Derived Equivalences of Abelian Varieties II

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1 Introduction and Recapitulation

This talk follows the exposition of [1], section 9.4. We want to understand, maybe in a more geometric way, when two abelian varieties are derived equivalent. Sadly the constructions and arguments just work for the abelian case. The general line goes as follows: we start with a derived equivalence of abelian varieties $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ given as a Fourier-Mukai transform with kernel $\mathcal{E}$. Gabriel proved us (Proposition 4.1 in [1]) that $\dim(A) = \dim(B) = g$. We call $\mathcal{E}_R = \mathcal{E}^\vee[g]$ and $\Phi_{\mathcal{E}_R}$ is a quasi-inverse of $\Phi_{\mathcal{E}}$. We pass to a derived equivalence $F_{\mathcal{E}} : D^b(A \times \widehat{A}) \to D^b(B \times \widehat{B})$ as in the diagram.

$$\begin{array}{ccc} D^b(A \times \widehat{A}) & \xrightarrow{F_{\mathcal{E}}} & D^b(B \times \widehat{B}) \\ \downarrow{\scriptstyle id\times\Phi_{\mathcal{P}_A}} & & \uparrow{\scriptstyle (id\times\Phi_{\mathcal{P}_B})^{-1}} \\ D^b(A \times A) & & D^b(B \times B) \\ \downarrow{\scriptstyle \mu_A *} & & \uparrow{\scriptstyle \mu_B^*} \\ D^b(A \times A) & \xrightarrow[\Phi_{\mathcal{E}}\times\Phi_{\mathcal{E}_R}]{} & D^b(B \times B). \end{array}$$

The map $\Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$ respect composition. We will show that, for some reason, the situation in $A \times \widehat{A}$ and $B \times \widehat{B}$ is much more geometric, actually the equivalence $F_{\mathcal{E}}$ is induced by an isomorphism $f_{\mathcal{E}} : A \times \widehat{A} \xrightarrow{\sim} B \times \widehat{B}$. The construction goes backwards for some special isomorphisms $f \in Iso(A \times \widetilde{A}, B \times \widetilde{B})$ which allows us to translate the question wether two abelian varieties are derived equivalence in a question of geometric nature. The hard bone of this talk is to prove that $F_{\mathcal{E}}$ is given by an isomorphism as we explained. For this we will use Corollary 5.23 in [1] that Daniele proved for us.

Corollary 1.1. (5.23 in [1]) Let $\Phi : D^b(X) \to D^b(Y)$ be a derived equivalence such that for every close point $x \in X$ there is a close point $f(x) \in Y$ with

$$\Phi(k(x)) \simeq k(f(x)).$$

Then $f : X \to Y$ defines an isomorphism and Φ is given by the composition of f_* and the twist by some line bundle $M \in Pic(Y)$, i.e.,

$$\Phi \simeq (M \otimes (\cdot)) \circ f_*.$$

The strategy will be the following:

- Prove that for some close point $x_0 \in A \times \widehat{A}$, $F_{\mathcal{E}}(k(x_0))$ is a close point in $B \times \widehat{B}$.
- Show that the property of sending close points to close points can be extended to a neighborhood of x_0.
- Use the abelian group structure to extend it to the whole variety.

Finally we will try to understand which isomorphisms can occur. We will make extensive use of the following two propositions.

Proposition 1.2. (5.10 in [1]) The composition

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(Z)$$

is isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$ where

$$\mathcal{R} = \pi_{XZ*}(\pi_{XY}^{*}\mathcal{P} \otimes \pi_{YZ}^{*}\mathcal{Q}).$$

Proposition 1.3. (Exercise 5.13 in [1]) A special case of before is when $\mathcal{P}_i \in D^b(X_i \times Y_i)$ for $i = 1,2$ and $\mathcal{R} \in D^b(X_1 \times X_2)$ then the composition

$$D^b(Y_1) \xrightarrow{\Phi_{\mathcal{P}_1}} D^b(X_1) \xrightarrow{\Phi_{\mathcal{R}}} D^b(X_2) \xrightarrow{\Phi_{\mathcal{P}_2}} D^b(Y_2)$$

is equal to $\Phi_{\mathcal{S}} : D^b(Y_1) \to D^b(Y_2)$ where

$$\mathcal{S} = \Phi_{\mathcal{P}_1} \times \Phi_{\mathcal{P}_2}(\mathcal{R}) \in D^b(Y_1 \times Y_2).$$

2 Main Result

Lemma 2.1. Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ be a Fourier-Mukai equivalence. Suppose there is a close point $x_0 \in X$ such that

$$\Phi_{\mathcal{P}}(k(x_0)) \simeq k(y_0)$$

for some close point $y_0 \in Y$. The there is an open neighborhood $x_0 \in U \subset X$ and a morphism $f : U \to Y_0$ with $f(x_0) = y_0$ and such that

$$\Phi_{\mathcal{P}}(k(x)) \simeq k(f(x))$$

for all $x \in U$.

