Fourier–Mukai transforms II
Orlov's criterion

Gregor Bruns

07.01.2015

1 Orlov's criterion

In this note we're going to rely heavily on the projection formula, discussed earlier in Rostislav's talk) and the following derived analogue of a standard algebraic geometry fact:

Lemma 1.1 (Flat base change, [3] p. 85, compatibility v))**.** Suppose we are given a fiber diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{v} & Y \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{u} & Z \end{array}$$

where u is flat and f is proper. Then for all $\mathcal{F} \in D^b(Y)$ we have

$$u^* f_* \mathcal{F} \cong g_* v^* \mathcal{F}$$

We start off with two theorems, given without proofs, which shed some light on the specialness (or generality) of Fourier–Mukai transforms. Last time we saw that such a functor always admits left and right adjoints. In fact, this doesn't seem to be very special, since every exact functor has this property:

Theorem 1.2 (Bondal–van den Bergh)**.** *Let X, Y be smooth projective varieties, $F\colon D^b(X) \to D^b(Y)$ exact. Then F admits right and left adjoints.*

On the other hand, all fully faithful functors already are of Fourier–Mukai type. The result in its original form requires the existence of adjoint functors. This however we already got for free.

Theorem 1.3 (Orlov; [3] Theorem 5.14)**.** *Let X, Y be smooth projective varieties and $F\colon D^b(X) \to D^b(Y)$ be a fully faithful and exact functor. Then F is a Fourier–Mukai transform, i.e. there is a complex $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \simeq \Phi_{\mathcal{P}}$.*

Now we come to the central result of this note, a criterion to test full faithfulness of Fourier–Mukai transforms. In practice it is surprisingly easy to check.

Theorem 1.4 (Orlov's criterion; Bondal–Orlov, [3] Proposition 7.1). *Let X, Y be smooth projective varieties over an algebraically closed field k of characteristic 0 and $\mathcal{P} \in D^b(X \times Y)$ any complex. The functor $\Phi_{\mathcal{P}} \colon D^b(X) \to D^b(Y)$ is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$\operatorname{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(y))[i]) = \begin{cases} k, & x = y \textit{ and } i = 0 \\ 0, & x \neq y \textit{ or } i < 0 \textit{ or } i > \dim(X) \end{cases} \qquad (*)$$

We want to apply the facts we proved earlier about spanning classes (because the collection of $k(x)[i]$ is spanning, [3] Proposition 3.17). Remember the following proposition:

Proposition 1.5 ([3] Proposition 1.49). *Let $F \colon \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories with left and right adjoints $G \dashv F \dashv H$. Suppose Ω is a spanning class of $\mathcal{D}$ such that for all objects $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms*

$$F \colon \operatorname{Hom}(A, B[i]) \to \operatorname{Hom}(F(A), F(B)[i])$$

are bijective. Then F is fully faithful.

Although theorem 1.4 seems like a direct translation of this proposition, it is in fact much more convenient to check, since we don't have to know the spaces

$$\operatorname{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(x))[i]) \cong \operatorname{Ext}^i(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(x)))$$

for $i \in \{1, \ldots, \dim(X)\}$, which are in general difficult to calculate.

Proof of theorem 1.4. Let $F = \Phi_{\mathcal{P}}$. Remember that

$$\operatorname{Hom}(k(x), k(y)[i]) = \begin{cases} k, & x = y \text{ and } i = 0 \\ 0, & x \neq y \text{ or } i < 0 \text{ or } i > \dim(X) \end{cases}$$

To see the last statement, use the fact

$$\operatorname{Hom}(k(x), k(x)[i]) \cong \operatorname{Ext}^i(k(x), k(x)) = 0$$

for $i < 0$ or $i > \dim(X)$. From this we immediately get that full faithfulness implies condition $(*)$. In the other direction, the case $x \neq y$ is clear, as well as $x = y$ and $i \notin \{1, \ldots, \dim(X)\}$.

We divide the proof of the remaining cases in several steps.

