Line bundles and cohomology of complex tori

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1 Introduction

This is more or less a summary of the first chapter of Mumford's book on abelian varieties [1]. At parts I have expanded the explanations to suit my taste, especially in the first two sections. But mostly I skipped intricate, yet often enlightening calculations that Mumford spared no effort to type in full detail. Hopefully, this approach gives a quick (but understandable!) overview of what his first chapter contains.

We begin by showing that a compact complex Lie group X is a complex torus. This takes up the next two sections. Then we turn to studying the cohomology of the exponential sequence on X to be able to explicitly describe the line bundles on X.

2 A compact complex Lie group is commutative

Let X be a compact complex manifold of dimension g with the structure of a Lie group. We will denote by V the tangent space of X at the identity e.

We take as a black box the following two crucial results.

Fact 2.1. *For every $v \in V$ there exists a* unique *homomorphism of groups $\gamma_v : \mathbb{C} \to X$ such that the unit tangent at 0 gets mapped to v, that is $(\mathrm{d}\gamma_v)_0(\frac{\mathrm{d}}{\mathrm{d}t}) = v$.*

Fact 2.2. *Define a map $\exp : V \to X$ in the following way: $\exp(v) = \gamma_v(1)$, using the notation above. Then, this so called* exponential map *is holomorphic.*

It is clear that $(\mathrm{d}\exp)_0 : V \to V$ is just the identity map. But then, using the inverse function theorem we conclude that there is a neighbourhood of $0 \in V$ which maps (complex) isomorphically onto its image in X.

Using these tools we will prove

Theorem 2.3. *X is commutative.*

This is equivalent to showing for any $x \in X$, the conjugation defined by

$$C_x : X \to X : y \mapsto xyx^{-1}$$

is the identity. We will use the exponential map to 'linearize' this operation. Here is how one could do this:

Fact 2.4. *Let $T : X_1 \to X_2$ be a homomorphism of Lie groups. Let $\exp_i : V_i \to X_i$ denote the exponential map for $i = 1, 2$. Then $T(\exp_1(v)) = \exp_2((\mathrm{d}T)_e(v))$.*

Proof. Let $\gamma : \mathbb{C} \to X_1$ be the unique homomorphism with tangent vector v at e. Then $T \circ \gamma : \mathbb{C} \to X_2$ is a homomorphism with tangent vector $(\mathrm{d}T)_e(v)$. By uniqueness of such homomorphisms, the result follows. $\square$

However this fact would allow us to work only with the image of the exponential map and all we could do would be to show that the image of exp lies in the center of X. It turns out that this is enough by the following:

Fact 2.5. *Let G be a connected topological group and U a non-empty neighbourhood of the identity. Then G is generated by U. In particular, G is commutative if and only if there is a neighbourhood of the identity lying in the center of G.*

Proof. To simplify notation replace U with $U \cap U^{-1}$. Let H be the group generated by U. Then $H = \bigcup_{n\geq 1} U^n$. But $U^n = \bigcup_{u\in U} u \cdot U^{n-1}$, and hence it is open by induction. Consequently, so is H.

On the other hand, all the cosets of H are also open, being translates of H. Since H is in the complement of all non-trivial cosets, it is closed. G is connected, so $H = G$. $\square$

Now since $\exp(V)$ contains a neighbourhood of the identity, we only need to show that $\forall v \in V$ and $\forall x \in X$ that $C_x(\exp(v)) = \exp(v)$. Combining this observation with Fact 2.4, we conclude:

$$X \text{ is commutative} \iff \forall v\, \forall x \quad \exp((\mathrm{d}C_x)_e(v)) = \exp v.$$

Therefore, commutativity of X will follow from the following:

Lemma 2.6. *$\forall x \in X$ the automorphism $(\mathrm{d}C_x)_e : V \to V$ is the identity.*

Proof. We consider all these automorphisms simultaneously. Consider the map

$$X \ni x \overset{\phi}{\mapsto} (\mathrm{d}C_x)_e \in \mathrm{Aut}(V) \subset \mathrm{End}(V) \simeq \mathbb{C}^{g^2}.$$

The only holomorphic maps from a compact complex manifold to an affine space are the constant ones. So in showing the map above is holomorphic, the constancy will imply

$$(\mathrm{d}C_x)_e = (\mathrm{d}C_e)_e = \mathrm{Id}_V$$

as foretold.

Showing ϕ is holomorphic follows from general principles. We have a holomorphic map

$$C : X \times X \to X : (x, y) \to xyx^{-1},$$

and we can restrict the holomorphic differential

$$\mathrm{d}C : T(X \times X) \to TX$$

to $X \times V \hookrightarrow TX \times TX \simeq T(X \times X)$. Therefore the map $(x, v) \mapsto (\mathrm{d}C_x)_e(v)$ is holomorphic. Using coordinates on V and the induced coordinates on the endomorphism ring of V, we see that $x \mapsto (\mathrm{d}C_x)_e$ is holomorphic as well.

□

3 A compact complex Lie group is a complex torus

Throughout this section we will show $\exp : V \to X$ establishes X as a complex torus. For this, we need to show that exp is a surjective homomorphism with a discrete kernel. So we begin:

Theorem 3.1. $\exp : V \to X$ *is a homomorphism.*

Proof. Fix $v, w \in V$, then we wish to show $\exp(v+w) = \exp(v)\exp(w)$. Since the map $\gamma_{(v+w)}(t) = \exp(t(v+w))$ is uniquely characterized as the homomorphism $\mathbb{C} \to X$ with tangent vector $v+w$ over e, we need only show that $\phi(t) := \exp(tv)\exp(tw) = \gamma_v(t)\gamma_w(t)$ has the same characterizations.

