Triangulated (and Derived) Categories

Nicolas A. Schmidt

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1 Motivation

Recall that our goal in this seminar is to study the *derived category*

$$\mathbf{D}(X) := \mathbf{D}(\mathrm{QCoh}(X))$$

of a scheme, when X is an abelian variety. The formalism of *derived categories* is an abstract machinery that given an *abelian category* $\mathcal{A}$ produces another category $\mathbf{D}(\mathcal{A})$, that is still *additive* but in general *no longer abelian*. This construction can be broken down as follows

$$\mathcal{A} \overset{\text{chain complexes}}{\rightsquigarrow} \underbrace{\mathrm{Ch}_\bullet(\mathcal{A})}_{\text{still abelian}} \overset{\text{localization}}{\rightsquigarrow} \underbrace{\mathrm{Ch}_\bullet(\mathcal{A})[\mathcal{W}^{-1}]}_{\text{only additive}}$$

We first go from the category $\mathcal{A}$ to the category $\mathrm{Ch}_\bullet(\mathcal{A})$ of chain complexes[1], and then invert[2] a certain class $\mathcal{W}$ of morphisms (the 'weak equivalences' or 'quasi-isomorphisms') in $\mathrm{Ch}_\bullet(\mathcal{A})$ to obtain the derived category

$$\mathbf{D}(\mathcal{A}) = \mathrm{Ch}_\bullet(\mathcal{A})[\mathcal{W}^{-1}]$$

of $\mathcal{A}$. The category $\mathbf{D}(\mathcal{A})$ may not exist for set-theoretical reasons, and even if it does it is hard to deal with in general. Luckily, in most (all ?) cases of interest, $\mathcal{A}$ has enough injective/projective objects and so $\mathrm{Ch}_\bullet(\mathcal{A})$ carries canonically the structure of a *model category*. The derived category then coincides with the *homotopy category* $\mathbf{Ho}(\mathrm{Ch}_\bullet(\mathcal{A}))$ in the sense of the theory of model categories (see [1]) and can therefore be effectively computed as a full subcategory

$$\mathbf{D}(\mathcal{A}) = \mathbf{Ho}(\mathrm{Ch}_\bullet(\mathcal{A})) \subseteq \pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$$

of the 'category of connected components' (or 'naive homotopy category') $\pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$.

By construction the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is just an ordinary (1-)category. However in order to make even basic homological constructions (in particular to get long exact sequences) one has to use an 'extra structure' that is present in $\mathbf{D}(\mathcal{A})$, but

[1] or to the category $\mathrm{Ch}^\bullet(\mathcal{A}) = \mathrm{Ch}_\bullet(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ of cochain complexes, according to convention

[2] more details (maybe) later

which cannot be recovered from the abstract category $\mathbf{D}(\mathcal{A})$ alone. In order to axiomatize this 'extra structure', Jean-Louis Verdier came up with the notion of a *triangulated category* in his 1967 thesis [2], in which he also introduced derived categories. Independently the same axioms (except the 'octahedral axiom') where found by Dieter Puppe (see [3], [4]) when he investigated the formal structure of the stable homotopy category of spaces. Even though it may appear that 'stable homotopy theory of topological spaces' and 'derived categories of abelian categories' are two completely different contexts, both may be treated on the same footing via the theory of model categories, and the 'extra structure' present in both cases can be constructed in terms of the structure of a model category (see [1, I.§3]).

A more modern and conceptually better approach to derived categories and stable homotopy theory is through the notion of a *stable* $(\infty, 1)$*-category* (see [5]), thus replacing the ordinary 1-categorical localization $\mathrm{Ch}_\bullet(\mathcal{A})[\mathcal{W}^{-1}]$ with the *simplicial* or *Bousfiel-Kan localization* of $\mathrm{Ch}_\bullet(\mathcal{A})$ at $\mathcal{W}$, which no longer gives an ordinary category but instead an $(\infty, 1)$-category. This has the advantage that the 'extra structure' can be recovered from this abstract $(\infty, 1)$-category alone. Of course this comes at the cost of having to deal with $(\infty, 1)$-categories instead of ordinary categories. Also this does not make the study of the derived category (which is the category of connected components of the $(\infty, 1)$-category) or the notion of a triangulated category or a model category obsolete. But it clarifies the conceptual meaning behind these notions.

With this motivation in mind, let's forget everything about model categories and $(\infty, 1)$-categories and let's look how this 'extra structure' works in an illustrative and important example of a triangulated category, before we introduce the axioms.

2 An example of a triangulated category

In this section let $\mathcal{A}$ always denote an additive category. Recall that

2.1 Definition. The category $\mathrm{Ch}_\bullet(\mathcal{A})$ of **chain complexes** on $\mathcal{A}$ is the category whose objects are sequences $A_\bullet = (A_k)_{k\in\mathbb{Z}}$ of objects in $\mathcal{A}$ together with morphisms

$$\partial_k : A_k \to A_{k-1}$$

for all $k \in \mathbb{Z}$ such that $\partial_{k-1} \circ \partial_k = 0$ for all $k \in \mathbb{Z}$. A morphism $f_\bullet : A_\bullet \to B_\bullet$ in this category is a sequence of morphisms

$$f_k : A_k \to B_k$$

that make all diagrams

$$\begin{tikzcd} A_k \arrow[r, "\partial_k"] \arrow[d, "f_k"'] & A_{k-1} \arrow[d, "f_{k-1}"] \\ B_k \arrow[r, "\partial_k"] & B_{k-1} \end{tikzcd}$$

commute.

Dually the category $\mathrm{Ch}^\bullet(\mathcal{A})$ of **cochain complexes** on $\mathcal{A}$ has as objects sequences $A^\bullet = (A^k)_{k\in\mathbb{Z}}$ with morphisms

$$d_k : A^k \to A^{k+1}$$

that *increase the degree* and satisfy $d_{k+1}\circ d_k = 0$ for all $k \in \mathbb{Z}$. Morphisms $f^\bullet : A^\bullet \to B^\bullet$ of cochain complexes are defined similiarly as sequences $f^\bullet = (f^k)_{k\in\mathbb{Z}}$ of morphisms making

$$\begin{tikzcd} A^k \arrow[r, "d_k"] \arrow[d, "f^k"'] & A^{k+1} \arrow[d, "f^{k+1}"] \\ B^k \arrow[r, "d_k"] & B^{k+1} \end{tikzcd}$$

commute.

2.2 Remark. (i) We have a canonical isomorphism (!) of categories

$$\mathrm{Ch}^\bullet(\mathcal{A})^{\mathrm{op}} \simeq \mathrm{Ch}_\bullet(\mathcal{A}^{\mathrm{op}})$$

so that we really only need to deal with either chain or cochain complexes. Beware however that established notations for chain and cochain complexes may be inconsistent wrt this isomorphism. Besides the above isomorphism one also uses the isomorphism

$$\mathrm{Ch}_\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ch}^\bullet(\mathcal{A})$$

sending a chain complex $A_\bullet$ to the cochain complex $A^\bullet$ with $A^n = A_{-n}$, which leads to differences in signs for certain constructions (e.g. the 'shift functor') depending on whether one deals with chain or cochain complexes.

(ii) Both $\mathrm{Ch}_\bullet(\mathcal{A})$ and $\mathrm{Ch}^\bullet(\mathcal{A})$ are again additive, which 'degreewise' addition of morphisms.

(iii) Limits and colimits in $\mathrm{Ch}_\bullet(\mathcal{A})$ and $\mathrm{Ch}^\bullet(\mathcal{A})$ can be computed 'degreewise'. It therefore follows easily that both categories are again abelian, if $\mathcal{A}$ is abelian.

The category $\mathrm{Ch}_\bullet(\mathcal{A})$ is naturally a closed monoidal category, whenever $\mathcal{A}$ is closed monoidal via the following construction.