1) We first prove that it is enough to show $GF(k(x)) \cong k(x)$ for all x. Lemma 1.21 of [3] states that if $F \colon \mathcal{A} \to \mathcal{B}$, G is the left adjoint of F and $g \colon G \circ F \to \mathrm{id}_{\mathcal{A}}$ is the canonical morphism induced by adjunction, then we have the following commutative diagram for every two objects $A, B \in \mathcal{A}$:

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & & \\ \downarrow \circ g_A & \overset{F}{\searrow} & \\ \mathrm{Hom}(GF(A),B) & \xrightarrow{\cong} & \mathrm{Hom}(F(A),F(B)) \end{array}$$

So the bijectivity of

$$\mathrm{Hom}(k(x),k(x)[i]) \to \mathrm{Hom}(F(k(x)),F(k(x))[i])$$

is equivalent to the bijectivity of

$$\mathrm{Hom}(k(x),k(x)[i]) \xrightarrow{\circ g_{k(x)}} \mathrm{Hom}(GF(k(x)),k(x)[i])$$

Note that the sheaf $k(x)$ is isomorphic to k concentrated at x, since k is algebraically closed. Suppose now we know $GF(k(x)) \cong k(x)$. Then either $g_{k(x)}\colon k(x) \to k(x)$ is an isomorphism (which implies the wanted bijectivity for all i) or it is the zero morphism. But this we can exclude since by Exercise 1.19 of [3] the composition

$$FGF(k(x)) \xrightarrow{F(g_{k(x)})} F(k(x)) \xrightarrow{h_{F(k(x))}} FGF(k(x))$$

is the identity and $\mathrm{End}(F(k(x))) = k$ by assumption ($x = y$ and $i = 0$), implying $F(k(x)) \neq 0$, so the identity is not zero.

2) We prove $GF(k(x)) \cong k(x)$ under the additional two assumptions
   i) $GF(k(x))$ is a sheaf
   ii) $\mathrm{Hom}(k(x),k(x)[1]) \to \mathrm{Hom}(GF(k(x)),k(x)[1])$ is an injection.

   Set $\mathcal{F} = GF(k(x))$. Due to assumption i) $\mathcal{F}$ is a sheaf. Our "gloabel" assumption implies for $y \neq x$ another closed point that

$$\mathrm{Hom}(\mathcal{F},k(y)) \cong \mathrm{Hom}(F(k(x)),F(k(y))) = 0$$

   Therefore $\mathcal{F}$ is concentrated in x. As in the last step, the adjunction morphism $\delta = g_{k(x)}\colon \mathcal{F} \to k(x)$ is nontrivial and hence surjective. Consider the short exact sequence

$$0 \to \ker\delta \to \mathcal{F} \to k(x) \to 0$$

   which shows that $\ker\delta$ is also concentrated in x (since $\mathcal{F}$ and $k(x)$ are). We want $\ker\delta = 0$ and it is enough to show

$$\mathrm{Hom}(\ker\delta,k(x)) = 0$$

   Apply the left-exact contravariant functor $\mathrm{Hom}(-,k(x))$ to the short exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}(k(x),k(x)) \to \mathrm{Hom}(\mathcal{F},k(x)) \\ &\to \mathrm{Hom}(\ker\delta,k(x)) \to \mathrm{Hom}(k(x),k(x)[1]) \xrightarrow{\circ\delta} \mathrm{Hom}(\mathcal{F},k(x)[1]) \end{aligned}$$

The first two terms are just k, so we actually get

$$0 \to \mathrm{Hom}(\ker \delta, k(x)) \to \mathrm{Hom}(k(x), k(x)[1]) \xrightarrow{\circ\delta} \mathrm{Hom}(\mathcal{F}, k(x)[1])$$

where $\circ\delta$ is injective by assumption ii), hence $\mathrm{Hom}(\ker \delta, k(x)) = 0$.

3) We verify hypothesis i) of step 2). This is done by [3] Lemma 7.2, a variation of the spanning class argument. It is proved by a spectral sequence and apart from being technical not very interesting.

4) We verify hypothesis ii) of step 2) for general points x. We already know that the composition of two Fourier–Mukai transforms is again a Fourier–Mukai transform (by convoluting kernels). So $G \circ F$ is again a FMT, say with kernel $\mathcal{Q}$. This is the only step in the proof where we actually use the concrete description of $\mathcal{F}$.

   Let p and q be the two projections of $X \times X$. Now

$$GF(k(x)) = \Phi_{\mathcal{Q}}(k(x)) = q_*(\mathcal{Q} \otimes p^* k(x)) = q_*(\mathcal{Q}|_{\{x\} \times X}) \cong i_x^* \mathcal{Q}$$

   is a sheaf (step 3) concentrated in x (step 2). Here we denote by $i_x \colon \{x\} \times X \to X \times X$ the inclusion and consider $i_x^* \mathcal{Q}$ as a sheaf on the second X via the projection q_*.

