Derived equivalences of abelian varieties - I

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1 Aim of the talks

Unless otherwise stated, we will always work over an algebraically closed field k of characteristic zero. Moreover, since we will be working on the derived categories, we will not denote explicitly the derived functors.

We have seen before (see [Huy06, Prop 4.11] and Gabriel's talk) that if X, Y are smooth projective varieties with ample (anti)canonical bundle, then they are derived equivalent if and only if they are isomorphic.

In the case of abelian varieties, where the canonical bundle is trivial, the situation is much more involved. More precisely, if A, B are two abelian varieties we define a subset of isomorphisms

$$U(A \times \widehat{A}, B \times \widehat{B}) \subseteq \operatorname{Iso}(A \times \widehat{A}, B \times \widehat{B})$$

in this way: any isomorphism $f\colon A \times \widehat{A} \longrightarrow B \times \widehat{B}$ is of the form

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

and then, using the isomorphisms $A \cong \widehat{\widehat{A}}$, $B \cong \widehat{\widehat{B}}$, we can associate to any such f an isomorphism $\widetilde{f}\colon B \times \widehat{B} \xrightarrow{\sim} A \times \widehat{A}$

$$\widetilde{f} \overset{\text{def}}{=} \begin{pmatrix} \widehat{f_4} & -\widehat{f_2} \\ -\widehat{f_3} & \widehat{f_1} \end{pmatrix}$$

and then we can define $U(A \times \widehat{A}, B \times \widehat{B})$ as

$$U(A \times \widehat{A}, B \times \widehat{B}) \overset{\text{def}}{=} \{ f \in \operatorname{Iso}(A \times \widehat{A}, B \times \widehat{B}) \mid \widetilde{f} = f^{-1} \}$$

The aim of this and Ignacio's talk is to present the following result (see [Huy06, Cor 9.49]):

Theorem 1.1 (Polishchuck). *Let A and B be two abelian varieties. Then*

$$D^b(A) \cong D^b(B) \iff U(A \times \widehat{A}, B \times \widehat{B}) \neq \emptyset$$

In particular, we are going to associate to each derived equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$ an element of $U(A \times \widehat{A}, B \times \widehat{B})$. To do this, we begin with a brief digression about derived equivalences sending closed points to closed points.

2 Derived equivalences preserving closed points

In this section we work over an algebraically closed field k of any characteristic. We begin by describing a particular kind of Fourier-Mukai transform:

Example 2.1. Let X, Y be two smooth projective varieties, $f\colon X \longrightarrow Y$ a morphism and $N \in \mathrm{Pic}(Y)$ a line bundle. If we denote by $\Gamma \subseteq X \times Y$ the graph of f and we let $\mathcal{E} \overset{\text{def}}{=} \mathcal{O}_\Gamma \otimes (\mathcal{O} \boxtimes N)$, then we have seen in Ana's talk that the corresponding Fourier-Mukai transform is given by

$$\Phi_{\mathcal{E}} = f_*(-) \otimes N\colon D^b(X) \longrightarrow D^b(Y)$$

We can repeat the computation as a warm-up: let

$$j\colon X \hookrightarrow X \times Y \qquad x \mapsto (x, f(x))$$

be the closed embedding corresponding to Γ and let $\mathcal{F} \in D^b(X)$, then:

$$\begin{aligned}
\Phi_{\mathcal{E}}(\mathcal{F}) &= (pr_Y)_*(pr_X^*\mathcal{F} \otimes \mathcal{O}_\Gamma \otimes (\mathcal{O}_X \boxtimes N)) \\
&= (pr_Y)_*(j_*\mathcal{O}_X \otimes \mathcal{F} \boxtimes N) && [\text{ projection formula }] \\
&= (pr_Y)_* j_*(j^* pr_X^* \mathcal{F} \otimes j^* pr_Y^* N) \\
&= f_*(\mathcal{F} \otimes f^* N) && [\text{ projection formula }] \\
&= f_*(\mathcal{F}) \otimes N
\end{aligned}$$

In particular, we see that for every closed point $x \in X$ we have

$$\Phi_{\mathcal{E}}(\kappa(x)) = \kappa(f(x))$$

What we want to prove now is a partial inverse to this:

Proposition 2.1. *Let X and Y be two smooth projective varieties over k and let*

$$\Phi\colon D^b(X) \longrightarrow D^b(Y)$$

be a derived equivalence such that for every closed point $x \in X$ there is another closed point $f(x) \in Y$ for which

$$\Phi(\kappa(x)) \cong \kappa(f(x))$$

Then the map on closed points $x \mapsto f(x)$ extends to an isomorphism $f\colon X \longrightarrow Y$ and there exists a line bundle $N \in Pic(Y)$ such that

$$\Phi \simeq N \otimes f_*(-)$$

Moreover, such f and N are unique up to isomorphism.