Proof. First thing to notice is that the assumption says that the fiber over x_0 of the morphism $Supp(\mathcal{P}) \to X$ is zero dimensional, since by [1], Lemma 3.29, $Supp(\mathcal{P}) \cap (\{x_0\} \times Y) = Supp(\mathcal{P} \mid_{\{x_0\}\times Y})$ and

$$\Phi_{\mathcal{P}}(k(x_0)) \simeq Rp_{2*}(\mathcal{P} \mid_{\{x_0\}\times Y}) \simeq \mathcal{P} \mid_{\{x_0\}\times Y} .$$

Let us assume for our proposes that the morphism $Supp(\mathcal{P}) \to X$ is surjective, proper and the fibers are connected (Lemmas 6.11 and 6.4 in [1]). Then Chevalley's upper semi-continuity theorem (see EGA IV 13.1.5) says us that

$$y \mapsto \dim(X_y)$$

is upper semi-continuous. Hence, for some open neighborhood U of x_0 one has that the fibers are points, i.e., for any $x \in U$ the complex $\Phi_{\mathcal{P}}(k(x))$ is concentrated in a point. Notice that $Hom(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(x))[i]) = 0$ for $i < 0$.

We claim that due to this last statement $\Phi_{\mathcal{P}}(k(x))$ is of the form $k(y)[m]$ for some integer m. Let's call $\mathcal{F}^\bullet$ the complex $\Phi_{\mathcal{P}}(k(x))$, $\mathcal{H}^i$ the corresponding cohomology sheaves and m_0, m_1 the biggest (resp. smallest) number such that $\mathcal{H}^i \neq 0$. One can check that there is a sequence of morphisms

$$\mathcal{F}^\bullet[m_0] \to \mathcal{H}^{m_0} \to \mathcal{H}^{m_1} \to \mathcal{F}^\bullet[m_1]$$

whose composition is non trivial which contradicts the fact that $Hom(\mathcal{F}, \mathcal{F}[i]) = 0$ for $i < 0$ unless $m_1 = m_0$. Thus, $\mathcal{F}$ is a sheaf, concentrated at one point and one can check that is indecomposable. The only candidate is $k(y)[m]$. See [1], Lemma 4.5. Finally by semi-continuity the shift has to be constant around $x_0 \in U$. The fact that f is a morphism follows from the proof of Corollary 5.23 in [1] that Daniele proved last time. ∎

There are some comments to be made.

- Subjectivity follows from the fact that if x lies in the complement of $p_1(Supp(\mathcal{P}))$ then the derived tensor product $\mathcal{P} \otimes^L p_1^* k(x)$ would be trivial which means that $\Phi_{\mathcal{P}}(k(x)) \simeq 0$ contradicting the fact that $\Phi_{\mathcal{P}}$ is an equivalence. See Lemma 6.4 in [1] for a complete proof. For connectedness of the fibers we use first Lemma 3.29 in [1] saying in our case that $supp(\mathcal{P}) \cap (\{x\} \times Y) = Supp(\mathcal{P} \mid_{\{x\}\times Y})$. Now if $Supp(\mathcal{P} \mid_{\{x\}\times Y})$ has disconnected support then $\Phi_{\mathcal{P}}(k(x)) \simeq \mathcal{F}_1^\bullet \oplus \mathcal{G}_2^\bullet$ with $\mathcal{F}_1$ and $\mathcal{F}_2$ having disjoint supports. But this contradicts the fact that $Hom(\Phi(k(x)), \Phi(k(x)))$ is a field.