2.3 Definition. Let $\otimes$ and $\mathbf{Hom}_{\mathcal{A}}$ denote the tensor product and inner hom functors on $\mathcal{A}$ respectively. For two chain complexes A, B let $\mathbf{Hom}(A, B)$ denote the chain complex with

$$\mathbf{Hom}(A,B)_n = \prod_{k\in\mathbb{Z}} \mathbf{Hom}_{\mathcal{A}}(A_k, B_{k+n})$$

and differential as follows. For sake of concreteness and simplicity we assume now that $\mathcal{A}$ is the category of modules over a ring or some such thing, and define the differential using elements, leaving to the reader the task to define it in a general context. Given an element $f \in \mathbf{Hom}_{\mathcal{A}}(A,B)_n$ of degree n and writing it as a family $f = (f_k)_k$ with $f_k \in \mathbf{Hom}_{\mathcal{A}}(A_k, B_{k+n})$ we let $\partial_n f \in \mathbf{Hom}_{\mathcal{A}}(A,B)_{n-1}$ componentwise via the formula

$$(\partial_n f)_k = \partial \circ f_k + (-1)^{n-1} f_{k-1} \circ \partial \quad \forall k \in \mathbb{Z}$$

It's easy to check that this defines a complex $\mathbf{Hom}(A,B)$ that is functorial in both arguments. Similarly we can define the tensor product $A \otimes B$ of chain complexes A and B via

$$(A \otimes B)_m = \bigoplus_{n,k:\ n+k=m} A_n \otimes B_k$$

and differential given on elements by

$$\partial(a \otimes b) = (\partial a) \otimes b + (-1)^n a \otimes \partial b, \quad a \in A_n,\ b \in B_k$$

Again it's easy to check that this defines a complex $A \otimes B$ that is functorial in both arguments. Still easy but slighty more cumbersome (cf. [6, Theorem V.1.1]) is to check that we have an adjunction

$$\mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(A \otimes B, C) \simeq \mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(A, \mathbf{Hom}(B,C))$$

By abstract nonsense we then have for every A and B a canonical evaluation morphism

$$\mathrm{ev}_{A,B} : \mathbf{Hom}(A,B) \otimes A \to B$$

which corresponds to the identity under the bijection

$$\mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(\mathbf{Hom}(A,B) \otimes A, B) \simeq \mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(\mathbf{Hom}(A,B), \mathbf{Hom}(A,B))$$

One can check (actually I haven't) that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(A,B)_j \otimes A_i \arrow[r] \arrow[d] & B_{i+j} \arrow[d, "="] \\
(\mathbf{Hom}(A,B) \otimes A)_{i+j} \arrow[r, "\mathrm{ev}_{A,B}"] & B_{i+j}
\end{tikzcd}$$

where the top row is given by the composition

$$\mathbf{Hom}(A,B)_{i+j} \to \mathbf{Hom}_{\mathcal{A}}(A_i, B_{i+j}) \overset{\mathrm{ev}_{A_i,B_{i+j}}}{\to} B_{i+j}$$

where the first morphism is the projection, and the second morphism is the internal evaluation in the closed category $\mathcal{A}$.

Using these evaluation morphisms and the bijection

$$\mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(\mathbf{Hom}(B,C) \otimes \mathbf{Hom}(A,B), \mathbf{Hom}(A,C)) \simeq \mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(\mathbf{Hom}(B,C) \otimes \mathbf{Hom}(A,B) \otimes A, C)$$

we get also get canonical internal composition morphisms

$$\mathbf{Hom}(B,C) \otimes \mathbf{Hom}(A,B) \to \mathbf{Hom}(A,C)$$

natural in A, B and C. One can check (again I haven't) that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(B,C)_j \otimes \mathbf{Hom}(A,B)_i \arrow[r] \arrow[d] & \mathbf{Hom}(A,C)_{i+j} \arrow[d, "="] \\
(\mathbf{Hom}(B,C) \otimes \mathbf{Hom}(A,B))_{i+j} \arrow[r] & \mathbf{Hom}(A,C)_{i+j}
\end{tikzcd}$$

where the lower horizontal map fits into a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(B,C)_j \otimes \mathbf{Hom}(A,B)_i \arrow[r] \arrow[d] & \mathbf{Hom}(A,C)_{i+j} \arrow[d] \\
\mathbf{Hom}_{\mathcal{A}}(B_{k+i}, C_{k+i+j}) \otimes \mathbf{Hom}_{\mathcal{A}}(A_k, B_{k+i}) \arrow[r] & \mathbf{Hom}_{\mathcal{A}}(A_k, C_{k+i+j})
\end{tikzcd}$$

for all $k \in \mathbb{Z}$, where the lower horizontal map is the internal composition in $\mathcal{A}$.

2.4 Remark. If $\mathcal{A}$ is *symmetric* monoidal then $\mathrm{Ch}_\bullet(\mathcal{A})$ is also symmetric. But beware that the symmetry isomorphism

$$c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

for chain complexes is given by the *nontrivial* formula

$$(c_{A,B})_{i+j}(a \otimes b) = (-1)^{i \cdot j} b \otimes a, \quad \forall a \in A_i,\ b \in B_j$$

One therefore has to be careful about the order of tensor products of chain complexes, when giving explicit formulas (e.g. for the differential of the mapping cone below).

From now on assume $\mathcal{A}$ to be abelian, unless otherwise stated. On the category of chain complexes exists an important functor, that really gets the whole business of homological algebra started:

2.5 Definition. The functor

$$H_0 : \mathrm{Ch}_\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$$

given on objects by

$$H_0(A_\bullet) = \frac{\ker(A_0 \to A_{-1})}{\mathrm{im}(A_1 \to A_0)} = \frac{\ker(\partial_0)}{\mathrm{im}(\partial_1)}$$

and on morphisms in the obvious way, is called the **zeroth homology functor**, and the object $H_0(A_\bullet)$ is called the **zeroth homology** of $A_\bullet$. More generally for any $i \in \mathbb{Z}$ we can consider the **i-th homology functor**

$$H_i : \mathrm{Ch}_\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$$

given by

$$H_i(A_\bullet) = \frac{\ker(A_i \to A_{i-1})}{\mathrm{im}(A_{i+1} \to A_i)} = \frac{Z_i}{B_i}$$

and calls

$$Z_i = \ker(A_i \to A_{i-1}), \quad B_i = \mathrm{im}(A_{i+1} \to A_i)$$

the **'group' of i-cycles** resp. the **'group' of i-boundaries**.

Dually one considers the **i-th cohomology functor**

$$H^i : \mathrm{Ch}^\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$$

on cochain complexes given by

$$H^i = \frac{Z^i}{B^i}$$

where

$$Z^i = \ker(A^i \to A^{i+1}), \quad B^i = \mathrm{im}(A^{i-1} \to A^i)$$

are the 'groups' of **i-cocycles** and **i-coboundaries** respectively.

The functor H_0 interacts with the tensor product of chain complexes as follows:

2.6 Proposition. *The functor*

$$H_0 : \mathrm{Ch}_\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$$

is weakly monoidal wrt the tensor product of chain complexes and the given tensor product on $\mathcal{A}$. More precisely, there exists a family of morphisms

$$\phi_{A,B} : H_0(A) \otimes H_0(B) \longrightarrow H_0(A \otimes B)$$

natural in $A, B \in \mathrm{Ob}(\mathrm{Ch}_\bullet(\mathcal{A}))$ given as follows. In the following commutative diagram with the obvious maps, the left vertical morphism is well-defined and both rows are exact, hence inducing the dashed vertical arrow:

$$\begin{tikzcd}
B_0(A) \otimes Z_0(B) \oplus Z_0(A) \otimes B_0(A) \arrow[r] \arrow[d] & Z_0(A) \otimes Z_0(B) \arrow[r] \arrow[d] & H_0(A) \otimes H_0(B) \arrow[r] \arrow[d, dashed] & 0 \\
B_0(A \otimes B) \arrow[r] & Z_0(A \otimes B) \arrow[r] & H_0(A \otimes B) \arrow[r] & 0
\end{tikzcd}$$

Proof. The only nonobvious thing is the well-definedness of the left vertical map and the exactness of the top row. The exactness of the top row follows easily by diagram chasing imaginary elements in

$$\begin{tikzcd}
B_0(A) \otimes B_0(B) \arrow[r] \arrow[d] & Z_0(A) \otimes B_0(B) \arrow[r] \arrow[d] & H_0(A) \otimes B_0(B) \arrow[d] \\
B_0(A) \otimes Z_0(B) \arrow[r] \arrow[d] & Z_0(A) \otimes Z_0(B) \arrow[r] \arrow[d] & H_0(A) \otimes Z_0(B) \arrow[d] \\
B_0(A) \otimes H_0(B) \arrow[r] & Z_0(A) \otimes H_0(B) \arrow[r] & H_0(A) \otimes H_0(B)
\end{tikzcd}$$

in which every row and column is a cokernel sequence, thanks to the right exactness of the tensor product on $\mathcal{A}$ (which follows as for any closed monoidal category from the adjunction $(-) \otimes A \dashv \mathbf{Hom}_{\mathcal{A}}(A, -)$. To show the well-definedness of the left vertical map, note that[3] for any $b \in Z_0(B)$ and any $a \in A_0$ we have

$$\partial(a \otimes b) = \partial a \otimes b + a \otimes \partial b = \partial a \otimes b$$

So if $a = \partial a'$ we have

$$a \otimes b = \partial(a' \otimes b)$$

and hence $B_0(A) \otimes Z_0(B)$ maps into $B_0(A \otimes B)$ under the map $Z_0(A) \otimes Z_0(B) \to Z_0(A \otimes B)$. □

The functor H_0 and the inner hom-functor **Hom** together yield another category, the 'naive homotopy category' or 'category of connected components of $\mathrm{Ch}_\bullet(\mathcal{A})$ mentioned earlier.