Before going to the proof, we need some technical preparation: in the next lemmas we will consider a morphism $\pi\colon X \longrightarrow T$ between two smooth projective varieties over k. For every closed point $t \in T$ we denote by i_t the closed embedding of the fiber:

$$i_t\colon X_t \overset{\text{def}}{=} \pi^{-1}(t) \hookrightarrow X$$

Lemma 2.1. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\mathcal{H}^{-1}(i_t^*\mathcal{F}) = 0$ for every closed point $t \in T$. Then $\mathcal{F}$ is flat over T.*

Proof. Since everything is local we can reduce ourselves to the affine local setting, in which we have a local homomorphism of local rings $A \longrightarrow B$ and a finitely generated module M over B such that $Tor_A^1(M, A/\mathfrak{m}_A) = 0$. Then the thesis follows from the local criterion for flatness: see [TS15, Tag 00MK]. □

Lemma 2.2. *Let $\mathcal{F} \in D^b(X)$ be an object such that for every closed point $t \in T$ the derived pullback $i_t^*\mathcal{F}$ is a complex whose cohomology is concentrated in degree 0, i.e. a sheaf. Then $\mathcal{F}$ is isomorphic to a sheaf which is flat over T.*

Proof. See also [Huy06, Lemma 3.31]. To prove that the complex $\mathcal{F}$ is isomorphic to a sheaf we need to show that its cohomology is concentrated in degree 0. Our main tool will be the spectral sequence [Huy06, (3.10)]

$$E_2^{p,q} = \mathcal{H}^p(i_t^* \mathcal{H}^q(\mathcal{F})) \rightsquigarrow \mathcal{H}^{p+q}(i_t^* \mathcal{F})$$

Now, let $m \in \mathbb{Z}$ be the biggest integer such that $\mathcal{H}^m(\mathcal{F}) \neq 0$: then there must be a closed point $t \in T$ such that $\mathcal{H}^0(Li_t^* \mathcal{H}^m(\mathcal{F})) \neq 0$ (note that this is just the usual pullback). Then, if we draw the second page of the spectral sequence we get

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & E_2^{-1,m} & E_2^{0,m} & 0 & 0 \\ * & * & * & * & 0 & 0 \end{matrix}$$

observe that $\mathcal{H}^p(i_t^* \mathcal{H}^q(\mathcal{F})) = 0$ for every $p > 0$, since $\mathcal{H}^q(\mathcal{F})$ is a sheaf (i.e. concentrated in degree 0). This shows that $E_2^{0,m} \neq 0$ survives in the spectral sequence, and since $\mathcal{H}^m(i_t^* \mathcal{F}) = 0$ unless $m = 0$, it must be that $m = 0$.

Now, we want to prove that $\mathcal{H}^0(\mathcal{F})$ is flat over T: from above, we see that $E_2^{-1,0}$ survives in the spectral sequence for every closed $t \in T$, and since $\mathcal{H}^{-1}(i_t^* \mathcal{F}) = 0$, this gives us that $\mathcal{H}^{-1}(i_t^* \mathcal{H}^0(\mathcal{F})) = 0$. Then $\mathcal{H}^0(\mathcal{F})$ is flat from Lemma 2.1. Now, since $\mathcal{H}^0(\mathcal{F})$ is flat over T, it follows that $\mathcal{H}^p(i_t^* \mathcal{H}^0(\mathcal{F})) = 0$ for every $p < 0$ and for every $t \in T$ closed point. Using again the above spectral sequence this tells us that $\mathcal{H}^0(i_t^* \mathcal{H}^{-1}(\mathcal{F})) = 0$ for every $t \in T$ closed point, so that $\mathcal{H}^{-1}(\mathcal{F}) = 0$. In particular this implies that $\mathcal{H}^p(i_t^* \mathcal{H}^{-1}(\mathcal{F})) = 0$ for every $p < 0$ and for every $t \in T$ closed point, so that we can proceed in this way and show that $\mathcal{H}^r(\mathcal{F}) = 0$ for every $r < 0$, and this concludes the proof. □

