n \neq g \\ k(e_{A^\vee}) & \text{if } n = g \end{cases}$$

where $k(e_{A^\vee})$ denotes the skyscraper sheaf at $e_{A^\vee}$; and

$$\mathrm{H}^n(A \times A^\vee, \mathcal{P}) = \begin{cases} 0 & \text{if } n \neq g \\ k & \text{if } n = g \end{cases}.$$

Note that this gives us the derived direct image $Rp_{2*}(\mathcal{P})$.

Proof. We divide the proof in three steps:

i Claim: For all n, we have $\mathrm{Supp}\,(R^n p_{2*}(\mathcal{P})) \subset \{e_{A^\vee}\}$.

Proof. Let $\xi \neq e_{A^\vee} \in A^\vee$. Then $\mathcal{P}|_{A\times\xi}$ is a non trivial line bundle on A with class in Pic^0. We know that such sheaves have zero cohomology. Hence, applying Cohomology and base Change Theroem [Har77][Theorem 12.11, Chapter III], we get canonical isomorphisms

$$R^n p_{2*}\mathcal{P} \otimes k(\xi) \simeq \mathrm{H}^n(A \times \xi, \mathcal{P}|_{A\times\xi}) = 0$$

for all n. Moreover, we also get from the Theorem that the higher direct images $R^n p_{2*}\mathcal{P}$ are locally free in a neighberhood of ξ. Hence, the equation above implies that the stalk is also 0. Since this is valid for all $\xi \neq e_{A^\vee}$, our claim follows. □

ii Claim:

$$\begin{cases} R^n p_{2*}\mathcal{P} = 0 & \text{for all } n \neq g \\ \mathrm{H}^n(A \times A^\vee, \mathcal{P}) = 0 & \text{for all } n \neq g. \end{cases}$$

Proof. Since $e_{A^\vee}$ is a zero dimensional subscheme of $A^\vee$, [Har77][Theorem 2.7 and Lemma 2.10, Chapter III] imply that

$$\mathrm{H}^i(A^\vee, R^n p_{2*}\mathcal{P}) = 0 \text{ for all } i \geq 1.$$

Now, applying the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(A^\vee, R^q p_{2*}\mathcal{P}) \Rightarrow \mathrm{H}^{p+q}(A \times A^\vee, \mathcal{P})$$

we get

$$\mathrm{H}^n(A \times A^\vee, \mathcal{P}) \simeq \mathrm{H}^0(A^\vee, R^n p_{2*}\mathcal{P}). \tag{1}$$

Now, p_2 is a projective morphism of dimension g, hence it follows from [Har77][Corollary 11.2, Chapter III] that

$$R^n p_{2*}\mathcal{P} = 0 \text{ for all } n > g.$$

Hence, by 1, we het

$$\mathrm{H}^n(A \times A^\vee, \mathcal{P}) = 0 \text{ for all } n > g.$$

If we consider the morphisms $(1,-1), (-1,1)\colon A^\vee \times A^\vee \to A^\vee \times A^\vee$, from last talk, we get that $\mathcal{P}^{-1} \simeq (1,-1)^*\mathcal{P} \simeq (-1,1)^*\mathcal{P}$. It follows that $\mathcal{P}$ and $\mathcal{P}^{-1}$ have the same cohomology. Hence, by Serre duality we get

$$\mathrm{H}^n(A \times A^\vee, \mathcal{P}) \simeq \mathrm{H}^{2g-n}(A \times A^\vee, \mathcal{P}^{-1})^\vee \simeq \mathrm{H}^{2g-n}(A \times A^\vee, \mathcal{P})^\vee.$$

Therefore

$$\mathrm{H}^n(A \times A^\vee, \mathcal{P}) = 0 \text{ for all } n < g$$

and by 1 and the fact that $\mathrm{Supp}(R^n p_{2*}\mathcal{P}) \subset e_{A^\vee}$ we conclude that

$$R^n p_{2*}\mathcal{P} = 0 \text{ for all } n < g \text{ as well.}$$

This finishes the proof of the claim. □

iii Claim: $R^g p_{2*}\mathcal{P} = k(e_{A^\vee})$.

Proof. Since $\mathrm{Supp}(R^n p_{2*}\mathcal{P}) \subset e_{A^\vee}$ te question now becomes local. Let $R = \mathcal{O}_{A^\vee, e_{A^\vee}}$ be the local ring at $e_{A^\vee}$. This is a regular ring of dimension g. We perform the base change for

$$\mathrm{Spec}(R) \to A^\vee.$$

It is known that $Rp_{2*}\mathcal{P}|_{\mathrm{Spec}(R)}$, seen as R-modules, are of finite length. Hence, by the proper base change Theorem in [Pol03][Appendix C], these are calculated by a finite complex

$$K^\bullet \colon 0 \to K_0 \to \cdots \to K_g \to 0$$

of free finitely generated R-modules with

$$\mathrm{H}^i(K^\bullet) = 0 \text{ for } 0 \le i < g.$$

We calculate $\mathrm{H}^g(K^\bullet) \simeq R^g p_{2*}\mathcal{P}|_{\mathrm{Spec}(R)}$:
We have an exact sequence