   Now we apply the following
   Lemma 1.6 ([3] Lemma 3.31)**.** Let $S \to X$ be an X-scheme, $i_x \colon S_x \to S$ the inclusion of the fiber over x and suppose $\mathcal{Q} \in D^b(S)$ is such that for all $x \in X$ the derived pullback $i^* \mathcal{Q} \in D^b(S_x)$ is a sheaf. Then $\mathcal{Q}$ is a sheaf flat over X.

   So we know our kernel $\mathcal{Q}$ is a sheaf flat over the first factor.

   Now choose a generic $x \in X$. We compose the map

$$\mathrm{Hom}(k(x), k(x)[1]) \to \mathrm{Hom}(F(k(x)), F(k(x))[1])$$

   with the functor G to obtain the map

$$\kappa(x)\colon\ \mathrm{Hom}(k(x), k(x)[1]) \to \mathrm{Hom}(GF(k(x)), GF(k(x))[1])$$

   and we will give an intuitive argument why this is injective (which proves injectivity of the first map).

   Since $\mathcal{Q}$ is flat, we may view $\kappa(x)$ as the Kodaira–Spencer map of the flat family $\mathcal{Q}$ over $X \times X$ ([3] Example 5.4 vii)). The map $f \colon x \mapsto \mathcal{Q}_x$ is injective since every $\mathcal{Q}_x$ is concentrated in x. Hence $\kappa(x)$ (which is the differential of f at x) is generically injective. Here we have used $\mathrm{char}\, k = 0$.

5) It is actually enough to consider general points x, since $\mathcal{Q}$ is flat over X. Hence the Hilbert polynomial of $\mathcal{Q}_x$ is constant "under deformations", i.e. the same for all x. We also know that $\mathcal{Q}_x$ is concentrated in x for any $x \in X$, so this leaves $\mathcal{Q}_x \cong k(x)$ as the only possibility for all x.

□

Corollary 1.7 ([3] Corollary 7.5)**.** Let $\mathcal{P}$ be a coherent sheaf on $X \times Y$, flat over X. Then $\Phi_{\mathcal{P}}$ is fully faithful if and only if

i) For all $x \in X$ one has $\mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_x) \cong k$

ii) If $x \neq y$ then $\mathrm{Ext}^i(\mathcal{P}_x, \mathcal{P}_y) = 0$ for all i.

Proposition 1.8 (Mukai; [3] Proposition 9.19)**.** *Let A be an abelian variety. If $\mathcal{P}$ is the Poincaré bundle on $A \times A^\vee$ then*

$$\Phi_{\mathcal{P}} \colon D^b(A^\vee) \to D^b(A)$$

is an equivalence. Moreover, the composition

$$D^b(A^\vee) \xrightarrow{\Phi_{\mathcal{P}}} D^b(A) \xrightarrow{\Phi_{\mathcal{P}}} D^b(A^\vee)$$

is isomorphic to $(-1)^* \circ [-g]$ where $g = \dim A$.

Proof. Let $\alpha, \beta \in A^\vee$ be closed points. Then the transforms

$$\Phi_{\mathcal{P}}(k(\alpha)) \cong \mathcal{P}_\alpha, \quad \Phi_{\mathcal{P}}(k(\beta)) \cong \mathcal{P}_\beta$$

are line bundles on A. In the case $\alpha \neq \beta$ we need to check $\mathrm{Ext}^i(\mathcal{P}_\alpha, \mathcal{P}_\beta) = 0$ for all i. But

$$\mathrm{Ext}^i(\mathcal{P}_\alpha, \mathcal{P}_\beta) \cong H^i(A, \mathcal{P}_\alpha^\vee \otimes \mathcal{P}_\beta) = 0$$

To see the first equality, combine propositions III.6.3 and III.6.7 from [2]. To see the second, observe $\mathcal{P}_\alpha \not\cong \mathcal{P}_\beta$ and remember that nontrivial line bundles in $\mathrm{Pic}^0(A)$ have no cohomology. This already implies that $\Phi_{\mathcal{P}}$ is fully faithful by corollary 1.7. We don't prove the second assertion since Ana did something similar already in her talk. □

We encounter here already an easy case of the fact that the Fourier–Mukai transform sends torsion sheaves to vector bundles, i.e. we have

$$k(\alpha) \mapsto \mathcal{P}_\alpha$$

This works in greater generality and can for instance be used to interpret facts about vector bundles on elliptic curves (which were completely classified first by Atiyah).

