# Autoequivalences of the bounded derived category of an abelian variety

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In the last two talks we have described the relation between derived equivalent abelian varieties. For this, we started with a derived equivalence $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ and turned it into a derived equivalence $F_{\mathcal{E}} : D^b(A \times \widehat{A}) \to D^b(B \times \widehat{B})$:

$$\begin{array}{ccc}
D^b(A \times \widehat{A}) & \xrightarrow{F_{\mathcal{E}}} & D^b(B \times \widehat{B}) \\
\downarrow{\scriptstyle \mathrm{id} \times \Phi_{\mathcal{P}_A}} & & \downarrow{\scriptstyle \mathrm{id} \times \Phi_{\mathcal{P}_B}} \\
D^b(A \times A) & & D^b(B \times B) \\
\downarrow{\scriptstyle \mu_{A*}} & & \downarrow{\scriptstyle \mu_{B*}} \\
D^b(A \times A) & \xrightarrow{\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}} & D^b(B \times B)
\end{array}$$

The map $\mu_A : A \times A \to A \times A$ is given by $\mu_A(a_1, a_2) = (a_1 + a_2, a_2)$, and every arrow in this diagram is an equivalence, so the arrow at the top is defined. In this talk, we are going to apply this to the case $B = A$, which lets us compute the group of autoequivalences of $D^b(A)$. We start with a general result by Bondal and Orlov, which we will only state (and not use):

**Theorem 1.** *Let $X$ be a smooth projective variety with ample (anti-)canonical bundle. Then the group of autoequivalences $D^b(X)$ is generated by shifts, automorphisms of $X$ and twists by line bundles. More explicitly, we have an isomorphism*

$$D^b(X) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$$

In the case of an abelian variety, especially if it is principally polarized, we will get a more concrete description, which is not very compatible with this description.

**Lemma 1.** *The map $\mathrm{Aut}(D^b(A)) \to \mathrm{Aut}(D^b(A \times \widehat{A})), \Phi_{\mathcal{E}} \mapsto F_{\mathcal{E}}$ is a group homomorphism.*

*Proof.* We only have to show that if $\mathcal{G}$ is the kernel of $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}}$, we have $\Phi_{\mathcal{G}} \times \Phi_{\mathcal{G}_R} = \Phi_{\mathcal{F}} \times \Phi_{\mathcal{F}_R} \circ \Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}$. This follows from the composition formula of Fourier-Mukai kernels. □

We also proved the following theorem of Orlov:

**Theorem 2.** *Let $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ be an equivalence, then the induced equivalence $F_{\mathcal{E}} : D^b(A \times \widehat{A}) \to D^b(B \times \widehat{B})$ is of the form $F_{\mathcal{E}} = (N_{\mathcal{E}} \otimes -) \circ f_{\mathcal{E}*}$*

This means in our case that we have maps $\mathrm{Aut}(D^b(A)) \to \mathrm{Aut}(A \times \widehat{A}), \Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ and $\mathrm{Aut}(D^b(A)) \to \mathrm{Pic}(A \times \widehat{A}), \Phi_{\mathcal{E}} \mapsto N_{\mathcal{E}}$, and that we can identify $\Phi_{\mathcal{E}}$ with the pair $(N_{\mathcal{E}}, f_{\mathcal{E}})$. We make/recall the following definition:

**Definition 1.** *We let $U(A \times \widehat{A})$ denote the subgroup of $\mathrm{Aut}(A \times \widehat{A})$ given by those automorphisms $f$ that satisfy $f^{-1} = \tilde{f}$, where*

$$\tilde{f} = \begin{pmatrix} \widehat{f_4} & -\widehat{f_2} \\ -\widehat{f_3} & \widehat{f_1} \end{pmatrix}, \text{ if } f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

We also have shown that the map $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ is in fact a map $\mathrm{Aut}(D^b(A)) \mapsto U(A \times \widehat{A})$, and by a Theorem by Orlov and Polishchuk this map is surjective. It is also a morphism of groups:

**Proposition 1.** *Let $\Phi_{\mathcal{E}}, \Phi_{\mathcal{F}} \in \mathrm{Aut}(D^b(A))$. Then $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}} = (N_{\mathcal{F}} \otimes f_{\mathcal{F}*} N_{\mathcal{E}}, f_{\mathcal{F}} \circ f_{\mathcal{E}})$*

*Proof.* Writing the formula down explicitly and using the fact that derived tensor commutes with derived pushforward gives

$$(N_{\mathcal{F}} \otimes -) \circ f_{\mathcal{F}*} \circ (N_{\mathcal{E}} \otimes -) \circ f_{\mathcal{E}*} = (N_{\mathcal{F}} \otimes -) \circ (f_{\mathcal{F}*} N_{\mathcal{E}} \otimes f_{\mathcal{F}*}(-)) \circ f_{\mathcal{E}*}$$