Lemma 2.3. *Let $\mathcal{F} \in D^b(X)$ and suppose that for every closed point $t \in T$ there exists a closed point $x_t \in X_t$ such that*

$$i_t^* \mathcal{F} \cong \kappa(x_t)$$

Then if we consider Supp($\mathcal{F}$) with its natural scheme structure, the following are true:

1. *the restriction $\pi\colon Supp(\mathcal{F}) \longrightarrow T$ is an isomorphism.*
2. *$\mathcal{F}$ is a line bundle over Supp($\mathcal{F}$).*

Proof. From Lemma 2.2 we see that $\mathcal{F}$ is a sheaf, flat over X; in particular the derived pullbacks $i_t^* \mathcal{F}$ coincide with the usual pullbacks.

Now we observe that for every $t \in T$ closed point we have the set-theoretic equality

$$\mathrm{Supp}(\mathcal{F}) \cap X_t = x_t$$

so that $\pi\colon \mathrm{Supp}(\mathcal{F}) \longrightarrow T$ is a proper and quasi-finite morphism, hence finite. Now, if we manage to prove that the above equality actually holds scheme-theoretically, then we know that $\mathrm{Supp}(\mathcal{F}) \longrightarrow T$ is a finite flat morphism of degree 1, i.e. an isomorphism (this can be easily checked affine-locally).

To prove this, we can reduce ourselves to the local situation by restricting to an affine neighborhood of x_t and then taking the stalk at x_t. In this setting we have a Noetherian local ring $(A, \mathfrak{m})$ a finitely generated A-module $M \neq 0$ and a proper ideal $I \subsetneq A$ such that $M/IM \cong A/\mathfrak{m}$ as A/I-modules. What we want to prove is that

$$\mathrm{Spec}\ A/I \cap \mathrm{Spec}\ A/Ann(M) = \mathrm{Spec}\ A/\mathfrak{m}$$

that is

$$I + Ann(M) = \mathfrak{m}$$

It is clear that $I + Ann(M) \subseteq \mathfrak{m}$; for the converse, fix an isomorphism of A/I-modules

$$\phi\colon A/\mathfrak{m} \longrightarrow M/IM \qquad 1 \mapsto m$$

then it follows that $M = \langle m \rangle + IM$ as A-modules and Nakayama's Lemma implies $M = \langle m \rangle$. Now take $x \in \mathfrak{m}$: since ϕ is a well-defined map, we know that $xm = im$ for a certain $i \in I$ and then it follows that $x - i \in Ann(M)$, proving the result. Moreover, we can also observe that the map

$$A/Ann(M) \longrightarrow M \qquad 1 \mapsto m$$

is an isomorphism of $A/Ann(M)$-modules, so that M is free of rank one over $A/Ann(M)$ and this proves also the second assertion (since being a line bundle is a local property). □

Now we can prove Proposition 2.1:

Proof of Proposition 2.1. By Orlov's Theorem [Huy06, Cor 5.17], there exists an object $\mathcal{P} \in D^b(X \times Y)$ such that $\Phi = \Phi_{\mathcal{P}}$. Then, using flat base change (see [Huy06, (3.18)] and Gregor's talk), we see that for every closed point $x \in X$

$$\Phi_{\mathcal{P}}(\kappa(x)) = i_x^* \mathcal{P} \cong \kappa(f(x))$$

where $i_x\colon \{x\} \times Y \hookrightarrow X \times Y$ is the standard closed embedding. Then we can use Lemma 2.3 to see that $\mathcal{P}$ is a line bundle on $\Gamma = \operatorname{Supp}(\mathcal{P})$ and that that the projection

$$pr_{X|\Gamma}\colon \Gamma \longrightarrow X$$

is an isomorphism. Then we can define a morphism $f \overset{\text{def}}{=} pr_Y \circ (pr_{X|\Gamma})^{-1}$ and Γ is exactly the graph of f: in particular f extends the map $x \mapsto f(x)$ on closed points.