- The existence of the non trivial composition

$$\mathcal{F}^\bullet[m_0] \to \mathcal{H}^{m_0} \to \mathcal{H}^{m_1} \to \mathcal{F}^\bullet[m_1]$$

is not hard to check. the middle map follows from the fact that for a finite module M over a local noetherian ring $(A, \mathfrak{m})$ where $Supp(M) = \{\mathfrak{m}\}$ there is always an injection $k(\mathfrak{m}) \hookrightarrow M$ and a surjection $M \twoheadrightarrow k(\mathfrak{m})$. Indeed we can define $\mathcal{H}^{m_0} \twoheadrightarrow k(x) \hookrightarrow \mathcal{H}^{m_1}$. For the first is given by the roof

$$
\begin{array}{cccccccccc}
\mathcal{F}^\bullet : & \cdots & \longrightarrow & F^{m_0-1} & \longrightarrow & F^{m_0} & \xrightarrow{d^m} & F^{m_0+1} & \longrightarrow & \cdots \\
{\scriptstyle qis}\uparrow & & & \uparrow & & \hookuparrow & & \uparrow & & \\
\widetilde{\mathcal{F}^\bullet} : & \cdots & \longrightarrow & F^{m_0-1} & \longrightarrow & Ker(d^m) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}^{m_0}[-m_0] : & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{H}^{m_0} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

and we can see that is the identity in cohomology. And the same construction works for the last arrow.

- In the last semi-continuity argument. The question is if

$$\mathcal{H}^i(\mathcal{P} \mid_{\{x_0\} \times Y}) = 0$$

then for some neighborhood V of x_0

$$\mathcal{H}^i(\mathcal{P} \mid_{\{x\} \times Y}) = 0.$$

Consider the following situation.

$$
\begin{array}{ccc}
\{x\} \times Y & \xrightarrow{j_x} & X \times Y \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
x & \xrightarrow{i_x} & X.
\end{array}
$$

Where i_x is a close immersion and π the projection into the first factor. Then

$$Li_x^*(R\pi_*\mathcal{P}) \simeq R\pi'_*(Lj_x^*\mathcal{P}),$$

which can be translated in to the fact that restricting and then taking cohomology is equivalent to taking cohomology and then restricting. There is semi-continuity for complexes of coherent sheaves so our statement holds since one can use semi-continuity in the left hand side. The local version of this is proven in a survey on semicontinuity theorems, [2], Lemma 1.7.

Proposition 2.2. (Orlov, 2002) Let $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ be an equivalence. Then the associated equivalence $F_{\mathcal{E}} : D^b(A \times \widehat{A}) \to D^b(B \times \widehat{B})$ is of the form

$$F_{\mathcal{E}} \simeq (N_{\mathcal{E}} \otimes (\cdot)) \circ f_{\mathcal{E}*}$$

with $N_{\mathcal{E}} \in Pic(B \times \widehat{B})$ and $f_{\mathcal{E}} : A \times \widehat{A} \to B \times \widehat{B}$ an isomorphism of abelian varieties.

Proof. As we discussed we are going to prove that F send close points to close points. We divide the proof in three steps.

1) First we show $F_{\mathcal{E}}(k(e) \boxtimes k(\hat{e})) \simeq k(e) \boxtimes k(\hat{e})$. Recall that Daniele compute the image of a close point $(a, \alpha) \in A \times \widehat{A}$ under $\mu_{A*} \circ (id \times \Phi_{\mathcal{P}_A})$,

$$\mu_{A*}(Id \times \phi_{\mathcal{P}}(A(k(a,\alpha))) = \mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O}_A \boxtimes \mathcal{P}_{\alpha}).$$

In particular if $(a, \alpha) = (e, \hat{e})$ then

$$\mu_{A*}(Id \times \phi_{\mathcal{P}}(A(k(e,\hat{e}))) = \mathcal{O}_{\triangle_A}.$$

Let $\mathcal{G} = (\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})(\mathcal{O}_{\triangle_A})$. We want to prove that $\mathcal{G} = \mathcal{O}_{\triangle_B}$. But by our triple composition formula 1.3, $\Phi_{\mathcal{G}}$ is equal to the composition

$$B^b(B) \xrightarrow{\Phi_{\mathcal{E}}} D^b(A) \xrightarrow{\Phi_{\mathcal{O}_{\triangle_A}}} D^b(A) \xrightarrow{\Phi_{\mathcal{E}_R}} D^b(B)$$

but $\Phi_{\mathcal{O}_{\triangle_A}} = id$ hence $\Phi_{\mathcal{G}} \simeq id$ and $\mathcal{G} = \mathcal{O}_{\triangle_B}$. Thus,

$$F_{\mathcal{E}}(k(e,\hat{e})) \simeq k(e,\hat{e}).$$

2) By the previous lemma there is an open neighborhood $(e, \hat{e}) \in U \subset A \times \widehat{A}$ such that for any close point $(a, \alpha) \in U$,

$$F(k((a,\alpha))) = k((b,\beta))$$

and the map $(a, \alpha) \mapsto (b, \beta)$ is a morphism.