2 Index of line bundles

Throughout the next sections A will be an abelian variety. We denote by $p_1 \colon A \times A^\vee \to A$ and $p_2 \colon A \times A^\vee \to A^\vee$ the projections.

A nondegenerate line bundle (i.e. one for which $K(L)$ is finite) on A has an interesting interesting integral invariant associated to it: its index $i(L)$. This index governs the vanishing of the cohomology groups of L. To prove these things, we first need some preparatory facts.

Proposition 2.1 ([4] Proposition 11.9). *Let* L *be a nondegenerate line bundle on* A *Then one has a canonical isomorphism*

$$\phi_L^* \Phi_{\mathcal{P}}(L) \cong (\pi^* \pi_* L) \otimes L^\vee$$

where $\pi\colon A \to \mathrm{Spec}(k)$ *is the projection to a point.*

Proof. We use flat base change (Lemma 1.1) in the diagram

$$\begin{CD} A \times A @>{\mathrm{id} \times \phi_L}>> A \times A^\vee \\ @V{p_2}VV @VV{p_2}V \\ A @>{\phi_L}>> A^\vee \end{CD}$$

and compute

$$\phi_L^* \Phi_{\mathcal{P}}(L) \cong \phi_L^* p_{2*}(\mathcal{P} \otimes p_1^* L) \cong p_{2*}((\mathrm{id} \times \phi_L)^* \mathcal{P} \otimes p_1^* L)$$

Remember the definition of the Mumford line bundle associated to L:

$$\Lambda(L) = m^* L \otimes p_1^* L^\vee \otimes p_2^* L^\vee$$

and the previously proven fact

$$\Lambda(L) \cong (\mathrm{id} \times \phi_L)^* \mathcal{P}$$

Plugging in yields

$$\phi_L^* \Phi_{\mathcal{P}}(L) \cong p_{2*}(m^* L \otimes p_2^* L^\vee) \cong p_{2*} m^* L \otimes L^\vee$$

using the projection formula. Now we apply flat base change again, this time for the diagram

$$\begin{CD} A \times A @>{m}>> A \\ @V{p_2}VV @VV{\pi}V \\ A @>{\pi}>> k \end{CD}$$

to get

$$p_{2*} m^* L \otimes L^\vee \cong \pi^* \pi_* L \otimes L^\vee$$

which is the result we wanted. □

Theorem 2.2 ([4] Theorem 11.11). *Let* L *be a nondegenerate line bundle. Then there exists an integer* $i(L) \in \{0, \dots, g\}$ *(called the* index *of* L*) and a vector bundle* E *on* $A^\vee$ *such that*

$$\Phi_{\mathcal{P}}(L) \cong E[-i(L)]$$

Proof. Consider the product $A \times A$ and denote by p_1, p_2 the projections and by m the multiplication. We apply the convolution formula

$$\Phi_{\mathcal{P}}(\mathcal{F}) \otimes \Phi_{\mathcal{P}}(\mathcal{G}) \cong \Phi_{\mathcal{P}}(\mathcal{F} \star \mathcal{G})$$

with $\mathcal{F} * \mathcal{G} = m_*(\mathcal{F} \boxtimes \mathcal{G})$ to the sheaves L and $(-1)^* L^\vee$:

$$\Phi_{\mathcal{P}}(L) \otimes \Phi_{\mathcal{P}}((-1)^* L^\vee) \cong \Phi_{\mathcal{P}}(L \star (-1)^* L^\vee) \cong \Phi_{\mathcal{P}}(m_*(L \boxtimes (-1)^* L^\vee))$$

We know make a change of variables on $A \times A$ by setting $x' = x + y$, $y' = -y$, so that m_* becomes p_{1*}, $(-p_2)^*$ becomes p_2^* and p_1^* becomes m^*:

$$m_*(p_1^* L \otimes (-p_2)^* L^\vee) \rightsquigarrow p_{1*}(m^* L \otimes p_2^* L^\vee)$$

Substituting the Mumford line bundle yields

$$p_{1*}(m^* L \otimes p_2^* L^\vee) \cong p_{1*}(\Lambda(L) \otimes p_1^* L) \cong p_{1*}\Lambda(L) \otimes L$$

where we have used the projection formula. We use again the fact

$$\Lambda(L) \cong (\phi_L \times \mathrm{id})^* \mathcal{P}$$

to write

$$p_{1*}\Lambda(L) \cong p_{1*}(\phi_L \times \mathrm{id})^* \mathcal{P} \cong \phi_L^* p_{1*} \mathcal{P} \cong \phi_L^* \mathcal{O}_{\{0\}}[-g] \cong \mathcal{O}_{K(L)}[-g]$$

where we have used flat base change to exchange ϕ_L and p_1 and used the result about p_{1*} from Ana's talk. Putting everything together we obtain

$$L \star (-1)^* L^\vee \cong L_{K(L)}[-g]$$

For our Fourier–Mukai transforms this means

$$\Phi_{\mathcal{P}}(L) \otimes \Phi_{\mathcal{P}}((-1)^* L^\vee) \cong \Phi_{\mathcal{P}}(L_{K(L)})[-g]$$