Now,we prove that f is bijective on closed points: recall that the closed points $\kappa(x)$ are a spanning class of $D^b(X)$, and since Φ is an equivalence, the points $\kappa(f(x))$ must give a spanning class of $D^b(Y)$. In particular, for every closed point $y \in Y$ there must be a closed point $x \in X$ and an $m \in \mathbb{Z}$ such that

$$\operatorname{Hom}(\kappa(f(x)), \kappa(y)[m]) \neq 0$$

since this is possible only if $\kappa(f(x))$ and $\kappa(y)$ have intersecting supports (see [Huy06, proof of Lemma 3.9]), it must be that $f(x) = y$. For the same reason, if $x_1 \neq x_2$ are two distinct closed points in X, then

$$\operatorname{Hom}(\kappa(f(x_1)), \kappa(f(x_2))) \overset{\Phi}{\cong} \operatorname{Hom}(\kappa(x_1), \kappa(x_2)) = 0$$

so that $f(x_1) \neq f(x_2)$, otherwise there would be the identity morphism. This shows that f is bijective on closed points: in characteristic zero then f must be an isomorphism by Zariski's Main Theorem, in characteristic p one can argue as before with the quasi-inverse Φ^{-1} to produce a morphism $f^{-1}\colon Y \longrightarrow X$ and then it is easy to show that this is the inverse of f.

Now, consider the isomorphism

$$pr_{|\Gamma}\colon \Gamma \longrightarrow Y$$

and define $N \overset{\text{def}}{=} (pr_{|\Gamma})_*(\mathcal{P}) \in \operatorname{Pic}(Y)$. Then it is clear that $\mathcal{P} = \mathcal{O}_\Gamma \otimes (\mathcal{O}_X \boxtimes N)$ and from Example 2.1 we see that

$$\Phi \simeq N \otimes f_*(-)$$

To show uniqueness, suppose that $g\colon X \longrightarrow Y$ is another isomorphism and $M \in \operatorname{Pic}(Y)$ another line bundle such that

$$N \otimes f_*(-) \simeq M \otimes g_*(-)$$

then

$$N \cong N \otimes \mathcal{O}_Y \cong N \otimes f_*(\mathcal{O}_X) \cong M \otimes g_*(\mathcal{O}_X) \cong M \otimes \mathcal{O}_Y \cong M$$

and now this gives $f_*(-) \cong g_*(-)$ that implies $f \simeq g$. □

3 Derived equivalences of abelian varieties

For the rest of the talk, we will work over an algebraically closed field k of characteristic zero, even if many of the following results hold also in arbitrary characteristic. We follow [Huy06, Section 5.4].

Consider a derived equivalence of abelian varieties

$$\Phi\colon D^b(A) \longrightarrow D^b(B)$$

We want to associate to it an isomorphism $f \in U(A \times \widehat{A}, B \times \widehat{B})$: the first step will be associating to it a derived equivalence $D^b(A \times \widehat{A}) \longrightarrow D^b(B \times \widehat{B})$.

To do this, first observe that Orlov's Theorem [Huy06, Cor 5.17] tells us that there exists an unique $\mathcal{E} \in D^b(A \times B)$ such that $\Phi = \Phi_{\mathcal{E}}$. We also know that A and B have the same dimension g (see [Huy06, Prop 4.1] and Gabriel's talk) so that the complex $\mathcal{E}_R$ (see [Huy06, Def 5.7]) giving the quasi-inverse $\Phi^{-1}\colon D^b(B) \longrightarrow D^b(A)$ is $\mathcal{E}_R = \mathcal{E}^\vee[g]$. However we will be more interested in the associated Fourier-Mukai transform in the opposite direction

$$\Phi_{\mathcal{E}_R}\colon D^b(B) \longrightarrow D^b(A)$$

Remark 3.1. From the fact that $\Phi_{\mathcal{E}}\colon D^b(A) \longrightarrow D^b(B)$ is an equivalence, it follows that $\Phi_{\mathcal{E}_R}\colon D^b(A) \longrightarrow D^b(B)$ is an equivalence as well. Indeed, Bondal and Orlov's criterion (see [Huy06, Prop 7.1] and Gregor's talk) asserts that $\Phi_{\mathcal{E}_R}$ is fully faithful if and only if for any two closed points $a_1, a_2 \in A$ we have

$$\operatorname{Hom}(\Phi_{\mathcal{E}_R}(\kappa(a_1)), \Phi_{\mathcal{E}_R}(\kappa(a_2))[i]) = \begin{cases} k & \text{if } a_1 = a_2 \text{ and } i = 0 \\ 0 & \text{if } a_1 \neq a_2 \text{ or } i < 0 \text{ or } i > g \end{cases}$$