$$0 \to K_0 \to \cdots \to K_g \to \mathrm{H}^g(K^\bullet) \to 0.$$

On the other hand, dualizing and applying the same argument we get an exact sequence

$$0 \to K_g^\vee \to \cdots \xrightarrow{d} K_0^\vee \to K \to 0$$

where $K_i^\vee = \mathrm{Hom}_R(K_i, R)$ and $K = \mathit{coker}(d)$. Last talk, it was shown that $\mathit{coker}(d) \simeq R/I$ where I is the ideal corresponding to the maximal subsecheme S of $A^\vee$ such that $\mathcal{P}|_{S \times A}$ is trivial. But this is $e_{A^\vee}$ and hence $R/I = R/m_{e_{A^\vee}} = k$. Hence, we get that $K^{\vee\bullet}$ is a free resolution of k. Dualizing, we obtain

$$0 \to K_0 \to \cdots \to K_g \to \mathrm{Ext}'(k, R) \to 0$$

and thus

$$R^g p_{2*}\mathcal{P}|_{\mathrm{Spec}(R)} \simeq \mathrm{H}^g(K^\bullet) \simeq \mathrm{Ext}'(k, R) \simeq k.$$

□

This finishes the proof of the Proposition. □

Theorem/Corollary 2.3. The natural adjunction morphism

$$\mathrm{id}_{\mathrm{D}(A^\vee)} \to \phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^{-1}[g]}$$

is an isomorphism.

Proof. We have

$$\phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^{-1}[g]} \simeq \phi_{\mathcal{P}^{-1}[g] * \mathcal{P}}.$$

Hence, by 1.6 it suffices to show that the convolution $\mathcal{P}^{-1}[g] * \mathcal{P}$ is isomorphic to $R\Delta_*\mathcal{O}_{A^\vee}$, the derived push forward of the structure sheaf of $A^\vee$ under the diagonal morphism $\Delta \colon A^\vee \to A^\vee \times A^\vee$. We look at the following diagrams:

$$\begin{CD} A \times A^\vee \times A^\vee @>{p_{13}}>> A \times A^\vee \\ @V{p_{12}}VV @VV{p_1}V \\ A \times A^\vee @>>{p_2}> A \end{CD} \qquad\qquad \begin{CD} A \times A^\vee \times A^\vee @>{p_{23}}>> A^\vee \times A^\vee \\ @V{(\mathrm{id} \times d)}VV @VV{d}V \ \Delta \\ A \times A^\vee @>>{p_2}> A^\vee \end{CD}$$

where $d\colon A^\vee \times A^\vee \to A^\vee$ is the difference morphism $(a, \bar{a}) \mapsto a - \bar{a}$.
We have

$$\begin{aligned}
\mathcal{P}^{-1}[g] * \mathcal{P} &= Rp_{23*}(Lp_{13}^*\mathcal{P}^{-1}[g] \otimes^{\mathbb{L}} Lp_{12}^*\mathcal{P}) \\
&= Rp_{23*}(d \times \mathrm{id})^*\mathcal{P}[g] \quad \text{by the theorem of the cube} \\
&= Ld^* Rp_{2*}\mathcal{P}[g] \quad \text{by change of coordinates on } A^\vee \times A^\vee,\ (a, \bar{a}) \mapsto (a, d(a, \bar{a})) \\
&= Ld^* \mathcal{O}_{e_{A^\vee}} \quad \text{by the previous Proposition} \\
&= R\Delta_* \mathcal{O}_{A^\vee}.
\end{aligned}$$

□

Remark 2.4. The isomorphism above also holds for families, i.e. for every scheme S, we have a natural isomorphism of functors

$$\mathrm{id}_{\mathrm{D}(A^\vee \times S)} \to \phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^{-1}[g]}$$

where now $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{P}^{-1}[g]}$ are functors on families.

Remark 2.5. Similarly, we can show that there is a natural isomorphism of functors

$$\phi_{\mathcal{P}^{-1}[g]} \circ \phi_{\mathcal{P}} \simeq \mathrm{id}_{\mathrm{D}(A \times S)}.$$

It follows that there is an equivalence of categories

$$\mathrm{D}(A^\vee \times S) \simeq \mathrm{D}(A \times S).$$

3 Universal property of the Poncaré bundle

We will use our previous results to prove the universal property of the Poincaré bundle.

Theorem 3.1. *Let A be an abelian variety over an algebraically closed field k of characteristic 0. The dual abelian variety $A^\vee$ represents the functor from the category of schemes over k to the category of sets given by*

$$S \to \{L \in \mathrm{Pic}(A \times S) \mid L|_{A \times \{s\}} \in \mathrm{Pic}^0(A)\, \forall\, s \in S,\ L|_{\{e_A\} \times S} \simeq \mathcal{O}_S\}$$

so that $\mathcal{P} \in \mathrm{Pic}(A \times A^\vee)$ corresponds to the identity morphism $A^\vee \to A^\vee$. i.e. a familiy L as above corresponds to a morphism $S \xrightarrow{f} A^\vee$ such that $L \simeq (\mathrm{id} \times f)^*\mathcal{P}$.