2.7 Definition. Let

$$\mathcal{K}(\mathcal{A}) := \pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$$

denote the category which has the same objects as $\mathrm{Ch}_\bullet(\mathcal{A})$ but with hom-sets given as follows. The covariant hom-functor

$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$$

on the unit object $\mathbf{1}$ (wrt the monoidal structure on $\mathcal{A}$) induces a functor

$$\mathrm{Ch}_\bullet(\mathcal{A}) \longrightarrow \mathrm{Ch}_\bullet(\mathrm{Ab})$$

by 'termwise' application. In particular the inner hom object $\mathbf{Hom}(A, B)$ for two chain complexes in $\mathcal{A}$ gives rise to a chain complex

$$\underline{\mathrm{Hom}}(A, B) := \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \mathbf{Hom}(A, B))$$

of abelian groups. Let now

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, B) := [A, B] := H_0(\underline{\mathrm{Hom}}(A, B))$$

then it follows by abstract nonsense and the functoriality of **Hom** that this defines the structure of a category, the composition map

$$[B, C] \otimes [A, B] \longrightarrow [A, C]$$

being induced by the canonical map

$$H_0(\underline{\mathrm{Hom}}(B, C)) \otimes H_0(\underline{\mathrm{Hom}}(A, B)) \to H_0(\underline{\mathrm{Hom}}(A, C))$$

[3] Again I'm using elements even though $\mathcal{A}$ should be an abstract abelian category, because ...

and the internal composition (in chain complexes of abelian groups)

$$\underline{\mathrm{Hom}}(B,C)\otimes\underline{\mathrm{Hom}}(A,B)\longrightarrow\underline{\mathrm{Hom}}(A,C)$$

which is induced from

$$\mathbf{Hom}(B,C)\otimes\mathbf{Hom}(A,B)\longrightarrow\mathbf{Hom}(A,C)$$

and the fact that the hom-functor

$$\mathrm{Hom}_{\mathcal{A}}(\mathbf{1},-):\mathcal{A}\longrightarrow\mathrm{Ab}$$

'is' also weakly monoidal.

2.8 Remark. The set $[A,B]$ is more explicitly described as consisting of all chain maps $A\to B$ modulo *chain homotopy equivalence*. Indeed since

$$\mathrm{Hom}_{\mathcal{A}}(\mathbf{1},\mathbf{Hom}_{\mathcal{A}}(X,Y))\simeq\mathrm{Hom}_{\mathcal{A}}(X,Y)$$

it follows easily that the complex $\underline{\mathrm{Hom}}(A,B)=\mathrm{Hom}_{\mathcal{A}}(\mathbb{1},\mathbf{Hom}(A,B))$ is given in degree n by

$$\underline{\mathrm{Hom}}(A,B)_n=\prod_k\mathrm{Hom}_{\mathcal{A}}(A_k,B_{k+n})$$

with differential given by the same formula as for $\mathbf{Hom}(A,B)$. In particular the boundary of a degree 0 element $f=(f_k)_k\in\prod_k\mathrm{Hom}_{\mathcal{A}}(A_k,B_k)$ is given by

$$(\partial f)_k=\partial\circ f_k-f_{k-1}\circ\partial$$

and the differential of a degree 1 element $g=(g_k)_k\in\prod_k\mathrm{Hom}_{\mathcal{A}}(A_k,B_{k+1})$ is given by

$$(\partial g)_k=\partial\circ g_k+g_{k-1}\circ\partial$$

In particular the group Z_0 of 0-cycles identifies with the set $\mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(A,B)$ of chain maps $f:A\to B$, and the subgroup $B_0\subseteq Z_0$ of 0-boundaries coincides with

$$B_0=\{(f_k)_k:\exists(g_k)_k\in\prod_k\mathrm{Hom}_{\mathcal{A}}(A_k,B_{k+1})\quad f_k=\partial\circ g_k+g_{k-1}\circ\partial\}$$

By definition B_0 is the set of chain maps that are **chain homotopic to the zero map**. More generally, one calls two chain maps $f,f':A\to B$ **(chain) homotopic** if $f-f'$ is chain homotopic to zero. In that case an element $(g_k)_k\in\prod_k\mathrm{Hom}_{\mathcal{A}}(A_k,B_{k+1})$ with $\partial g=f-f'$ is called a **chain homotopy** between f and f'.

Maps of chain complexes that become isomorphisms in the category $\mathcal{K}(\mathcal{A})$ have a special name, given in the following definition.

2.9 Definition. A map $g : B \to A$ of chain complexes is said to be **homotopy left inverse** to $f : A \to B$ iff

$$g \circ f \sim \mathrm{id}_A$$

i.e. if $g \circ f$ is chain homotopic to the identity. Similarly, we say that g is **homotopy right inverse** to f iff

$$f \circ g \sim \mathrm{id}_B$$

If g is both homotopy left and homotopy right inverse to f, we say that g is a **homotopy inverse** to f. A map $f : A \to B$ that possesses a homotopy inverse is called a **(chain) homotopy equivalence**.

2.10 Remark. Note that a homotopy inverse of a map is itself only well-defined up to homotopy equivalence. Note also that the notion of a *quasi-isomorphism* (to be introduced down below) is weaker than the notion of a homotopy equivalence.

One is usually mostly interested in the higher homology, but the higher homology can be reduced to the zeroth homology using the 'shift functors':

2.11 Remark. For any $k \in \mathbb{Z}$ there is an automorphism[4]

$$[k] : \mathrm{Ch}_\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ch}_\bullet(\mathcal{A})$$

given on objects by the formula

$$A[k]_n = A_{n+k}$$

where the differential $\partial^{A[k]}$ on $A[k]$ is twisted[5] according to the formula

$$\partial_n^{A[k]}(a) = (-1)^k \partial_{n+k}^A(a), \quad a \in A[k]_n = A_{n+k}$$

We then have a canonical isorphism

$$H_i \simeq H_0 \circ [i]$$

One sometimes uses the special notation $\Sigma = [-1]$ and $\Omega = [1]$, and calls Σ, Ω the 'suspension functor' and the 'loop functor' respectively, because the usual suspension and loop functors on pointed topological spaces have the same effect on singular homology

$$H_{i,\mathrm{sing}}(\Sigma X) \simeq H_{i-1,\mathrm{sing}}(X), \quad H_{i,\mathrm{sing}}(\Omega X) \simeq H_{i+1,\mathrm{sing}}(X)$$

In the same way[6] one defines shift functors

$$[k] : \mathrm{Ch}^\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ch}^\bullet(\mathcal{A})$$

[4]so in particular an autoequivalence, but its really an automorphism

[5]This funny sign is not necessary in order to make $A[k]$ into a complex, but leads to nice formulas later on. There is probably also a conceptual reason for this lurking in the background ...

[6]Sometimes people define shift functors differently on cochain complexes than on chain complexes, so that $A[k]^n = A^{n-k}$ while $B[k]_n = B_{n+k}$.

on the category of cochain complexes, s.t.

$$H^i \simeq H^0 \circ [i]$$

Moreover the shift functor $[k]$ preserves the chain homotopy relation between maps, and therefore induces a functor

$$[k] : \mathcal{K}(\mathcal{A}) \xrightarrow{\sim} \mathcal{K}(\mathcal{A})$$

on the naive homotopy category. In particular the shift functors preserve chain homotopy equivalences. The additive automorphism $\Sigma = [-1]$ of $\mathcal{K}(\mathcal{A})$ will later be one part of the structure of a triangulated category.

The functor H_0 does not behave well wrt categorical limits/colimits. In particular it does not preserve cokernels (or kernels) as the next example shows.

2.12 Example. Consider the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "{(\mathrm{id}, -\mathrm{id})}"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "+"] \arrow[d, "+"] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z}
\end{tikzcd}$$

Interpreting each column as part of a two-term chain complex (the condition $\partial^2 = 0$ is trivial for two-term complexes), where we agree that the top row corresponds to degree 0 and the bottom row to degree -1, and designating these chain complexes by A,B and C from left to right, the above diagram corresponds to a cokernel sequence

$$A \to B \to C$$

Applying H_0 we get the sequence

$$\underbrace{H_0(A)}_{=0} \to \underbrace{H_0(B)}_{=\mathbb{Z}} \to \underbrace{H_0(C)}_{=0}$$

which is not exact in the middle. Note that the original version of this note contained a 'false example', where the complexes A, B and C had only nonzero terms in degree ≥ 0. It's easy to see, that the H_0-sequence is always exact when $C = \operatorname{coker}(A \to B)$ and A, B are concentrated in nonnegative degrees.

Fortunately the functor H_0 behaves well wrt 'homotopy versions' of the kernel and cokernel, namely the *homotopy fiber* and *homotopy cofiber*. In contrast to the (co-)kernel, the homotopy (co-)fiber is not well-defined up to isomorphism. Fortunately for us, the homotopy cofiber[7] has a nice model both in the category of topological space and the category of chain complexes, namely the *mapping cone*.

[7] What about the homotopy fiber?