Now, we observe that $\Phi_{\mathcal{E}_R}(\kappa(a)) = (pr_B)_*(\mathcal{O}_{\{a\}\times B} \otimes \mathcal{E}^\vee[-g]) = i_a^*\mathcal{E}^\vee[-g]$ for every closed point $a \in A$, where

$$i_a\colon \{a\} \times B = B \hookrightarrow A \times B$$

is the closed immersion of the fiber over a. Then, we see that

$$\begin{aligned}
\operatorname{Hom}(\Phi_{\mathcal{E}_R}(\kappa(a_1)), \Phi_{\mathcal{E}_R}(\kappa(a_2))[i]) &\cong \operatorname{Hom}(i_{a_1}^*\mathcal{E}^\vee[-g], i_{a_2}^*\mathcal{E}^\vee[-g][i]) \\
&\cong \operatorname{Hom}(i_{a_1}^*\mathcal{E}^\vee, i_{a_2}^*\mathcal{E}^\vee[i]) && [\text{ pullback commutes with dualizing }] \\
&\cong \operatorname{Hom}((i_{a_1}^*\mathcal{E})^\vee, (i_{a_2}^*\mathcal{E})^\vee[i]) \\
&\cong \operatorname{Hom}(i_{a_2}^*\mathcal{E}, i_{a_1}^*\mathcal{E}) \\
&\cong \operatorname{Hom}(\Phi_{\mathcal{E}}(\kappa(a_1)), \Phi_{\mathcal{E}}(\kappa(a_2))[i])
\end{aligned}$$

so that the criterion is satisfied, as it is satisfied by $\Phi_{\mathcal{E}}$. Now, since the canonical bundle on any abelian variety is trivial, we also know that $\Phi_{\mathcal{E}_R}$ is an equivalence, thanks to [Huy06, Cor 7.8].

Then, we can form the product of these Fourier-Mukai transform and get a map

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R} \overset{\text{def}}{=} \Phi_{\mathcal{E}\boxtimes\mathcal{E}_R}\colon D^b(A \times A) \longrightarrow D^b(B \times B)$$

Remark 3.2. It is easy to show in this case that the product $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}$ is an equivalence. Indeed, thanks to [Huy06, Cor 7.4] it is a fully faithful functor, and then it is an equivalence thanks to [Huy06, Cor 7.8].

Now, for every abelian variety A, consider the isomorphism

$$\mu_A\colon A \times A \longrightarrow A \times A \qquad (a_1, a_2) \mapsto (a_1 + a_2, a_2)$$

and the corresponding derived isomorphisms

$$(\mu_A)_*\colon D^b(A \times A) \longrightarrow D^b(A \times A) \qquad (\mu_A)^*\colon D^b(A \times A) \longrightarrow D^b(A \times A)$$

and, lastly, recall, that for every abelian variety A we have the derived equivalence

$$\Phi_{\mathcal{P}_A}\colon D^b(\widehat{A}) \longrightarrow D^b(A)$$

given by the Poincare bundle, that in turn induces a derived equivalence

$$\mathrm{id} \times \Phi_{\mathcal{P}_A}\colon D^b(A \times \widehat{A}) \longrightarrow D^b(A \times A)$$

Now we are ready to give the following definition:

Definition 3.1. Let $\Phi_{\mathcal{E}}\colon D^b(A) \longrightarrow D^b(B)$ be a derived equivalence between abelian varieties. Then we define the derived equivalence $F_{\mathcal{E}}\colon D^b(A \times \widehat{A}) \longrightarrow D^b(B \times \widehat{B})$ via the following commutative diagram:

$$\begin{array}{ccc} D^b(A \times \widehat{A}) & \xrightarrow{F_{\mathcal{E}}} & D^b(B \times \widehat{B}) \\ {\scriptstyle \mathrm{id} \times \Phi_{\mathcal{P}_A}} \downarrow & & \uparrow {\scriptstyle (\mathrm{id} \times \Phi_{\mathcal{P}_B})^{-1}} \\ D^b(A \times A) & & D^b(B \times B) \\ {\scriptstyle \mu_{A_*}} \downarrow & & \uparrow {\scriptstyle \mu_B^*} \\ D^b(A \times A) & \xrightarrow{\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}} & D^b(B \times B) \end{array}$$