Observe that $E := \Phi_{\mathcal{P}}(L_{K(L)})$ is a vector bundle of rank $|K(L)|$ on $A^\vee$. By proposition 2.1 we can write the left hand side as

$$\pi^*(\pi_* L \otimes \pi_*(L^\vee)) \cong \Phi_{\mathcal{P}}(L) \otimes \Phi_{\mathcal{P}}((-1)^* L^\vee) \cong E[-g]$$

Since π is the projection to a point, if the pullback of a complex is a vector bundle, then the original complex must have been a sheaf. So $\pi_* L$ is concentrated in one degree. We take the statement

$$\phi_L^* \Phi_{\mathcal{P}}(L) \cong \pi^* \pi_* L \otimes L^\vee$$

of proposition 2.1 and conclude that the RHS is locally free, which means that the LHS is locally free, which means that $\Phi_{\mathcal{P}}(L)$ already has to be locally free (ϕ_L is a covering). □

We now use this result to show that L only has cohomology in degree $i(L)$.

Lemma 2.3. For any homomorphism $f\colon A \to B$ of abelian varieties we have

$$\Phi_{\mathcal{P}_B} \circ f_* \simeq f^{\vee *} \circ \Phi_{\mathcal{P}_A}$$

Corollary 2.4. If L is a nondegenerate line bundle on A then $H^i(A, L) = 0$ for $i \neq i(L)$.

Proof. Apply the previous lemma to $f = \pi\colon A \to \operatorname{Spec} k$. □

Remark 2.5. We can also show that $h^{i(L)}(A, L) = \sqrt{|K(L)|}$. Furthermore, in the complex case $k = \mathbb{C}$, the number $i(L)$ is just the number of negative eigenvalues of the Hermitian form H associated to L (by its first Chern class). Polishchuk ([4]) proves the following facts on page 146:

1) If L and M are nondegenerate and algebraically equivalent then $i(L) = i(M)$.
2) Let $f\colon A \to B$ be an isogeny of abelian varieties and L a nondegenerate line bundle on B. Then f^*L is nondegenerate and we have $i(L) = i(f^*L)$.
3) For a nondegenerate line bundle L and every integer $n > 0$ we have $i(L) = i(L^{\otimes n})$.
4) If L is ample then $H^i(A, L) = 0$ for $i \neq 0$. If L defines a principal polarization then $h^0(A, L) = 1$.

3 WIT sheaves and homogeneous vector bundles

Combining the theorem and remarks from the last section, we get something usually called the index theorem:

Proposition 3.1 (Index theorem)**.** *Let* L *be a nondegenerate line bundle of index* i *on* A*. Then for all* $P \in \mathrm{Pic}^0(A)$ *and for all* $j \neq i$ *we have*

$$H^j(A, L \otimes P) = 0$$

Mukai then used this feature to define

Definition 3.2 (IT sheaf)**.** A coherent sheaf $\mathcal{F}$ on A is called an *IT-sheaf of index* i if for all $j \neq i$ and all $P \in \mathrm{Pic}^0(A)$ we have

$$H^j(A, \mathcal{F} \otimes P) = 0$$

IT-sheaves satisfy the following interesting property:

Lemma 3.3 ([1] Lemma 14.2.1)**.** Let $\mathcal{F}$ be an IT-sheaf of index i. Then

i) $R^j p_{2*}(\mathcal{P} \otimes p_1^*\mathcal{F}) = 0$ for all $j \neq i$
ii) $R^i p_{2*}(\mathcal{P} \otimes p_1^*\mathcal{F})$ is a locally free sheaf of finite rank on $A^\vee$.

Here the functors used are *not* the derived but rather the ordinary ones. In fact, the properties imply that the derived Fourier–Mukai transform is equivalent to the sheaf $R^i p_{2*}(\mathcal{P} \otimes p_1^* \mathcal{F})$. To prove the lemma, we just have to combine Grothendieck's coherence theorem (implying that the higher pushforwards are all coherent) and the "Cohomology and Base Change" theorem, e.g. from [2] chapter III.12.