Proof. Let S be a scheme and L a line bundle on $A \times S$ such that for all $s \in S$, $L|_{A \times \{s\}} \in \mathrm{Pic}^0(A)$ and $L|_{\{e_A\} \times S} \simeq \mathcal{O}_S$.
We want to construct a morphism $f : S \to A^\vee$ such that $L \simeq (\mathrm{id} \times f)^*\mathcal{P}$. For this, we will evaluate $\phi_{\mathcal{P}}(L) \in \mathrm{D}(A^\vee \times S)$. Now, for all $s \in S$, we have

$$L|_{A \times \{s\}} = \mathcal{P}_{A \times \{\xi\}} := \mathcal{P}_\xi \text{ for some } \xi \in A^\vee.$$

Using $\mathrm{id} \simeq \phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^{-1}[g]}$, we get

$$\begin{aligned}
\mathcal{O}_{-\xi}[-g] &\simeq \phi_{\mathcal{P}}(Rp_{1*}(Lp_2^*\mathcal{O}_{-\xi}[-g] \otimes^{\mathbb{L}} \mathcal{P}^{-1}[g]) \\
&\simeq \phi_{\mathcal{P}}(\mathcal{P}^{-1}|_{A \times \{-\xi\}}) \\
&\simeq \phi_{\mathcal{P}}(\mathcal{P}_\xi).
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi_{\mathcal{P}}(L)|_{A^\vee \times \{s\}} &\simeq \phi_{\mathcal{P}}(L|_{A^\vee \times \{s\}}) \\
&\simeq \mathcal{O}_{-\xi}[-g].
\end{aligned}$$

Therefore, we have that that $\phi_{\mathcal{P}}(L) = \mathcal{F}[-g]$ for some coherent sheaf $\mathcal{F}$ on $A^\vee \times S$ such that for all $s \in S$,

$$\mathcal{F}|_{A^\vee \times \{s\}} \simeq \mathcal{O}_{-\xi} \tag{2}$$

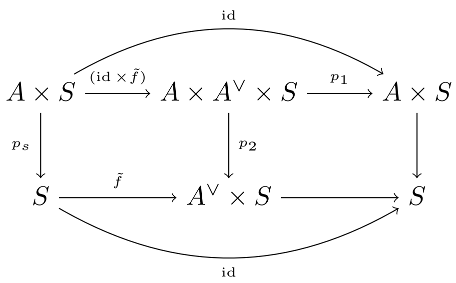
for a point $\xi \in A^\vee$.
Hence, $\mathcal{F}$ is finitely generated as an $\mathcal{O}_S$-modules.
On the other hand, one can show (see [Pol03][Chapter11]) that $\mathcal{F}$ is also flat over S.
We conclude that $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module and by 2, we get that $\mathcal{F}$ is a line bundle supported on the graph of a morphism

$$\tilde{f} \colon S \to A^\vee.$$

If we denote the graph morphism $S \to A^\vee \times S$ also by $\tilde{f}$, then the conclusion above means that we can write

$$\mathcal{F} \simeq \tilde{f}_* K$$

for some line bundle K on S. Consider the following commuting diagram



Set

$$f = [-1]_{A^\vee} \circ \tilde{f}.$$

We compute

$$\begin{aligned}
\phi_{\mathcal{P}^{-1}}(\mathcal{F}) &= Rp_{1*}(Lp_2^*\mathcal{F} \otimes^{\mathbb{L}} \mathcal{P}^{-1}) \\
&\simeq Rp_{1*}(Lp_2^* f_* K \otimes^{\mathbb{L}} \mathcal{P}^{-1}) \\
&\simeq Rp_{1*}(R(\mathrm{id} \times \tilde{f})_* Lp_s^* K \otimes^{\mathbb{L}} \mathcal{P}^{-1}) \\
&\overset{\text{P.F}}{\simeq} Rp_{1*}(R(\mathrm{id} \times \tilde{f})_*(Lp_s^* K \otimes^{\mathbb{L}} L(\mathrm{id} \times f)^* \mathcal{P}^{-1})) \\
&\simeq Lp_s^* K \otimes^{\mathbb{L}} L(\mathrm{id} \times \tilde{f})^* \mathcal{P}^{-1} \\
&\simeq p_s^* K \otimes (\mathrm{id} \times \tilde{f})^* \mathcal{P}^{-1} \\
&\simeq p_s^* K \otimes (\mathrm{id} \times f)^* \mathcal{P}.
\end{aligned}$$

On the other hand we have the isomorphism

$$\phi_{\mathcal{P}^{-1}}(\mathcal{F}) \simeq \phi_{\mathcal{P}^{-1}[g]} \circ \phi_{\mathcal{P}}(L) \simeq L.$$

Hence,

$$L \simeq \phi_{\mathcal{P}^{-1}}(\mathcal{F}) \simeq (\mathrm{id} \times f)^* \mathcal{P} \otimes p_2^* K$$

and restricting both sides to $\{e_A\} \times S$ we get $K \simeq \mathcal{O}_S$. Our claim follows.

□

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