Lemma 3.1. *The construction $\Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$ is compatible with composition. More precisely, if $\Phi_{\mathcal{G}}\colon D^b(A) \longrightarrow D^b(B)$ is the composition of*

$$\Phi_{\mathcal{E}}\colon D^b(A) \longrightarrow D^b(B) \qquad \Phi_{\mathcal{F}}\colon D^b(B) \longrightarrow D^b(C)$$

then $F_{\mathcal{G}} \simeq F_{\mathcal{F}} \circ F_{\mathcal{E}}$.

Proof. We need to prove that

$$(\Phi_{\mathcal{F}} \times \Phi_{\mathcal{F}_R}) \circ (\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}}) = (\Phi_{\mathcal{G}} \times \Phi_{\mathcal{G}})$$

bu this follows from the formula for the composition of Fourier-Mukai transforms: see [Huy06, Prop 5.10]. □

Corollary 3.1. *The map*

$$Aut(D^b(A)) \longrightarrow Aut(D^b(A \times \widehat{A})), \qquad \Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$$

is a group homomorphism.

Example 3.1. Now we are going to give some examples of this construction in some important cases. We begin by studying the composition

$$\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}})\colon D^b(A \times \widehat{A}) \longrightarrow D^b(A \times A)$$

and we try to understand what this does on closed points. Take then a closed point $(a, \alpha) \in A \times \widehat{A}$, where α corresponds to a line bundle $L = \mathcal{P}_a$ of degree 0 on A. We see that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\kappa(a, \alpha)) = \mu_*(\kappa(a) \boxtimes \mathcal{P}_a) = \mu_*(\kappa(a) \boxtimes L)$$

and now, if we take

$$j_a\colon A \longrightarrow A \times A \qquad x \mapsto (a, x)$$

by flat base change we get

$$\mu_*(\kappa(a) \boxtimes L) = \mu_*(j_{a*}\mathcal{O}_A \otimes pr_2^* L)$$

and now the projection formula tells us that

$$\mu_*(j_{a*}\mathcal{O}_A \otimes pr_2^*L) = \mu_* j_{a*}(j_a^* pr_2^* L) = (\mu \circ j_a)_* L$$

however

$$\mu \circ j_a \colon A \longrightarrow A \times A \qquad x \mapsto (a+x, x)$$

is nothing more than the embedding of the graph Γ_{-a} of t_{-a} into $A \times A$, so that we have

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\kappa(a,\alpha)) = (\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}$$

In particular, by Example 2.1, we see that the Fourier-Mukai transform induced by this object is precisely

$$\Phi_{(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\kappa(a,\alpha))} = t_{-a*}(-) \otimes M$$

Example 3.2. Let $M \in \mathrm{Pic}^0(A)$ be a line bundle of degree 0 and consider the equivalence

$$\Phi_{\mathcal{E}} \colon D^b(A) \longrightarrow D^b(A) \qquad \mathcal{F} \mapsto M \otimes \mathcal{F}$$

then we have seen in Ana's talk that $\mathcal{E} = \Delta_*(M)$, and since the inverse equivalence is obviously given by $M^\vee \otimes (-) = \Phi_{\Delta_* M^\vee}$ we see that $\mathcal{E}_R = \Delta_* M^\vee$ so that $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R} = (M \boxtimes M^\vee) \otimes (-)$. Now, let's look at the action of $F_{\mathcal{E}}$ on closed points: let $a \in A$ be a closed point and $\alpha \in \widehat{A}$ a closed point corresponding to a line bundle L on A. We know already from Example 3.1 that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\kappa(a,\alpha)) = (\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}$$

and then

$$(\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})((\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}) = (\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}} \otimes (M \boxtimes M^\vee)$$