Property i) in the lemma is certainly weaker than being an IT-sheaf. Still the class of sheaves satisfying this property is interesting in its own right:

Definition 3.4. A coherent sheaf $\mathcal{F}$ on A is called a *WIT-sheaf of index* i if

$$R^j p_{2*}(\mathcal{P} \otimes p_1^* \mathcal{F}) = 0$$

for all $j \neq i$. Here the acronym *WIT* stands for "Weak Index Theorem".

Now we want to study various classes of vector bundles on abelian varieties. The concept of a WIT-sheaf is going to play a role here. We begin with an analogue of the fact that the classical Fourier transform interchanges translation and multiplication by a character.

Lemma 3.5 ([1] Proposition 14.7.8)**.** For each $\mathcal{F} \in D^b(A)$ and all points $x \in A$ and $x^\vee \in A^\vee$ we have

i) $\Phi_{\mathcal{P}}(\mathcal{F} \otimes \mathcal{P}_{x^\vee}) \cong t_{x^\vee}^* \Phi_{\mathcal{P}}(\mathcal{F})$

ii) $\Phi_{\mathcal{P}}(t_x^* \mathcal{F}) \cong \mathcal{P}_{-x} \otimes \Phi_{\mathcal{P}}(\mathcal{F})$

where $\mathcal{P}_x$ denotes the line bundle $\mathcal{P}|_{\{x\} \times A^\vee}$ and similarly for the dual.

Proof. We show ii). The fact i) can be proved similarly, using the projection formula and flat base change (lemma 1.1). Let $\mathcal{F} \in D^b(A)$. Then

$$p_{2*}(\mathcal{P} \otimes p_1^* t_x^* \mathcal{F}) \cong p_{2*}(t_{(x,0)}^* p_1^* \mathcal{F} \otimes t_{(x,0)}^* \mathcal{P} \otimes p_2^* \mathcal{P}_{-x}) \cong p_{2*}(t_{(x,0)}^*(p_1^* \mathcal{F} \otimes \mathcal{P}) \otimes p_2^* \mathcal{P}_{-x}))$$

using the general fact

$$t_{(x,x^\vee)}^* \mathcal{P} \cong \mathcal{P} \otimes p_1^* \mathcal{P}_{x^\vee} \otimes p_2^* \mathcal{P}_x$$

(e.g. [1] Lemma 14.1.3). We can then use the projection formula to write

$$p_{2*}(t_{(x,0)}^*(p_1^* \mathcal{F} \otimes \mathcal{P}) \otimes p_2^* \mathcal{P}_{-x})) \cong \mathcal{P}_{-x} \otimes p_{2*} t_{(x,0)}^*(\mathcal{P} \otimes p_1^* \mathcal{F})$$

Now we use flat base change in the diagram

$$\begin{array}{ccc} A \times A^\vee & \xrightarrow{t_{(x,0)}} & A \times A^\vee \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ A^\vee & \xrightarrow{\text{id}} & A^\vee \end{array}$$

to obtain $p_{2*}t^*_{(x,0)}(-) \cong p_{2*}(-)$, so

$$\mathcal{P}_{-x} \otimes p_{2*}t^*_{(x,0)}(\mathcal{P} \otimes p_1^*\mathcal{F}) \cong \mathcal{P}_{-x} \otimes p_{2*}(\mathcal{P} \otimes p_1^*\mathcal{F}) \cong \mathcal{P}_{-x} \otimes \Phi_{\mathcal{P}}(\mathcal{F})$$

as required. □

Definition 3.6. A vector bundle U on A is called *unipotent* if it admits a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{r-1} \subset U_r = U$$

with $U_i/U_{i-1} \cong \mathcal{O}_A$ for all $i = 1, \dots, r$. In other words, U is a successive extension of $\mathcal{O}_A$ by $\mathcal{O}_A$, i.e. we have $U_1 \cong \mathcal{O}_A$ and exact sequences

$$0 \to \mathcal{O}_A \to U_2 \to \mathcal{O}_A \to 0$$

and

$$0 \to U_{i-1} \to U_i \to \mathcal{O}_A \to 0$$

for $i = 3, \dots, r$.

We will need a characterization of unipotent vector bundles in terms of its Fourier–Mukai transform. This is another instance of the more general phenomenon: Certain classes of vector bundles and sheafs with zero-dimensional support are Fourier–Mukai transforms of each other (see below).