2.13 Definition. Given a map $f : A \to B$ of chain complexes, the **mapping cone** of f is the chain complex $\mathrm{cone}(f)$ together with the canonical map

$$B \to \mathrm{cone}(f)$$

given as follows. For $n \in \mathbb{Z}$ we let

$$\mathrm{cone}(f)_n = A_{n-1} \oplus B_n$$

and let the differential be given by

$$\partial(a, b) = (-\partial a, \partial(b) - f(a)), \quad \forall (a, b) \in A_{n-1} \oplus B_n$$

The canonical map $B \to \mathrm{cone}(f)$ is termwise given simply by

$$B_n \longrightarrow \mathrm{cone}(f)_n = A_{n-1} \oplus B_n, \quad b \mapsto (0, b)$$

One checks easily that $\mathrm{cone}(f)$ is a chain complex, that $B \to \mathrm{cone}(f)$ is a chain map and that is natural in the sense that any commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & \mathrm{cone}(f) \arrow[d] \\ A' \arrow[r, "f'"] & B' \arrow[r] & \mathrm{cone}(f') \end{tikzcd}$$

2.14 Remark. The definition of the mapping cone for chain complexes looks rather unmotivated. But it is in fact given by the exact same formula as the mapping cone in algebraic topology, when one substitutes the unit interval $I = [0, 1]$ for the chain complex (again $\mathbb{1}$ is the unit object in the monoidal category $\mathcal{A}$)

$$I_\bullet : \qquad \ldots \longrightarrow 0 \longrightarrow \mathbb{1}e \xrightarrow{\partial_1} \mathbb{1}v_0 \oplus \mathbb{1}v_1 \longrightarrow 0$$

This chain complex (for $\mathcal{A} = \mathrm{Ab}$) is precisely the *normalized*[8] chain complex of the 1-simplex $\Delta^1 \in \mathrm{sSet}$ which has precisely one non-degenerate edge e and two non-degenerate vertices v_0, v_1 with differential given by

$$\partial e = v_0 - v_1$$

[8]That is, the chain complex obtained by throwing away all degenerate faces

Taking $A = B$ and f to be the identity, the mapping cone gives back the **cone** over A

$$\mathrm{cone}(A) := \mathrm{cone}(A \xrightarrow{\mathrm{id}} A)$$

The mapping cone for a general morphism $f : A \to B$ is given in terms of the cone as a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathrm{cone}(A) \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

which can be easily verified in terms of the explicit formulas. The same formula also relates the cone and the mapping cone in topology. Moreover both the cone of chain complexes and the cone of (pointed) topological spaces is given as a pushout

$$\begin{tikzcd} A \arrow[r, "A\otimes i_0"] \arrow[d] & A \otimes I \arrow[d] \\ * \arrow[r] & \mathrm{cone}(A) \end{tikzcd}$$

where[9] $i_0 : \mathbb{1} \to I$ is the inclusion of one of the vertices into the 'unit interval', and $* = 0$ stands for the zero object in both categories, and $\otimes$ is to be understood as the usual product of (pointed) topological spaces for the cone in algebraic topology.

2.15 Remark. Because of the pushout formula

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathrm{cone}(A) \arrow[r, "\phi"] & \mathrm{cone}(f) \end{tikzcd}$$

for the mapping cone and the fact that pushout squares 'compose', we see that the pushout square ('collapse the cone to a point')

$$\begin{tikzcd} \mathrm{cone}(A) \arrow[r, "\phi"] \arrow[d] & \mathrm{cone}(f) \arrow[d] \\ 0 \arrow[r] & \mathrm{coker}(\phi) \end{tikzcd}$$

identifies with a square

$$\begin{tikzcd} \mathrm{cone}(A) \arrow[r, "\phi"] \arrow[d] & \mathrm{cone}(f) \arrow[d, "\psi"] \\ 0 \arrow[r] & \mathrm{coker}(f) \end{tikzcd}$$

[9]Here an object of $\mathcal{A}$ is identified with the complex concentrated in degree zero it gives rise to.

The map

$$\psi : \mathrm{cone}(f) \longrightarrow \mathrm{coker}(f)$$

is explicitly given by

$$\begin{aligned} \psi_n : \mathrm{cone}(f)_n = A_{n-1} \oplus B_n &\longrightarrow \mathrm{coker}(f)_n = B_n / f(A_n) \\ (a, b) &\longmapsto [b] \end{aligned}$$

Thus the cokernel map $B \to \mathrm{coker}(f)$ factors as

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & \mathrm{coker}(f) \\ & \mathrm{cone}(f) \arrow[ur, "\psi"'] & \end{tikzcd}$$

so that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(f) \arrow[d, "\psi"] \\ A \arrow[r, "f"] & B \arrow[r] & \mathrm{coker}(f) \end{tikzcd}$$

The comparison map

$$\mathrm{cone}(f) \longrightarrow \mathrm{coker}(f)$$

is not a chain homotopy equivalence in general, even if we assume the map f to be a degreewise monomorphism (cf. [8, 1.5.8]). However in that case it *is* a quasi-isomorphism[10] (cf. loc. cit).

But there is one important case when the comparison map is always a chain homotopy equivalence, as we will see now.

2.16 Remark. Let us specialize the above discussion to the case where B is itself a mapping cone and f the canonical map of the mapping cone, i.e. let's consider the diagram

$$X \xrightarrow{g} Y \xrightarrow{i_1} \mathrm{cone}(g) \xrightarrow{i_2} \mathrm{cone}(i_1)$$

where both i_1 and i_2 are the canonical maps for the mapping cone construction. Since

$$\mathrm{cone}(g)_n = X_{n-1} \oplus Y_n$$

it is easy to see that

$$\mathrm{coker}(i_1) \simeq \Sigma X$$

[10] It would be interesting to know whether this map is a chain homotopy equivalence when f is a cofibration in the projective model structure on chain complexes, i.e. if f is a degreewise monomorphism with degreewise projective cokernels.

In fact
$$X_{n-1} \oplus Y_n \longrightarrow X_{n-1}, \quad (x,y) \mapsto x$$
defines a chain map[11]
$$\delta : \mathrm{cone}(g) \longrightarrow \Sigma X$$
identifying ΣX as the cokernel of i_1. Observing this, the commutative diagram from the end of the previous remark now takes the following form (extending it on the left)

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(g) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(i_1) \arrow[d] \\
X \arrow[r, "g"] & Y \arrow[r, "i_1"] & \mathrm{cone}(g) \arrow[r, "\delta"] & \Sigma X
\end{tikzcd}$$

We claim now that the rightmost vertical map, i.e. the comparison map
$$\mathrm{cone}(i_1) \longrightarrow \Sigma X$$
between $\mathrm{cone}(i_1)$ and $\mathrm{coker}(i_1) \simeq \Sigma X$ is a chain homotopy equivalence. To see this, first note that writing
$$\mathrm{cone}(i_1)_n = Y_{n-1} \oplus \mathrm{cone}(g)_n = Y_{n-1} \oplus (X_{n-1} \oplus Y_n)$$
the comparison map is explicitly given in degree n by
$$\begin{aligned} Y_{n-1} \oplus X_{n-1} \oplus Y_n &\longrightarrow X_{n-1} \\ (y, x, y') &\longmapsto x \end{aligned}$$
Define now a homotopy inverse
$$\Sigma X \longrightarrow \mathrm{cone}(i_1)$$
in degree n by the formula
$$\begin{aligned} X_{n-1} &\longrightarrow Y_{n-1} \oplus X_{n-1} \oplus Y_n \\ x &\longmapsto (-g(x), x, 0) \end{aligned}$$
This map is easily seen to be compatible with differentials. In fact unwinding the definitions, we see that the differential on $\mathrm{cone}(i_1)$ is explicitly given in terms of components by
$$\partial(y, x, y') = (-\partial y, -\partial x, \partial y' - g(x) - y)$$
Therefore in particular
$$\partial(-g(x), x, 0) = (\partial g(x), \partial x, 0) = (g(\partial x), \partial x, 0)$$

[11] Here we use that Σ twists the differential

Note that in the expression $g(\partial x)$ the differential refers to the differential of the chain complex X. Viewing $x \in X_{n-1}$ as an element of $(\Sigma X)_n$, the differential $\partial^{\Sigma X}$ the differential $\partial^X = \partial$ are related by $\partial x = -\partial^{\Sigma X} x$. Therefore we see that the above formula indeed defines a chain map

$$\Sigma X \longrightarrow \operatorname{cone}(i_1)$$

To see that this map is homotopy inverse to the comparison map, first note that by construction the composition

$$\Sigma X \longrightarrow \operatorname{cone}(i_1) \longrightarrow \Sigma X$$

is in fact *equal* to the identity. Composing these maps the other way round we get a map

$$\phi : \operatorname{cone}(i_1) \longrightarrow \operatorname{cone}(i_1)$$

of chain complexes, that is explicitly given in degree n by

$$\begin{aligned} Y_{n-1} \oplus X_{n-1} \oplus Y_n &\longrightarrow Y_{n-1} \oplus X_{n-1} \oplus Y_n \\ (y, x, y') &\longmapsto (-g(x), x, 0) \end{aligned}$$

This map is chain homotopic to the identity via the following explicit chain homotopy $h = (h_k)_k = \prod_k \operatorname{Hom}_{\mathcal{A}}(\operatorname{cone}(g)_k, \operatorname{cone}(g)_{k+1})$ given in degree n by

$$h_n(y, x, y') = (-y', 0, 0)$$

We compute that

$$\begin{aligned} \partial h_n(y, x, y') + h_{n-1}\partial(y, x, y') &= \partial(-y', 0, 0) + h_{n-1}(-\partial y, -\partial x, \partial y' - g(x) - y) \\ &= (\partial y', 0, y') + (y + g(x) - \partial y', 0, 0) = (y + g(x), 0, y') \end{aligned}$$

And therefore

$$\operatorname{id} - \phi = \partial h$$

which concludes the proof that

$$\operatorname{cone}(i_1) \longrightarrow \Sigma X$$

is a chain homotopy equivalence. The datum $(X, Y, \operatorname{cone}(g), g, i_1, \delta)$ will later be part of the triangulated structure on $\mathcal{K}(\mathcal{A})$, and the above argument will constitute a part of showing that axiom (**TR2**) holds.