now we observe that

$$\begin{aligned}
\mathcal{O}_{\Gamma_{-a}} \otimes (M \boxtimes M^\vee) &= j_{a*}\mathcal{O}_A \otimes pr_1^* M \otimes pr_1^* M^\vee && [\text{ projection formula }] \\
&= j_{a*}(j_a^* pr_1^* M \otimes j_a^* pr_2^* M^\vee) \\
&= j_{a*}(M \otimes t_{-a*} M^\vee) && [M \in \mathrm{Pic}^0(A)] \\
&= j_{a*}(M \otimes M^\vee) = \mathcal{O}_{\Gamma_{-a}}
\end{aligned}$$

so that

$$(\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})((\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}) = (\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}$$

and using Example 3.1 once again we see that

$$F_{\mathcal{E}}(\kappa(a,\alpha)) = \kappa(a,\alpha)$$

Then we can use Proposition 2.1 to see that $F_{\mathcal{E}} = \mathrm{id}_*(-) \otimes N = (-) \otimes N$, for a certain line bundle N on A. To determine N, we just need to compute $F_{\mathcal{E}}(\mathcal{O}_{A \times \widehat{A}})$: to do this, recall that for the Poincare' bundle $\mathcal{P}$ it holds that

$$pr_{A*}\mathcal{P} \cong \kappa(0)[-g]$$

(see [Huy06, proof of Prop 9.19] and Ana's talk), so that

$$(\mathrm{id} \times \Phi_{\mathcal{P}})(\mathcal{O}_{A \times \widehat{A}}) = \mathcal{O}_A \boxtimes \kappa(0)[-g]$$

We see by flat base change and projection formula that $\mu_*(\mathcal{O}_A \boxtimes \kappa(0)[-g]) = \mathcal{O}_A \boxtimes \kappa(0)[-g]$, and then

$$(\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})(\mathcal{O}_A \boxtimes \kappa(0)[-g]) = M \boxtimes \kappa(0)[-g]$$

Now, it is easy to show as above that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(M \boxtimes \mathcal{O}_A) = M \boxtimes \kappa(0)[-g]$$

and this proves that

$$F_{\mathcal{E}}(\mathcal{O}_{A \times \widehat{A}}) \cong M \boxtimes \mathcal{O}_A$$

In conclusion, this proves that

$$F_{\Delta_* M} = (M \boxtimes \mathcal{O}_A) \otimes (-)$$

Example 3.3. Now we consider the equivalence

$$\Phi_{\mathcal{E}}\colon D^b(A) \longrightarrow D^b(A) \qquad \mathcal{F} \mapsto t_{a_0}^*\mathcal{F}$$

in this case we know that $\mathcal{E} = \mathcal{O}_{\Gamma_{a_0}}$ and since the inverse equivalence is given by t_{-a_0*} (but in the opposite direction) we see that $\mathcal{E}_R = \mathcal{O}_{\Gamma_{a_0}}$ as well. Hence

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R} = t_{a_0*} \times t_{a_0*}$$

Now, let's consider the action of $F_{\mathcal{E}}$ on closed points: let $a \in A$ be a closed point and $\alpha \in \widehat{A}$ another closed point representing a line bundle $L = \mathcal{P}_\alpha \in \mathrm{Pic}^0(A)$: then we know from Example 2.1 that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\kappa(a,\alpha)) = \mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O}_A \boxtimes L)$$

and now it is easy to see that

$$(\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})(\mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O}_A \boxtimes L)) = \mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O}_A \boxtimes L)$$

so that

$$F_{\mathcal{E}}(\kappa(a,\alpha)) = \kappa(a,\alpha)$$

and using again Proposition 2.1, we see that $F_{\mathcal{E}} = F_{\mathcal{E}}(\mathcal{O}_{A\times\widehat{A}}) \otimes (-)$. To conclude, we need to compute $F_{\mathcal{E}}(\mathcal{O}_{A\times\widehat{A}})$: we know already that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathbb{P}}))(\mathcal{O}_{A\times\widehat{A}}) = \mathcal{O}_A \boxtimes \kappa(0)[-g]$$

so that

$$(\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})(\mathcal{O}_A \boxtimes \kappa(0)[-g]) = \mathcal{O}_A \boxtimes \kappa(a_0)[-g]$$