3) We want to extend this to the whole variety $A \times \widehat{A}$. Is a general fact of abelian varieties that every $(a, \alpha) \in A \times \widehat{A}$ can be written as $(a_1, \alpha_1) + (a_2, \alpha_2)$ with $(a_i, \alpha_i) \in U$ (for all $(a, \alpha) \in A \times \widehat{A}$, $t_{-(a,\alpha)} : U \to A \times \widehat{A}$ is an open immersion so the image intersects $(-1)(U)$). Let $(b_i, \beta_i) \in B \times \widehat{B}$ be the image points of (a_i, α_i) under F and M_i the line bundles on B corresponding to β_i. We define

$$\begin{aligned} \mathcal{G} &:= (\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})(\mu_{A*}(id \times \Phi_{\mathcal{P}_A})(k(a,\alpha))) \\ &\simeq (\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R})((\mathcal{O} \boxtimes L) \otimes \mathcal{O}_{\Gamma_{-a}}). \end{aligned}$$

Where as in the example that Daniele gave us $\mu_A : A \times A \to A \times A, (a_1, a_2) \mapsto (a_1 + a_2, a_2)$, Γ_{-a} is the graph of t_{-a} and L is the line bundle corresponding to α that can be written as $L_1 \otimes L_2$ where $L_i = \mathcal{P}_{A\times\{\alpha_i\}}$.

Recall that Ana Maria showed us that $\Phi_{\mathcal{P}}(k(\alpha)) = \mathcal{P}_{A\times\{\alpha\}}$ and by the universal property of the Poincaré bundle this is exactly the line bundle corresponding to α, moreover the map $\alpha \mapsto \mathcal{P}_{A\times\{\alpha\}}$ is a group isomomorphism between $\widehat{A}$ and Pic^0.

One can check (by 1.2 or 1.3) that the induced Fourier-Mukai transform $\Phi_{\mathcal{G}} : D^b(B) \to D^b(B)$ is isomorphic to the composition

$$D^b(B) \xrightarrow{\Phi_{\mathcal{E}}} D^b(A) \xrightarrow{(L\otimes(\cdot))\circ t_{-a*}} D^b(A) \xrightarrow{\Phi_{\mathcal{E}_R}} D^b(B).$$

Hence,

$$\begin{aligned} \Phi_{\mathcal{G}} &= \Phi_{\mathcal{E}_R} \circ (L_1 \otimes L_2 \otimes (\cdot)) \circ t_{-a_1-a_2*} \circ \Phi_{\mathcal{E}} \\ &= (\Phi_{\mathcal{E}_R} \circ (L_1 \otimes (\cdot)) \circ t_{-a_1*} \circ \Phi_{\mathcal{E}}) \circ (\Phi_{\mathcal{E}_R} \circ (L_2 \otimes (\cdot)) \circ t_{-a_2*} \circ \Phi_{\mathcal{E}}). \end{aligned}$$

But $(L_i \otimes (\cdot)) \circ t_{-a_i*} = \Phi_{(\mathcal{O}\boxtimes L_i)\otimes\mathcal{O}_{\Gamma_{-a_i}}}$ and we have our formula for the priple composition;

$$\Phi_{\mathcal{E}_R} \circ (L_i \otimes (\cdot)) \circ t_{-a_i*} \circ \Phi_{\mathcal{E}} = \Phi_{\Phi_{\mathcal{E}}\times\Phi_{\mathcal{E}_R}((\mathcal{O}\boxtimes L_i)\otimes\mathcal{O}_{\Gamma_{-a_i}})}$$

but by hypothesis

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}((\mathcal{O} \boxtimes L_i) \otimes \mathcal{O}_{\Gamma_{-a_i}}) = (M_i \otimes (\cdot)) \circ t_{-b_i*}.$$