Proposition 3.7 ([1] Proposition 14.2.6)**.** *A vector bundle U on A is unipotent if and only if U is a WIT sheaf of index g with* $\operatorname{supp} \Phi_{\mathcal{P}}(U) = \{0\} \subseteq A^\vee$.

Proof. Let U be unipotent. If $r = \operatorname{rk} U = 1$ then $U \cong \mathcal{O}_A$ and we are done. If $r > 1$ then we do induction on the rank and assume that we already have the result for rank $r - 1$. Since U is homogenenous, there is an exact sequence

$$0 \to U_{r-1} \to U \to \mathcal{O}_A \to 0$$

Pull this sequence back by p_1^* and tensor by the Poincaré bundle $\mathcal{P}$. The sequence stays exact. Now we take the long exact sequence associated to the pushforward p_{2*}.

$$0 \to p_{2*}(\mathcal{P} \otimes p_1^* U_{r-1}) \to p_{2*}(\mathcal{P} \otimes p_1^* U) \to p_{2*}\mathcal{P} \to R^1 p_{2*}(\mathcal{P} \otimes p_1^* U_{r-1}) \to \cdots$$

Since U_{r-1} is a WIT sheaf of index g, all $R^j p_{2*}(\mathcal{P} \otimes p_1^* U_{r-1})$ vanish except for $j = g$. Similarly, $R^j p_{2*}\mathcal{P}$ vanishes, except for $j = g$ when we have $R^g P_{2*}\mathcal{P} = k(0)$ is the skyscraper sheaf supported at 0. Thus the only remaining terms in the long exact sequence are

$$0 \to R^g p_{2*}(\mathcal{P} \otimes p_1^* U_{r-1}) \to R^g p_{2*}(\mathcal{P} \otimes p_1^* U) \to R^g p_{2*}\mathcal{P} \to 0$$

which shows that U is a WIT sheaf of index g and its Fourier–Mukai transform is supported at 0 (since both $R^g p_{2*}\mathcal{P}$ and $R^g p_{2*}(\mathcal{P} \otimes p_1^* U_{r-1})$ are).

Conversely, assume that U is a WIT sheaf of index g on A with supp $\Phi_{\mathcal{P}}(U) = \{0\}$. We do induction on the length n of the sheaf $\Phi_{\mathcal{P}}(U)$. If $n = 1$ then

$$\mathcal{O}_A \cong \Phi_{\mathcal{P}}(\Phi_{\mathcal{P}}(U)) \cong (-1)^* U$$

and this is homogeneous. So assume $n > 1$. Then we have an exact sequence

$$0 \to V \to \Phi_{\mathcal{P}}(U) \to k(0) \to 0$$

where $V = \Phi_{\mathcal{P}}(U_{r-1})$ with U_{r-1} a vector bundle. U_{r-1} is unipotent by induction hypothesis. As before, we get an exact sequence

$$0 \to \mathcal{P} \otimes p_1^* V \to \mathcal{P} \otimes p_1^* \Phi_{\mathcal{P}}(U) \to \mathcal{P} \otimes p_1^* k(0) \to 0$$

and by taking the long exact sequence associated to p_{2*} we see that U fits into the exact sequence

$$0 \to U_{r-1} \to U \to \mathcal{O}_A \to 0$$

i.e. is unipotent. □

Next we come to an important class of vector bundles on abelian varieties. They are specific to abelian varieties and encode a lot of their geometry, for instance the geometry of sets of (fat) points, as we shall see.

Definition 3.8. A vector bundle $\mathcal{F}$ on A is called *homogeneous* if $t_x^* \mathcal{F} \cong \mathcal{F}$ for all $x \in A$.

Example 3.9. We have already seen that all $L \in \mathrm{Pic}^0(A)$ are homogeneous. In fact, we first defined $\mathrm{Pic}^0(A)$ that way.

There is a very elegant description of homogeneous vector bundles, due to Mukai:

Theorem 3.10 ([1] Theorem 14.7.10)**.** *For a vector bundle $\mathcal{F}$ on A the following are equivalent:*

i) $\mathcal{F}$ is homogeneous

ii) $\mathcal{F} \cong \bigoplus_{i=1}^r (U_i \otimes P_i)$ with U_i unipotent and $P_i \in \mathrm{Pic}^0(A)$.