The terminology 'homotopy cofiber' or 'homotopy cokernel' for the mapping cone is appropriate, in view of the following proposition.

2.17 Proposition. *Given a (not necessarily commutative) diagram with solid arrows*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"] & B \arrow[r, "i"] \arrow[d, "\psi"] & \operatorname{cone}(f) \arrow[r, "\delta"] \arrow[d, dashed, "\alpha"] & \Sigma A \arrow[d, "\Sigma\phi"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "i'"] & \operatorname{cone}(f') \arrow[r, "\delta'"] & \Sigma A' \end{tikzcd}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$ *such that the leftmost inner square is commutative up to chain homotopy, i.e. such that there exists a chain homotopy* g *with*

$$\partial g = \psi \circ f - f' \circ \phi$$

there exists a dashed arrow α *making both the middle and right inner squares commutative up to chain-homotopy.*

Proof. Before we begin the proof we remark that this proposition gives axiom (**TR3**) for triangulated category[12]. We simply define a map α by the formula

$$\alpha(a,b) = (\phi(a), \psi(b) + g_n(a)), \quad \forall\ (a,b) \in \mathrm{cone}(f)_n = A_{n-1} \oplus B_n$$

First we have to verify that α is indeed a chain map. That is (TODO) an easy computation left to the reader. Moreover one checks that the middle and rightmost inner squares actually compute on the nose, again by an easy computation. $\square$

The mapping cone satisfies the following exactness property, which can be seen[13] as the essential reason why long exact sequences in (co-)homology exist.

2.18 Lemma. *For any map* $f : A \to B$ *and any chain complex* K *the diagram*

$$A \xrightarrow{f} B \to \mathrm{cone}(f)$$

induces an exact sequence

$$[A, K] \leftarrow [B, K] \leftarrow [\mathrm{cone}(f), K]$$

of abelian groups, that is the pullback $B \to K$ *along* $B \to \mathrm{cone}(f)$ *of every map* $\mathrm{cone}(f) \to K$ *restricts to a zero-homotopic map* $A \to K$*, and conversely every map* $B \to K$ *that restricts to a zero-homotopic map* $A \to K$ *arises as a pullback of a map* $\mathrm{cone}(f) \to K$*.*

Proof. We recall the following commutative (pushout) square that was given earlier relating the mapping cone to the cone over A

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathrm{cone}(A) \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

By the lemma below any chain map $\mathrm{cone}(A) \to K$ is zero-homotopic, hence by the commutativity of the diagram also the restriction to A of the pullback of any map $\mathrm{cone}(f) \to K$

[12] We also remark that the proof of the corresponding statement to be found in at least two popular graduate textbooks on homological algebra is invalid, since they assume the homotopy commutative diagrams to be strictly commutative and simply argue with the functoriality of the mapping cone.

[13] modulo the existence of (co-)fiber sequences

along $B \to \mathrm{cone}(f)$ is zero-homotopic. On the other hand if $\phi : B \to K$ restricts to a zero-homotopic map $f : A \to K$, then this map f extends to a map $\widehat{f} : \mathrm{cone}(A) \to K$ by the lemma below. Since the maps ϕ and $\widehat{f}$ agree on A by construction, we get a unique induced map $\psi : \mathrm{cone}(f) \to K$ extending both ϕ and $\widehat{f}$ by the universal property of the pushout. This map then is the extension of ϕ to be constructed. □

2.19 Lemma. *The cone* $\mathrm{cone}(A)$ *of a chain complex* A *is contractible, i.e. the unique map* $0 \to \mathrm{cone}(A)$ *is a chain-homotopy equivalence. Moreover a map* $f : A \to K$ *can be (possibly nonuniquely) extended to a map* $\mathrm{cone}(A) \to K$ *if and only if* f *is homotopic to zero.*

Proof. To prove the first statement, we have to show that the unique map $\mathrm{cone}(A) \to 0$ is homotopy inverse to $0 \to \mathrm{cone}(A)$. For this it suffices to show that the identity $\mathrm{id} : \mathrm{cone}(A) \to \mathrm{cone}(A)$ is homotopic to zero by giving an explicit chain homotopy $g = (g_k)_k \in \prod_k \mathrm{Hom}_{\mathcal{A}}(\mathrm{cone}(A)_k, \mathrm{cone}(A)_{k+1})$ with $\partial g = \mathrm{id}$. Let

$$g_k : \mathrm{cone}(A)_k = A_{k-1} \oplus A_k \to \mathrm{cone}(A)_{k+1} = A_k \oplus A_{k+1}$$

be given by the formula

$$g_k(a, a') = (-a', 0)$$

then

$$\begin{aligned}(\partial g)_k(a, a') &= (\partial \circ g_k + g_{k-1} \circ \partial)(a, a') = \partial(-a', 0) + g_{k-1}(-\partial a, \partial a' - a) \\ &= (\partial a', a') + (a - \partial a', 0) = (a, a')\end{aligned}$$

Hence $\mathrm{cone}(A)$ is contractible as claimed and it follows that any map $\mathrm{cone}(A) \to K$ is zero-homotopic, therefore any map $f : A \to K$ that extends to a map $\mathrm{cone}(A) \to K$ must be zero-homotopic. Now assume conversely, that $f : A \to K$ is chain-homotopic to zero, and let $g = (g_k)_k \in \prod_k \mathrm{Hom}_{\mathcal{A}}(A_k, K_{k+1})$ be a chain-homotopy between f and 0, i.e. $\partial g = f$. Then define

$$\widehat{f} : \mathrm{cone}(A) \to K$$

in degree k via the formula

$$\widehat{f}(a, a') = -g_{k-1}(a) + f(a'), \quad a \in A_{k-1}, a' \in A_k$$

Then $\widehat{f}$ obviously extends f and moreover it is a chain map since

$$\begin{aligned}\partial \widehat{f}(a, a') &= \partial(-g_{k-1}(a) + f(a')) = -\partial g_{k-1}(a) + \partial f(a') \\ &= g_{k-2}(\partial a) - f(a) + \partial f(a') = g_{k-2}(\partial a) - f(a) + f(\partial a')\end{aligned}$$

where we have used

$$f(a) = (\partial g)_{k-1}(a) = \partial g_{k-1}(a) + g_{k-2}(\partial a)$$

On the other hand

$$\widehat{f}\partial(a, a') = \widehat{f}(-\partial a, \partial a' - a) = g_{k-2}(\partial a) + f(\partial a' - a)$$

□

2.20 Remark. The 'unit-interval' chain complex I is not contractible, even though the unit interval $[0,1]$ is a contractible topological space. The reason why the intuition fails here, is that when working with pointed topological spaces one should really take the reduced singular chain complex instead of the ordinary chain complex. The reduced singular complex of a contractible topological space then is contractible in the above sense.

We have just seen above that any 'homotopy cofiber'

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f)$$

gives rise to a (very) short exact sequence

$$[A,K] \leftarrow [B,K] \leftarrow [\mathrm{cone}(f),K]$$

We can now simply iterate the procedure of taking homotopy cofibers to obtain an infinite sequence

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f) \xrightarrow{i_2} \mathrm{cone}(i_1) \xrightarrow{i_3} \mathrm{cone}(i_2) \to \dots$$

and so obtain a long exact sequence

$$[A,K] \leftarrow [B,K] \leftarrow [\mathrm{cone}(f),K] \leftarrow [\mathrm{cone}(i_1),K] \leftarrow [\mathrm{cone}(i_2),K] \leftarrow [\mathrm{cone}(i_3),K] \leftarrow \dots$$

by applying $[-,K]$. It may be a little surprising that this sequence is nontrivial in general, since the cokernel of a cokernel map is zero. But the homotopy cokernel of a homotopy cokernel is not zero in general. In fact long exact sequences in (co-)homology are precisely all obtained in the above way, via long homotopy (co-)fiber sequences. In order to recognize the long exact sequence above to something that looks like a long exact sequence in cohomology, we need the following lemma.

2.21 Lemma. *Let $f : A \to B$ be a map of chain complexes.*

(i) The diagram[14]

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(f) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \mathrm{cone}(i_2) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "i_1"] & \mathrm{cone}(f) \arrow[r, "\delta"] & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B \end{tikzcd}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$, whose two rightmost maps are the 'comparison maps' between the mapping cone and the cokernel from remark 2.16, is commutative up to chain homotopy, i.e. it gives a commutative diagram in $\pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$. Moreover the two rightmost vertical maps are chain homotopy equivalences.