Changing parantheses around gives the result. □

Now, if we describe the kernel of this map, we will have a full description of $\mathrm{Aut}(D^b(A))$. This will be the main theorem of this talk:

**Theorem 3.** *Let $A$ be an abelian variety. Then there is a short exact sequence:*

$$0 \to \mathbb{Z} \oplus A \times \widehat{A} \to \mathrm{Aut}(D^b(A)) \to U(A \times \widehat{A}) \to 1$$

*where $\mathbb{Z} \oplus A \times \widehat{A}$ is the subgroup of $\mathrm{Aut}(D^b(A))$ generated by the shifts and the maps $t_{a*}$ and $N \otimes -$*

To prove this, we first need a technical lemma. Given a Fourier-Mukai equivalence $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$, we assign to it the equivalence $F_{\mathcal{E}} : D^b(A \times \widehat{A}) \to D^b(B \times \widehat{B})$. This is again a Fourier-Mukai transform, with kernel $\mathcal{I}(\mathcal{E})$.

**Lemma 2.** *Let $\pi : A \times \widehat{A} \times B \times \widehat{B} \to A \times B$ be the canonical projection. Then*

$$\pi_* \mathcal{I}(\mathcal{E}) \simeq \mathcal{E}^{\vee}_{(e_A, e_B)} \otimes \mathcal{E}$$

*Proof.* ☐

Now, we can prove the previous theorem:

*Proof.* First, we have to show that these automorphisms lie in the kernel. For this, we will use the fact that $f_{\mathcal{E}}(a,\alpha) = (b,\beta)$ if and only if $F_{\mathcal{E}}(k(a) \boxtimes k(\alpha)) = k(b) \boxtimes k(\beta)$ and that $\mu_*((\mathrm{id} \times \Phi_{\mathcal{P}})(k(a) \boxtimes k(\alpha))) = (\mathcal{O} \boxtimes L) \otimes \Gamma_{-a}$.

1. For the shift functor, the induced functor $D^b(A \times A) \to D^b(A \times A)$ is isomorphic to the identity.

2. Let $a_0 \in A$ and consider the translation by $a_0$, given by the Fourier-Mukai kernel $\mathcal{O}_{\Gamma_{a_0}}$. Now the opposite kernel, $\mathcal{O}_{\Gamma_{a_0 R}}$, is again given by $\mathcal{O}_{\Gamma_{a_0}}$ (because taking the adjoint gives the pullback and inverting the direction gives us back the original map) so we have the following:

$$(t_{a_0 *} \times t_{a_0 *})((\mathcal{O} \boxtimes \mathcal{P}_\alpha) \otimes \Gamma_{-a}) = (t_{a_0 *}\mathcal{O} \boxtimes t_{a_0 *}\mathcal{P}_\alpha) \otimes \Gamma_{-a}$$

   Because $t_{a_0 *}\mathcal{P}_\alpha \simeq \mathcal{P}_\alpha$, the result follows.

3. Let $N \in \widehat{A}$, and $\Phi_{\mathcal{E}} = N \otimes -$. Then, $\mathcal{E} = \Delta_* N$ and because the inverse of the map $N \otimes -$ is $N^* \otimes -$, we also have $\mathcal{E}_R = \Delta_* N^*$. So $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R} = (N \boxtimes N^*) \otimes -$. So, we want to show that $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}((\mathcal{O} \boxtimes \mathcal{P}_\alpha) \otimes \Gamma_{-a}) = (\mathcal{O} \boxtimes \mathcal{P}_\alpha) \otimes \Gamma_{-a}$. Now, evaluating gives us:

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}((\mathcal{O} \boxtimes \mathcal{P}_\alpha) \otimes \Gamma_{-a}) = (N \boxtimes N^*) \otimes \mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O} \boxtimes P_\alpha)$$

   We have that

$$(N \boxtimes N^*) \otimes \mathcal{O}_{\Gamma_{-a}} = \mathcal{O}_{\Gamma_{-a}} \Leftrightarrow N \in \mathrm{Pic}^0(A)$$

   So, because everything appearing in these equations are line bundles (and thus they have inverses), we get that

$$\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}((\mathcal{O} \boxtimes \mathcal{P}_\alpha) \otimes \Gamma_{-a}) = \mathcal{O}_{\Gamma_{-a}} \otimes (\mathcal{O} \boxtimes P_\alpha) \Leftrightarrow N \in \mathrm{Pic}^0(A)$$