Now if we denote by $L_{a_0} = \mathcal{P}_{|\{a_0\}\times\widehat{A}}$ the line bundle of degree 0 on $\widehat{A}$ corresponding to a_0, one can prove as in [Huy06, proof of Prop 9.19] and Ana's talk, that

$$\Phi_{\mathcal{P}}(L_{a_0}^\vee) = \kappa(a_0)[-g]$$

then it follows that

$$(\mu_* \circ (\mathrm{id} \times \Phi_{\mathcal{P}}))(\mathcal{O}_A \boxtimes L_{a_0}^\vee) = \mathcal{O}_A \boxtimes \kappa(a_0)[-g]$$

so that

$$F_{\mathcal{E}}(\mathcal{O}_{A\times\widehat{A}}) = \mathcal{O}_A \boxtimes L_{a_0}^\vee$$

and

$$F_{\mathcal{O}_{\Gamma_{a_0}}} = (\mathcal{O}_A \boxtimes L_{a_0}^\vee) \otimes (-)$$

Example 3.4. Consider now for any closed points $a \in A$ and $\alpha \in \widehat{A}$ the equivalence

$$\Phi_{(a,\alpha)} = \mathcal{P}_\alpha \otimes t_{a*}(-)\colon D^b(A) \longrightarrow D^b(A)$$

then we can write this as

$$\Phi_{(a,\alpha)} = (\mathcal{P}_\alpha \otimes (-)) \circ t_{a*}(-)$$

and using Corollary 3.1 and the previous Examples we see that the corresponding $F_{(a,\alpha)}$ is given by

$$F_{(a,\alpha)} = ((\mathcal{P}_\alpha \boxtimes \mathcal{O}_{\widehat{A}}) \otimes (-)) \circ ((\mathcal{O}_A \boxtimes \mathcal{P}_a^\vee) \otimes (-)) = (\mathcal{P}_\alpha \boxtimes \mathcal{P}_a^\vee) \otimes (-)$$

Example 3.5. Consider the derived autoequivalence given by the shift functor

$$\Phi_{\mathcal{E}} = T^m\colon D^b(A) \longrightarrow D^b(A) \qquad \mathcal{F} \mapsto \mathcal{F}[m]$$

for a certain $m \in \mathbb{Z}$. Then we know that $\mathcal{E} = \mathcal{O}_\Delta[m]$ and since the inverse is given by T^{-m} we see that $\mathcal{E}_R = \mathcal{O}_\Delta[-m]$ so that

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R} = \mathcal{O}_\Delta[m] \boxtimes \mathcal{O}_\Delta[-m] = \mathcal{O}_\Delta \boxtimes \mathcal{O}_\Delta = \mathrm{id}$$

and then $F_{\mathcal{E}} = \mathrm{id}$. In particular we see that under the mapping

$$\mathrm{Aut}(D^b(A)) \longrightarrow \mathrm{Aut}(D^b(A \times \widehat{A})) \qquad \Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$$

the situation becomes "more geometrical" (in the sense that "non geometrical" automorphisms are sent to the identity).

In all there examples we have seen that the derived equivalence $F_{\mathcal{E}}$ sends closed points to closed points, so that it is of the form $F_{\mathcal{E}} = f_{\mathcal{E}*}(-) \otimes N_{\mathcal{E}}$. Actually one can prove that this principle holds in general:

Theorem 3.1. *Let $\Phi_{\mathcal{E}} \colon D^b(A) \longrightarrow D^b(B)$ be a derived equivalence between abelian varieties. Then the associated equivalence $F_{\mathcal{E}} \colon D^b(A \times \widehat{A}) \longrightarrow D^b(B \times \widehat{B})$ is of the form*

$$F_{\mathcal{E}} = N_{\mathcal{E}} \otimes f_{\mathcal{E}*}(-)$$

for an unique isomorphism $f_{\mathcal{E}} \colon A \times \widehat{A} \longrightarrow B \times \widehat{B}$ and an unique line bundle $N_{\mathcal{E}} \in Pic(B \times \widehat{B})$.

Moreover $f_{\mathcal{E}} \in U(A \times \widehat{A}, B \times \widehat{B})$.

Proof. It will be given in the next talk. □

References

[Huy06] Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.

[TS15] The Stacks Project Authors, *Stacks Project* (2015), http://stacks.math.columbia.edu.