[14]Note the sign in front of $-\Sigma f$. It's really there, even if you wish it wasn't.

(ii) There exists a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(f) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \operatorname{cone}(i_2) \arrow[r, "i_4"] \arrow[d] & \operatorname{cone}(i_3) \arrow[r, "i_5"] \arrow[d] & \operatorname{cone}(i_4) \arrow[r, "i_6"] \arrow[d] & \dots \\
A \arrow[r, "f"] & B \arrow[r, "i_1"] & \operatorname{cone}(f) \arrow[r, "\delta"] & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B \arrow[r, "-\Sigma i_1"] & \Sigma \operatorname{cone}(f) \arrow[r, "-\Sigma i_2"] & \Sigma \operatorname{cone}(i_1) \arrow[r, "-\Sigma i_3"] & \dots
\end{tikzcd}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$*, that is commutative up to chain homotopy, whose vertical maps are chain homotopy equivalences, given as follows. The top row is obtained by iterating the mapping cone construction, starting from the map* $f : A \to B$*. The bottom row is obtained by concatenating the sequence*

$$A \xrightarrow{f} B \xrightarrow{i_1} \operatorname{cone}(f) \xrightarrow{\delta} \Sigma A$$

with the result of applying the functor Σ *to the top row and then changing the signs. The vertical maps are the comparison maps between the mapping cone and the cokernel from remark 2.16.*

(iii) There exists a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(f) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \operatorname{cone}(i_2) \arrow[r, "i_4"] \arrow[d] & \operatorname{cone}(i_3) \arrow[r, "i_5"] \arrow[d] & \operatorname{cone}(i_4) \arrow[r, "i_6"] \arrow[d] & \dots \\
A \arrow[r, "f"] & B \arrow[r, "i_1"] & \operatorname{cone}(f) \arrow[r, "\delta"] & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B \arrow[r, "-\Sigma i_1"] & \Sigma \operatorname{cone}(f) \arrow[r, "-\Sigma \delta"] & \Sigma^2 A \arrow[r, "\Sigma^2 f"] & \dots
\end{tikzcd}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$*, that is commutative up to chain homotopy, whose vertical maps are chain homotopy equivalences, that is given as follows. The top row is given by iterating the mapping cone construction starting from* $f : A \to B$*. The bottom row is given by 'periodically' continuing the sequence*

$$A \xrightarrow{f} B \xrightarrow{i_1} \operatorname{cone}(f) \xrightarrow{\delta} \Sigma A$$

by applying the functor Σ *and changing signs. The vertical maps are obtained by the following recipe. Take the diagram obtained in (ii), and glue the result of applying* Σ *and changing the signs of the horizontal maps to it below as indicated by the following diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(f) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \operatorname{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \operatorname{cone}(i_2) \arrow[r, "i_4"] \arrow[d] & \operatorname{cone}(i_3) \arrow[r, "i_5"] \arrow[d] & \operatorname{cone}(i_4) \arrow[r, "i_6"] \arrow[d] & \dots \\
A \arrow[r, "f"] & B \arrow[r, "i_1"] & \operatorname{cone}(f) \arrow[r, "\delta"] & \Sigma A \arrow[r, "-\Sigma f"] \arrow[d, "\mathrm{id}"] & \Sigma B \arrow[r, "-\Sigma i_1"] \arrow[d, "\mathrm{id}"] & \Sigma \operatorname{cone}(f) \arrow[r, "-\Sigma i_2"] \arrow[d, "\mathrm{id}"] & \Sigma \operatorname{cone}(i_1) \arrow[r, "-\Sigma i_3"] \arrow[d] & \dots \\
 & & & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B \arrow[r, "-\Sigma i_1"] & \Sigma \operatorname{cone}(f) \arrow[r, "-\Sigma \delta"] & \Sigma^2 A \arrow[r, "\Sigma^2 f"] & \dots
\end{tikzcd}$$

Iterate this procedure to obtain the claimed diagram.

Proof. Claim (iii) follows immediately from (ii) observing (cf. remark 2.11) that Σ preserves chain homotopy equivalences as well as the relation of chain homotopy between maps. Claim (ii) follows immediately by applying (i) iteratively. Therefore it remains to show (i). The fact that the two rightmost vertical maps are chain homotopy equivalences was already proved in remark 2.16. It was also seen there that the second rightmost inner square commutes on the nose. It therefore only remains to show that the square

$$\begin{tikzcd} \operatorname{cone}(i_1) \arrow[r, "i_3"] \arrow[d, "\phi"] & \operatorname{cone}(i_2) \arrow[d, "\psi"] \\ \Sigma A \arrow[r, "\Sigma f"] & \Sigma B \end{tikzcd}$$

is homotopy-commutative. In degree n this diagram is given by

$$\begin{tikzcd} B_{n-1} \oplus A_{n-1} \oplus B_n \arrow[r, "(i_3)_n"] \arrow[d, "\phi_n"] & A_{n-2} \oplus B_{n-1} \oplus B_{n-1} \oplus A_{n-1} \oplus B_n \arrow[d, "\psi_n"] \\ A_{n-1} \arrow[r, "f_{n-1}"] & B_{n-1} \end{tikzcd}$$

where the top row morphism is given by

$$(b, a, b') \mapsto (0, 0, b, a, b')$$

The left vertical map is given by $(b, a, b') \mapsto a$ and the right vertical map is given by

$$(a_1, b_1, b_2, a_2, b_3) \mapsto b_2$$

This square therefore is *not* commutative on the nose. However

$$\psi \circ i_3 - \Sigma f \circ \phi = \partial h$$

for the following chain homotopy $h = (h_k)_k \in \prod_k \operatorname{Hom}_{\mathcal{A}}(\operatorname{cone}(i_1)_k, (\Sigma B)_{k+1})$ given by

$$h_n(b, a, b') = b'$$

Indeed we compute

$$\begin{aligned} \partial^{\Sigma X} h_n(b, a, b') + h_{n-1}\partial^{\operatorname{cone}(i_1)}(b, a, b') &= \partial^{\Sigma X} b' + h_{n-1}(-\partial b, -\partial a, \partial b' - f(a) - b) \\ &\overset{(!)}{=} -\partial b' + \partial b' - f(a) - b \\ &= -f(a) - b \\ &= (\psi \circ i_3 - \Sigma f \circ \phi)_n(b, a, b') \end{aligned}$$

noting again that $\partial^{\Sigma X} b' = -\partial b'$. $\square$

2.22 Corollary. *For any map $f : A \to B$ of chain complexes, and any chain complex K the diagram*

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f) \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma i_1} \Sigma\,\mathrm{cone}(f) \xrightarrow{\Sigma\delta} \Sigma^2 A \xrightarrow{\Sigma^2 f} \dots$$

induces a long exact sequence

$$[A, K] \longleftarrow [B, K] \longleftarrow [\mathrm{cone}(f), K] \longleftarrow [\Sigma A, K] \longleftarrow [\Sigma B, K] \longleftarrow [\Sigma\,\mathrm{cone}(f), K] \longleftarrow \dots$$

of abelian groups.

Proof. For

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f) \xrightarrow{\delta} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \xrightarrow{-\Sigma i_1} \Sigma\,\mathrm{cone}(f) \xrightarrow{-\Sigma\delta} \Sigma^2 A \xrightarrow{\Sigma^2 f} \dots$$

instead of

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f) \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma i_1} \Sigma\,\mathrm{cone}(f) \xrightarrow{\Sigma\delta} \Sigma^2 A \xrightarrow{\Sigma^2 f} \dots$$

this follows immediately from the above lemma and from lemma 2.18. However the signs don't matter. □

2.23 Remark. The way we have proved the above corollary was way too complicated, even though it was informative from a conceptual point of view. We will rederive the exactness of the above sequence in the next section in the abstract setting of triangulated categories. Of course that makes it necesarry to prove that $\pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$ carries the structure of a triangulated category. But we only need a few of the results we have shown. In particular we don't need lemma 2.18 or 2.21.