Next, we show that this is in fact the whole kernel. Let $\mathcal{E}$ be such that $f_{\mathcal{E}} = \mathrm{id}$, so that $F_{\mathcal{E}} = N_{\mathcal{E}} \otimes -$. Then, if we regard $F_{\mathcal{E}}$ as a Fourier-Mukai transform $\Phi_{\mathcal{I}(\mathcal{E})}$, we have that $\mathcal{I}(\mathcal{E}) = \Delta_* N_{\mathcal{E}}$. By the previous lemma, we know that $\pi_*\Delta_* N_{\mathcal{E}} \simeq \mathcal{E}^\vee_{(e,e)} \otimes \mathcal{E}$, and because the left side is supported on the diagonal, the right side is too. Now, we can assume $\mathcal{E}^\vee_{(e,e)} \neq 0$, because there exists $(a,b)$ such that $\mathcal{E}^\vee_{(a,b)} \neq 0$. Now, if we define

$$\Phi_{\mathcal{F}^\vee} := t_{-a*} \circ \Phi_{\mathcal{E}^\vee} \circ t_{-b*}$$

then $\mathcal{F}^\vee_{(e,e)} \neq 0$, because (using triple compositions), $\mathcal{F}^\vee = t_{(-a,-b)_*}\mathcal{E}^\vee$, and we can replace $\mathcal{E}$ by $\mathcal{F}$ (because the maps $t_{-a*}$ are in the kernel). Because $\mathcal{E}^\vee_{(e,e)} \neq 0$ and $\mathcal{E}^\vee_{(e,e)} \otimes \mathcal{E}$ is supported at the diagonal, $\mathcal{E}$ is supported at the diagonal. Because every Fourier-Mukai equivalence comes from a (possibly shifted) sheaf, $\mathcal{E}$ is a sheaf up to shift. Putting this together, $\Phi_{\mathcal{E}} = E \otimes -$ for some sheaf $E$ which is in fact a shifted line bundle (because it maps $k(a) \boxtimes k(\alpha)$ to itself). If $E \notin \mathrm{Pic}^0(A)$, then the map $E \otimes -$ does not induce the identity map. ☐

This theorem tells us that the only thing we need to do to understand the autoequivalences of $D^b(A)$ is computing the group $U(A \times \widehat{A})$. In the case of a principally polarized abelian variety we can do this very explicitly:

**Proposition 2.** *Let $(A, \varphi_L)$ be a generic principally polarized abelian variety. Then*

$$U(A \times \widehat{A}) \simeq \mathrm{SL}_2(\mathbb{Z})$$

First of all, we want to find generators of $\mathrm{SL}_2(\mathbb{Z})$ in $U(A \times \widehat{A})$. We will use the following result:

**Proposition 3.** *Let $\Phi_{\mathcal{E}} : D^b(A) \to D^b(B)$ be an equivalence, and for $(a, \alpha) \in A \times \widehat{A}$, define $\Phi_{(a,\alpha)} = (\mathcal{P}_\alpha \otimes -) \circ t_{a*}$. Then, for the induced morphism $f_{\mathcal{E}}$, we have $f_{\mathcal{E}}(a, \alpha) = (b, \beta)$ if and only if*

$$\Phi_{(b,\beta)} \circ \Phi_{\mathcal{E}} = \Phi_{\mathcal{E}} \circ \Phi_{(a,\alpha)}$$

In case of the Poincare bundle, we also have the following fact:

$$(\mathcal{P}_\alpha^* \otimes -) \circ \Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{P}} \circ t_\alpha^*$$

$$\Phi_{\mathcal{P}} \circ (\mathcal{P}_\alpha \otimes -) \simeq t_\alpha^* \circ \Phi_{\mathcal{P}}$$

Piecing this together, we get that

$$\Phi_{(-\alpha,a)} \circ \Phi_{\mathcal{P}} = (\mathcal{P}_a \otimes -) \circ t_{-\alpha *} \circ \Phi_{\mathcal{P}} = (\mathcal{P}_a \otimes -) \circ {t_\alpha}^* \circ \Phi_{\mathcal{P}} = (\mathcal{P}_a \otimes -) \circ \Phi_{\mathcal{P}} \circ (\mathcal{P}_\alpha \otimes -) = \ldots$$

which means that $f_{\mathcal{P}}(a, \alpha) = (-\alpha, a)$. Using the isomorphism $\varphi_L : A \to \widehat{A}$, we get that $f_{\mathcal{P}}$ induces the map

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : A \times A \to A \times A$$

Analogously (this is an exercise), the map $L \otimes -$ induces the map

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : A \times A \to A \times A$$

Those two matrices generate $\mathrm{SL}_2(\mathbb{Z})$.