In order to prove this, we first characterize homogeneous bundles in terms of their Fourier–Mukai transforms. This is in itself a very interesting fact:

Theorem 3.11 ([1] Corollary 14.7.13)**.** *The Fourier–Mukai transform $\Phi_{\mathcal{P}}$ yields a bijection between the set of homogeneous vector bundles and the set of coherent sheaves with finite support.*

First we need a lemma.

Lemma 3.12 ([1] Lemma 14.7.11)**.** For a coherent sheaf $\mathcal{E}$ on A the following are equivalent:

i) $\mathrm{supp}(\mathcal{E})$ is finite

ii) $\mathcal{E} \otimes P \cong \mathcal{E}$ for all $P \in \mathrm{Pic}^0(A)$

Proof. The implication i) $\Rightarrow$ ii) is clear. Suppose therefore ii) and suppose also that $\operatorname{supp}(\mathcal{E})$ contains a curve C. Denote by $\widetilde{C} \to C$ its normalization and by $\iota\colon \widetilde{C} \to X$ the composition of normalization and embedding of C. Note that $\widetilde{C}$ has nonzero genus since A as an abelian variety does not contain a rational curve.

The sheaf $\mathcal{G} := \iota^*\mathcal{E}/\operatorname{Tor}(\iota^*\mathcal{E})$ is torsion free and supported on the whole of $\widetilde{C}$, therefore a vector bundle. Let r be its rank. Our assumption ii) implies

$$\mathcal{G} \otimes \iota^* P \cong \iota^*(\mathcal{E} \otimes P)/\operatorname{Tor}(\iota^*(\mathcal{E} \otimes P)) \cong \mathcal{G}$$

for all $P \in \operatorname{Pic}^0(A)$. Taking determinants we get

$$\det \mathcal{G} \cong \det \mathcal{G} \otimes (\iota^* P)^{\otimes r}$$

which implies $(\iota^* P)^{\otimes r} \cong \mathcal{O}_{\widetilde{C}}$ for all $P \in \operatorname{Pic}^0(A)$. On the other hand, the morphism

$$\operatorname{Pic}^0(A) \xrightarrow{\iota^*} \operatorname{Pic}^0(\widetilde{C}) \xrightarrow{(-)^{\otimes r}} \operatorname{Pic}^0(\widetilde{C})$$

is nonzero (in fact it is dominant), which is a contradiction. $\square$

Proof of theorem 3.11. A vector bundle $\mathcal{F}$ is homogeneous if $t_x^*\mathcal{F} \cong \mathcal{F}$ for all $x \in A$. This is equivalent by Lemma 3.5 to $\Phi_{\mathcal{P}}(\mathcal{F}) \cong P_{-x} \otimes \Phi_{\mathcal{P}}(\mathcal{F})$ for all $x \in A$.

By our previous lemma, this is equivalent to all terms in the complex $\Phi_{\mathcal{P}}(\mathcal{F})$ having finite support. Since

$$\Phi_{\mathcal{P}}(\Phi_{\mathcal{P}}(\mathcal{F})) \cong (-1)^*\mathcal{F}[-g]$$

is concentrated in degree g, $\Phi_{\mathcal{P}}(\mathcal{F})$ has to be a sheaf. $\square$

Proof of theorem 3.10. We show that $\mathcal{E} \in \operatorname{Coh}(A)$ has finite support if and only if $\Phi_{\mathcal{P}}(\mathcal{E}) = \bigoplus_{i=1}^r (U_i \otimes P_i)$. But if $\mathcal{E}$ has finite support, we can write it in the form

$$\mathcal{E} = \bigoplus_{i=1}^r t^*_{x_i^\vee} M_i$$

where $x_i^\vee \in A^\vee$ and M_i are coherent sheaves with support $\{0\}$. Applying $\Phi_{\mathcal{P}}$ we get

$$\Phi_{\mathcal{P}}\left(\bigoplus_{i=1}^r t^*_{x_i^\vee} M_i\right) \cong \bigoplus_{i=1}^r \left(\Phi_{\mathcal{P}}(M_i) \otimes \mathcal{P}_{-x_i^\vee}\right)$$

and we know $\phi_{\mathcal{P}}(M_i)$ is unipotent by proposition 3.7. $\square$

References

[1] C. Birkenhake and H. Lange. *Complex abelian varieties*. Springer, 2004.

[2] R. Hartshorne. *Algebraic geometry*. Springer, 1977.

[3] D. Huybrechts. *Fourier–Mukai transforms in algebraic geometry*. Oxford University Press, 2006.

[4] A. Polishchuk. *Abelian varieties, theta functions and the Fourier transform*. Cambridge University Press, 2003.