2.24 Remark. The sequence

$$A \xrightarrow{f} B \xrightarrow{i_1} \mathrm{cone}(f) \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma i_1} \Sigma\,\mathrm{cone}(f) \xrightarrow{\Sigma\delta} \Sigma^2 A \xrightarrow{\Sigma^2 f} \dots$$

is sometimes called **cofiber sequence** or even **Puppe sequence**. This terminology is appropriate, since it is up to homotopy equivalence (and signs) just given by iterated homotopy cofibers. The same construction can be carried out in the category Top_* of pointed topological spaces (cf. [9, Theorem 4.58]) instead of $\mathrm{Ch}_\bullet(\mathcal{A})$. There is also a dual version of the cofiber sequence, the **fiber sequence**. In terms of the 'loop functor' Ω it looks like

$$\dots \longrightarrow \Omega\,\mathrm{path}(f) \xrightarrow{\Omega\alpha} \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\alpha} \mathrm{path}(f) \longrightarrow A \xrightarrow{f} B$$

in either Top_* or $\mathrm{Ch}_\bullet(\mathcal{A})$. The role of the mapping cone is played here by a certain 'mapping path space' $\mathrm{path}(f)$, which has the property of a homotopy fiber of f. In particular a dual version of lemma 2.18 holds. Writing

$$B^A = \mathbf{Hom}(A, B)$$

for the 'inner hom space' in either Top_* or $\mathrm{Ch}_\bullet(\mathcal{A})$, and using the interval object I together with the two canonical maps

$$i_0, i_1 : * \longrightarrow I$$

the 'mapping path space' $\mathrm{path}(f)$ is given as a pullback

$$\begin{tikzcd} \mathrm{path}(f) \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ \mathrm{path}(B) \arrow[r] & B \end{tikzcd}$$

similar for the pushout formula for the mapping cone in terms of the cone. The analogue of the cone is played by the 'path space'

$$\mathrm{path}(B) = \mathrm{path}(\mathrm{id}_B)$$

that comes equipped with a canonical map $\mathrm{path}(B) \to B$. This map is the top horizontal map in the following pullback square

$$\begin{tikzcd} \mathrm{path}(B) \arrow[r] \arrow[d] & B \arrow[d] \\ B^I \arrow[r, "B^{i_0}"] & * \end{tikzcd}$$

In words this diagram says that '$\mathrm{path}(B)$ consists of all path $\gamma : I \to B$ with $\gamma(0) = *$ the distinguished point of B'. The pullback formula for the 'mapping path space' $\mathrm{path}(f)$ therefore says that '$\mathrm{path}(f)$ consists of pairs (x, γ) were $x \in A$ and $\gamma : I \to B$ a path with $\gamma(0) = *$ and $\gamma(1) = f(x)$'. An easy computation shows that for chain complexes the mapping path space is degreewise given by

$$\mathrm{path}(f)_n = A_n \oplus B_{n+1}$$

with a differential that's also easy to figure out. The fiber sequence of a map $f : A \to B$ (in Top_* of $\mathrm{Ch}_\bullet(\mathcal{A})$) also induces a long exact sequence *covariantly*

$$\ldots \longrightarrow [\Omega A, K] \longrightarrow [\Omega B, K] \longrightarrow [\mathrm{path}(f), K] \longrightarrow [A, K] \longrightarrow [B, K]$$

Because of the adjointness $\Sigma \dashv \Omega$ there exists a comparison diagram (cf. [10, 8.7]) between the fiber and cofiber sequences. In particular there are comparison maps

$$\varepsilon : \Sigma\,\mathrm{path}(f) \longrightarrow \mathrm{cone}(f), \quad \eta : \mathrm{path}(f) \longrightarrow \Omega\,\mathrm{cone}(f)$$

between the mapping cone and the mapping path space. The difference between Top_* and $\mathrm{Ch}_\bullet(\mathcal{A})$ is that Σ and Ω are only adjoint functors in Top_*, whereas on $\mathrm{Ch}_\bullet(\mathcal{A})$ these functors form an adjoint autoequivalence. That implies that the naive homotopy category $\pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$

is a *stable* homotopy category, i.e. a triangulated category, whereas $\pi_0(\mathrm{Top}_*)$ is not. In particular the comparison maps

$$\varepsilon : \Sigma\,\mathrm{path}(f) \longrightarrow \mathrm{cone}(f), \quad \eta : \mathrm{path}(f) \longrightarrow \Omega\,\mathrm{cone}(f)$$

are homotopy equivalences (in fact isomorphisms) in $\mathrm{Ch}_\bullet(\mathcal{A})$. That also means that we don't even have to bother with introducing[15] path(f) and fiber sequences separately, since they can be recovered from cone(f) and the cofiber sequence.

3 Triangulated Categories

In this section we will introduce the abstract notion of a triangulated category, and deduce some elementary properties from the axioms.

Before we introduce the axioms, let's recall some facts about additive categories.

3.1 Remark. An **additive category** $\mathcal{A}$ is an Ab-category, i.e. a category enriched over the monoidal category Ab of abelian groups, that has a zero object and binary products. Equivalently (see [7, VIII.2]) it is an Ab-category with zero object in which every pair of objects $A, B \in \mathrm{Ob}(\mathcal{A})$ has a **binary product**, i.e. an object $A \oplus B$ together with maps

$$i_1 : A \to A \oplus B, \quad i_2 : B \to A \oplus B$$

and

$$p_1 : A \oplus B \to a, \quad p_2 : A \oplus B \to B$$

such that

$$p_1 \circ i_1 = \mathrm{id}_A, \quad p_2 \circ i_2 = \mathrm{id}_B, \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A\oplus B} \tag{3.1}$$

Moreover in an additive category binary coproducts exist, and the canonical map

$$A \amalg B \to A \times B$$

is an isomorphism, making every (co-)product canonically into a biproduct. Also it is justified to talk about a category being additive or not, since an additive category has a unique Ab-structure. Also the opposite category $\mathcal{A}^{\mathrm{op}}$ of an additive category $\mathcal{A}$ is again additive.

3.2 Example. The Ab-category $\mathcal{K}(\mathcal{A}) = \pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$ for an abelian category $\mathcal{A}$ introduced in the previous section is additive, a zero object being given by the zero object in $\mathrm{Ch}_\bullet(\mathcal{A})$. Moreover the biproduct of chain complexes induces a biproduct on $\mathcal{K}(\mathcal{A})$, as the equations 3.1 are preserved.

[15] I couldn't even find a textbook introducing the mapping path space for complexes.

3.3 Definition. A **triangulated category** is an additive category $\mathcal{A}$ together with an additive endofunctor

$$\Sigma : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$$

that is an equivalence, and a class $\mathcal{T}$ of diagrams of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

A diagram of this shape is called a **triangle**, and is denoted by (A, B, C, u, v, W). A triangle is called **distinguished** iff it belongs to $\mathcal{T}$. A **morphism of triangles**

$$(A, B, C, u, v, w) \to (A', B', C', u', v', w')$$

is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\phi"] & B \arrow[r, "v"] \arrow[d] & C \arrow[r, "w"] \arrow[d] & \Sigma A \arrow[d, "\Sigma\phi"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma A' \end{tikzcd}$$

This data is subject to the following axioms.

(**TR1**) For any object $A \in \mathrm{Ob}(\mathcal{A})$ the triangle

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

is distinguished. Moreover every triangle isomorphic to a distinguished triangle is distinguished, and every morphism $A \xrightarrow{u} B$ can be completed to a distinguished triangle

$$A \xrightarrow{u} B \longrightarrow C \longrightarrow \Sigma A$$

(**TR2**) If

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

is distinguished, then both

$$B \xrightarrow{v} C \xrightarrow{w} \Sigma A \xrightarrow{\Sigma u} \Sigma B$$

and

$$\Sigma^{-1} C \xrightarrow{\Sigma^{-1} w} A \xrightarrow{u} B \xrightarrow{v} C$$

are distinguished.

(**TR3**) Given the solid arrows in the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma A \arrow[d, "\Sigma\phi"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \end{tikzcd}$$

where the top rows form distinguished triangles, and the lefthand inner square commutes, there exists a dashed arrow making both the middle and right inner squares commute.

(TR4) The so called octahedral axiom. It goes as follows. Given any commutative diagram

$$\begin{tikzcd} & X_2 \arrow[dr, "u_1"] & \\ X_1 \arrow[ur, "u_3"] \arrow[rr, "u_2"] & & X_3 \end{tikzcd}$$

and any triple

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} \Sigma X_1$$

$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} \Sigma X_2$$

$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} \Sigma X_1$$

of distinguished triangles, there exist morphisms

$$m_1 : Z_3 \to Z_2$$

$$m_2 : Z_2 \to Z_1$$

such that $(\mathrm{id}_{X_1}, u_1, m_1)$ and $(u_3, \mathrm{id}_{X_3}, m_3)$ are morphisms of triangles and

$$Z_3 \xrightarrow{m_3} Z_2 \xrightarrow{m_2} Z_1 \xrightarrow{(\Sigma v_3)\circ w_1} \Sigma Z_3$$

is distinguished.

We now deduce some properties from the axiom. The first one is very useful for reduction arguments

3.4 Proposition. *(cf. [2, II.1.1.7]) Given a triangulated category $\mathcal{A} = (\mathcal{A}, \Sigma, \mathcal{T})$, the opposite category $\mathcal{A}^{\mathrm{op}}$ is canonically a triangulated category in the following way. Let*

$$T : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}$$

denote the functor given by

$$T = (\Sigma^{-1})^{\mathrm{op}}$$

which identifies with Σ^{-1} on $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$. Let $\mathcal{T}'$ denote the set of diagrams

$$A \longrightarrow B \longrightarrow C \longrightarrow TA$$

in $\mathcal{A}^{\mathrm{op}}$, such that the corresponding diagram

$$\Sigma^{-1}A \longrightarrow C \longrightarrow B \longrightarrow A$$

in $\mathcal{A}$ is an anti-distinguished triangle, that is if ϕ denotes the map $A \to B$ in the above diagram, then

$$\Sigma^{-1}A \longrightarrow C \longrightarrow B \xrightarrow{-\phi} A$$

is a distinguished triangle. Then $(\mathcal{A}^{\mathrm{op}}, T, \mathcal{T}')$ is a triangulated category.