Thus,

$$\begin{aligned} \Phi_{\mathcal{G}} &= (\Phi_{\mathcal{E}_R} \circ (L_1 \otimes (\cdot)) \circ t_{-a_1*} \circ \Phi_{\mathcal{E}}) \circ (\Phi_{\mathcal{E}_R} \circ (L_2 \otimes (\cdot)) \circ t_{-a_2*} \circ \Phi_{\mathcal{E}}) \\ &= (M_1 \otimes (\cdot)) \circ t_{-b_1*} \circ (M_2 \otimes (\cdot)) \circ t_{-b_2*} \\ &= (M_1 \otimes M_2 \otimes (\cdot)) \circ t_{-b_1-b_2*} \end{aligned}$$

which means that

$$F_{\mathcal{E}}(k(a,\alpha)) = k(b_1 + b_2, \beta_1 + \beta_2).$$

This finishes the proof. ∎

Corollary 2.3. For any abelian variety A there exist, up to isomorphisms, a finite number of derived equivalent abelian varieties B.

Proof. If B is an abelian variety such that $D^b(A) \simeq D^b(B)$, then $A \times \widehat{A} \simeq B \times \widehat{B}$. Then B is a factor of $A \times \widehat{A}$ and an abelian variety has a finite number of direct factors up to automorphisms. See Theorem 18.7 in [3]. ∎

Corollary 2.4. (Corollary 9.44 in [1]) Let $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ be an equivalence with induced isomorphism $f_{\mathcal{E}} : A \times \widehat{A} \to B \times \widehat{B}$. Then $f_{\mathcal{E}}(a, \alpha) = (b, \beta)$ if and only if

$$\Phi_{(b,\beta)} \circ \Phi_{\mathcal{E}} \simeq \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)}.$$

Where $\Phi_{(a,\alpha)} = (L \otimes (\cdot)) \circ t_{a*}$ (resp. $\Phi_{(b,\beta)}$) and L is the line bundle corresponding to α.

This corollary give us a characterization of isomorphisms that can come from a derive equivalence. We want to find all of them.

Any isomorphism $f : A \times \widehat{A} \to B \times \widehat{B}$ can be written as

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

and we can associate to f the isomorphism $\tilde{f} : B \times \widehat{B} \to A \times \widehat{A}$ given by

$$\tilde{f} = \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix}.$$

Of course here we are implicitly identifying the dual of $\widehat{A}$ with A and the same for B.

Definition 2.5. By $U(A \times \widehat{A}, B \times \widehat{B})$ we denote the subgroup of isomorphisms $f : A \times \widehat{A} \to B \times \widehat{B}$ such that $f^{-1} = \tilde{f}$.

Corollary 2.6. The isomorphism $f_{\mathcal{E}}$ associated to $\Phi_{\mathcal{E}}$ is contained in $U(A \times \widehat{A}, B \times \widehat{B})$.

Proof. By 2.4 and the fact that $\Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$ is a group homomorphism we have

$$F_{\mathcal{E}} \circ F_{(a,\alpha)} = F_{f_{\mathcal{E}}(a,\alpha)} \circ F_{\mathcal{E}}.$$

Daniele showed us that

$$F_{(a,\alpha)} \simeq L \boxtimes L_0^{\vee} \otimes (\cdot) : D^b(A \times \widehat{A}) \to D^b(A \times \widehat{A})$$

where $L_0 = \mathcal{P}\,|_{\{a\} \times \widehat{A}}$. Hence, the previous equality says that

$$(N_{\mathcal{E}} \otimes (\cdot)) \circ f_{\mathcal{E}*} \circ \left(L \boxtimes L_0^{\vee} \otimes (\cdot)\right) \simeq \left(M \boxtimes M_0^{\vee} \otimes (\cdot)\right) \circ (N_{\mathcal{E}} \otimes (\cdot)) \circ f_{\mathcal{E}*}$$

by projection formula and, since $f_{\mathcal{E}}$ is an isomorphisms, this is equivalent to

$$f_{\mathcal{E}*}\left(L \boxtimes L_0^{\vee}\right) \simeq M \boxtimes M_0^{\vee}$$

which is exactly $\widehat{f_{\mathcal{E}}}(\beta, -b) = (\alpha, -a)$ and on the other hand $f_{\mathcal{E}}(a, \alpha) = (b, \beta)$. Thus, $\tilde{f}_{\mathcal{E}}(b, \beta) = (a, \alpha)$ which means that $\tilde{f}_{\mathcal{E}} = f_{\mathcal{E}}^{-1}$. ∎

Proposition 2.7. (Orlov, Polishchuk. See [4] and [5]) Consider two abelian varieties A and B. Any $f \in U(A \times \widehat{A}, B \times \widehat{B})$ is of the form $f = f_{\mathcal{E}}$ for some equivalence $\Phi_{\mathcal{E}} : D^b(A \to D^b(B))$.