Proof. According to Verdier it's easy, but I haven't checked it. $\square$

3.5 Remark. One simple consequence of the axioms of a triangulated category is, that for every distinguished triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

we have $vu = 0$. This follows by applying (**TR3**) to the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed] & \Sigma A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & \Sigma A \end{tikzcd}$$

noting that the upper triangle is distinguished by (**TR1**). The existence of the dashed arrow then implies that $vu = 0$.

Using the axioms of a triangulated category, we can now recover the property of the homotopy cofiber/mapping cone in our abstract setting. To ease the notation and to make it more suggestive, let's put

$$[A, B] := \mathrm{Hom}_{\mathcal{A}}(A, B)$$

for objects A, B of $\mathcal{A}$ [16]

3.6 Proposition. *Given a distinguished triangle*

$$A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow \Sigma A$$

an an object K of $\mathcal{A}$, the induced sequences

$$[A, K] \leftarrow [B, K] \leftarrow [C, K]$$

$$[K, A] \rightarrow [K, B] \rightarrow [K, C]$$

are exact.

Proof. By arguing using the opposite triangulated category $\mathcal{A}^{\mathrm{op}}$, it suffices to show the exactness of the second sequence[17]. But in fact the proof of both statements is similar. Since $vu = 0$ we have that the composite of the two maps

$$[K, A] \rightarrow [K, B] \rightarrow [K, C]$$

[16] But please don't get confused later when we talk about the native homotopy category $\mathcal{K}(\mathcal{A})$ of chain complexes as an example of a triangulated category!

[17] Says Verdier, but again I haven't checked this. Most people don't even care to mention the opposite triangulated category.

is zero. Let now $f \in [K, B]$ be a map with $vf = 0$. Applying axiom (**TR3**) to the diagram

$$\begin{tikzcd} \Sigma^{-1}K \arrow[r] \arrow[d, "\Sigma^{-1}f"] & 0 \arrow[r] \arrow[d] & K \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "\widehat{f}"] & K \arrow[d, "f"] \\ \Sigma^{-1}B \arrow[r] & \Sigma^{-1}C \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

where both rows form distinguished triangles (by (**TR1**) and (**TR2**) repeatedly), we get a dashed arrow $\widehat{f}$ making this diagram commute by axiom (**TR3**), and thus obtain a preimage of f under $[K, A] \to [K, B]$. $\square$

3.7 Corollary. *Given a distinguished triangle*

$$A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow \Sigma A$$

and any object K of $\mathcal{A}$ the very long sequence

$$\ldots \longleftarrow [\Sigma^{-1}C, K] \longleftarrow [A, K] \longleftarrow [B, K] \longleftarrow [C, K] \longleftarrow [\Sigma A, K] \longleftarrow [\Sigma B, K] \longleftarrow \ldots$$

is exact.

Another consequence of the axioms and the proposition above is that a version of the five-lemma holds in a triangulated category.

3.8 Lemma. *Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma A \arrow[r, "\Sigma f"] \arrow[d, "\Sigma\alpha"] & \Sigma B \arrow[d, "\Sigma\beta"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A' \arrow[r, "\Sigma f'"] & \Sigma B' \end{tikzcd}$$

where (A, B, C, f, g, h) and (A', B', C', f', g', h') are distinguished triangles and α and β are isomorphisms, then also γ is an isomorphism.

Proof. This can easily (for details see [2, II.1.2.3]) be reduced to the normal five lemma using the exact sequences induced by $[K, -]$ and by $[-, K]$. $\square$

The five lemma itself has a noteworthy corollary.

3.9 Corollary. *A distinguished triangle (A, B, C, u, v, w) depends up to isomorphism only on (A, B, u, v). That is, given a second distinguished triangle (A, B, C', u, v, w') there exists an isomorphism of triangles*

$$(f, g, h) : (A, B, C, u, v, w) \xrightarrow{\sim} (A, B, C', u, v, w')$$

where the maps f and g can be taken to be the identity on A resp. on B.

Proof. By (**TR3**) we get a map h making the following diagram commute

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & C \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma A \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C' \arrow[r, "w'"] & \Sigma A
\end{tikzcd}$$

By the five lemma the map h has to be an isomorphism. $\square$

Finally let us show that the naive homotopy category $\mathcal{K}(\mathcal{A}) = \pi_0(\mathrm{Ch}_\bullet(\mathcal{A}))$ of chain complexes over an abelian category $\mathcal{A}$ is canonically triangulated, by letting Σ be the functor induced by the suspension of chain complexes, and letting the class of distinguished triangles consist of all triangles that are isomorphic to triangles of the form

$$A \xrightarrow{[u]} B \xrightarrow{[v]} \mathrm{cone}(u) \xrightarrow{[\delta]} \Sigma A$$

where $u : A \to B$ is a map of chain complexes (denoting by $[u]$ the corresponding morphism in $\mathcal{K}(\mathcal{A})$) and

$$A \xrightarrow{u} B \xrightarrow{v} \mathrm{cone}(u) \xrightarrow{\delta} \Sigma A$$

is the mapping cone sequence in $\mathrm{Ch}_\bullet(\mathcal{A})$ associated to u.

3.10 Theorem. *(cf. [2, Ch. I.3]) The category $\mathcal{K}(\mathcal{A})$ together with the functor Σ and the class of distinguished triangles as defined above is a triangulated category.*

Proof. We have already remarked that $\mathcal{K}(\mathcal{A})$ is additive. What's left then is to verify the axioms (**TR1**)-(**TR4**). Since every map $f : A \to B$ can be fit into a mapping cone sequence, and obviously any triangle isomorphic to a triangle that's isomorphic to a mapping cone sequence is itself isomorphic to a mapping cone sequence, to verify (**TR1**) it only remains to show that the triangle $(A, A, 0, \mathrm{id}, 0, 0)$ is isomorphic to a mapping cone sequence in $\mathcal{K}(\mathcal{A})$. But we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "\mathrm{id}"] & A \arrow[r] & \mathrm{cone}(\mathrm{id}_A) \arrow[r, "\delta"] & \Sigma A
\end{tikzcd}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$. We have already seen that the mapping cone $\mathrm{cone}(A) = \mathrm{cone}(\mathrm{id}_A)$ over any object A is contractible, hence the third vertical map from the left is a chain homotopy equivalence, and hence this diagram gives an isomorphism of triangles in $\mathcal{K}(\mathcal{A})$. Axiom (**TR3**) was taken care of in proposition 2.17. We have also seen in remark 2.16 that any map $f : A \to B$ of chain complexes gives rise to a diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(g) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \mathrm{cone}(i_2) \arrow[d, "\phi"] \\
A \arrow[r, "g"] & B \arrow[r, "i_1"] & \mathrm{cone}(g) \arrow[r, "\delta"] & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B
\end{tikzcd}$$

that commutes up to chain homotopy, and where the two rightmost vertical maps are chain-homotopy equivalences. Replacing ϕ by $-\phi$ and $-\Sigma f$ by Σf gives another diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(g) \arrow[r, "i_2"] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(i_1) \arrow[r, "i_3"] \arrow[d] & \mathrm{cone}(i_2) \arrow[d, "-\phi"] \\ A \arrow[r, "g"] & B \arrow[r, "i_1"] & \mathrm{cone}(g) \arrow[r, "\delta"] & \Sigma A \arrow[r, "\Sigma f"] & \Sigma B \end{tikzcd}$$

that has the same properties. This proves one part of (**TR2**), at least for distinguished triangles of the special form

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{\delta} \Sigma A$$

But obviously it suffices to prove it in this case. To show the other part of (**TR2**), we have to verify that the triangle

$$\Sigma^{-1}\,\mathrm{cone}(f) \xrightarrow{\Sigma^{-1}\delta} A \xrightarrow{f} B \xrightarrow{i} \mathrm{cone}(f)$$

is again distinguished. But this follows easily by observing that we have an commutative diagram in $\mathrm{Ch}_\bullet(\mathcal{A})$

$$\begin{tikzcd} \Sigma^{-1}A \arrow[r, "\Sigma^{-1}f"] \arrow[d, "\mathrm{id}"] & \Sigma^{-1}B \arrow[r, "\Sigma^{-1}i"] \arrow[d, "\mathrm{id}"] & \Sigma^{-1}\,\mathrm{cone}(f) \arrow[r, "\Sigma^{-1}\delta"] \arrow[d, "\sim"] & A \arrow[d, "\mathrm{id}"] \\ \Sigma^{-1}A \arrow[r, "\Sigma^{-1}f"] & \Sigma^{-1}B \arrow[r, "i"] & \mathrm{cone}(\Sigma^{-1}f) \arrow[r, "\delta"] & A \end{tikzcd}$$

where i and δ in the bottom row also designate the corresponding canonical maps from the mapping cone construction for $\Sigma^{-1}f$. Therefore the top row is again a distinguished triangle. Applying the half of (**TR2**) that was already proven twice to the top row triangle, we get the claim. The proof of (**TR4**) is omitted. □

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