The following corollary answer our question or at leas translate the question of wether two abelian varieties are derived equivalent in something that has to do with the geometry of $A \times \widehat{A}$ and $B \times \widehat{B}$.

Corollary 2.8. Two abelian varieties A, B are derived equivalent is and only if there is an isomorphism $f : A \times \widehat{A} \to B \times \widehat{B}$ with $f^{-1} = \tilde{f}$. In other words $D^b(A) \simeq D^b(B)$ is and only if $U(A \times \widehat{A}, B \times \widehat{B}) \neq \varnothing$.

3 Cohomological Fourier-Mukai

The following discussion follows the exposition of section 5.2 in [1]. Let $\mathcal{F}^\bullet \in D^b(X)$ be a bounded complex of coherent sheaves over a smooth projective variety X over $\mathbb{C}$. We associate the element

$$[\mathcal{F}^\bullet] := \sum_i (-1)^i \left[\mathcal{F}^i\right] \in K(X)$$

where $K(X)$ is the Grothendieck group. Recall that $[\mathcal{E}^0] + [\mathcal{E}^2] = [\mathcal{E}^1]$ in $K(X)$ if there is an exact sequence

$$0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to 0.$$

Recall that every coherent sheaf admits a finite locally free resolution, which means that every element of $K(X)$ can be written as linear combinations of classes of locally free sheaves. This allow us to define a ring structure on $K(X)$ with multiplication

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes \mathcal{E}_2]$$

for locally free sheaves. We define the map

$$\begin{array}{ccc} D^b(X) & \to & K(X) \\ \mathcal{F}^\bullet & \mapsto & [\mathcal{F}^\bullet]. \end{array}$$

Notice that $[\mathcal{F}^\bullet[k]] = (-1)^k [\mathcal{F}^\bullet]$ and $[\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet] = [\mathcal{F}_1^\bullet] + [\mathcal{F}_2^\bullet]$. And also one can check that

$$[\mathcal{F}^\bullet] = \sum (-1)^i \left[\mathcal{H}^i(\mathcal{F}^\bullet)\right].$$

In particular two isomorphic objects in $D^b(X)$ land in the same object under this map. Also the derived tensor is just the normal tensor of complexes for complexes of locally free sheaves which means the map $[\cdot] : D^b(X) \to K(X)$ respect the additive and multiplicative structure on both sides. Now let

$f : X \to Y$ be any projective morphism. Then the pull-back defines a ring homomorphism $f^* : K(Y) \to K(X)$. On the other hand for a coherent sheaf we define the *generalized direct image* as

$$f_!(\mathcal{F}) := \sum (-1)^i \left[R^i f_*(\mathcal{F}) \right].$$

And it defines a group homomorphism $f_! : K(X) \to K(Y)$

The derived pull-back and push-forward are compatible with this two homomorphisms, i.e., the following diagrams commute

$$\begin{array}{ccc} D^b(Y) & \xrightarrow{Lf^*} & D^b(X) \\ {\scriptstyle [\cdot]}\downarrow & & \downarrow{\scriptstyle [\cdot]} \\ K(Y) & \xrightarrow[f^*]{} & K(X), \end{array} \qquad \begin{array}{ccc} D^b(X) & \xrightarrow{Rf_*} & D^b(Y) \\ {\scriptstyle [\cdot]}\downarrow & & \downarrow{\scriptstyle [\cdot]} \\ K(X) & \xrightarrow[f_!]{} & K(Y). \end{array}$$

We can define, in analogy to what we have done, the K-theoretic Fourier-Mukai transform. Let $e \in K(X \times Y)$ be some class in the Grothendieck group, then we define

$$\begin{array}{rcl} \Phi_e^K : K(X) & \to & K(Y) \\ E & \mapsto & p_!(e \otimes q^* E) \end{array}$$

and the compatibilities mentioned before give us the commutativity of the following diagram:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{\mathcal{E}}} & D^b(Y) \\ {\scriptstyle [\cdot]}\downarrow & & \downarrow{\scriptstyle [\cdot]} \\ K(X) & \xrightarrow[\Phi^K_{[\mathcal{P}]}]{} & K(Y). \end{array}$$

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