Since γ_v, γ_w are homomorphisms and X is commutative ϕ is a homomorphism. It remains to show $(\mathrm{d}\phi)_0(\frac{\mathrm{d}}{\mathrm{d}t}) = v + w$. Let $X \times X \overset{m}{\to} X$ be the group multiplication. Then $\phi = m \circ (\gamma_v, \gamma_w)$. Now the required result will follow from two basic differential geometric results below. The first one implies

$$\mathrm{d}(\gamma_v, \gamma_w)_0(\frac{\mathrm{d}}{\mathrm{d}t}) = ((\mathrm{d}\gamma_v)_0(\frac{\mathrm{d}}{\mathrm{d}t}), (\mathrm{d}\gamma_w)_0(\frac{\mathrm{d}}{\mathrm{d}t})) = (v, w)$$

and the second one implies

$$(\mathrm{d}m)_{(e,e)}(v, w) = (\mathrm{d}m(\cdot, e))_e(v) + (\mathrm{d}m(e, \cdot))_e(w) = v + w.$$

Here we used the fact that $m(\cdot, e) = m(e, \cdot) = \mathrm{Id}_X$. The chain rule gives

$$(\mathrm{d}\phi)_0 = (\mathrm{d}m)_{(e,e)} \circ \mathrm{d}(\gamma_v, \gamma_w)_0$$

and we are done modulo the results below. □

The facts below are meant as a reminder for those who know them, and as an exercise for those who do not.

Fact 3.2. *Let* $f_1 : M \to N_1$ *and* $f_2 : M \to N_2$ *be morphisms between (complex) manifolds. Then the map* $(f_1, f_2) : M \to N_1 \times N_2$ *has differential* $\mathrm{d}(f_1, f_2) = (\mathrm{d}f_1, \mathrm{d}f_2) : TM \to TN_1 \times TN_2$.

Fact 3.3. *Let* $g : N_1 \times N_2 \to L$ *be a morphism of (complex) manifolds. Fix a point* $(p_1, p_2) \in N_1 \times N_2$ *and the inclusions:*

$$\begin{aligned} i_1 &: N_1 \overset{\sim}{\to} N_1 \times \{p_2\} \hookrightarrow N_1 \times N_2 \\ i_2 &: N_2 \overset{\sim}{\to} \{p_1\} \times N_2 \hookrightarrow N_1 \times N_2 \,. \end{aligned}$$

Then the differential of g at (p_1, p_2) can be expressed in terms of the differentials of g restricted along these sections. To be more precise, we define the partial derivatives*:*

$$\begin{array}{rcl} \partial_1 g_{(p_1,p_2)} & := & (\mathrm{d}(g \circ i_1))_{p_1} \\ \partial_2 g_{(p_1,p_2)} & := & (\mathrm{d}(g \circ i_2))_{p_2} \end{array} .$$

Now, for any $(v, w) \in TN_{1,p_1} \times TN_{2,p_2}$ we have the following equality:

$$(\mathrm{d}g)_{(p_1,p_2)}(v, w) = \partial_1 g_{(p_1,p_2)}(v) + \partial_2 g_{(p_1,p_2)}(w).$$

The fact that exp is a homomorphism between V and X has the following immediate consequence:

Corollary 3.4. *The map* $\exp : V \to X$ *is surjective with discrete kernel.*

Proof. We remarked on the fact that $\exp(V)$ contains a neighbourhood W of the identity. Moreover, a neighbourhood of the identity generates all of X. The subgroup generated by W must lie in the subgroup $\exp(V)$, hence $\exp(W) = X$.

The discreteness of the kernel $U := \ker \exp$ follows from the fact that exp is locally an isomorphism around $0 \in V$ and $e \in X$, this isolates 0 from other points in the kernel. We may translate this neighbourhood to any other point $u \in U$ and clearly exp remains an isomorphism in this neighbourhood, isolating u. □

Now we finish the proof that X is a complex torus. But first the relevant definitions.

Definition 3.5. *A subgroup Λ of an $\mathbb{R}$-vector space V that is generated (as a group) by an $\mathbb{R}$-basis of V is called a lattice.*

Definition 3.6. *If $\Lambda \hookrightarrow \mathbb{C}^g$ is a lattice, then a complex Lie group isomorphic to the quotient $\mathbb{C}^g/\Lambda$ is called a complex torus.*

Corollary 3.7. *The kernel U defined above is a lattice. Hence X is a complex torus.*

Proof. First, we need to show there exists an $\mathbb{R}$-linearly independent set of generators for U. This follows from the discreteness of U, we have deferred the rather lengthy proof to Fact 3.8.

Secondly, we need to show that this linearly independent set of generators in fact forms a basis for V. Indeed, if they generated a proper subspace $V' \subset V$ of V then the quotient $X = V/U = V/V' \times V'/U$ can not be compact. □

The following can be skipped without breaking continuity, it is meant for the curious.

Fact 3.8. *A subgroup Λ in a real vector space V is discrete if and only if there is a linearly independent set of generators for Λ.*

Proof. Without loss of generality, we will assume $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq V$.
($\Longleftarrow$) Using the basis-generators for Λ we get an isomorphism of V to $\mathbb{R}^{\dim V}$ which maps Λ to the integral lattice $\mathbb{Z}^{\dim V}$ lying in $\mathbb{R}^{\dim V}$.

($\implies$) To begin our induction on dimension, assume $V \simeq \mathbb{R}^1$. We will show that Λ is generated by a single element.

Take v to be the smallest positive number in Λ. Then we claim that $\Lambda = \mathbb{Z}\langle v\rangle$. First note that such a v exists because λ is discrete. Pick any $w \in \Lambda$, wlog $w > 0$, then find the largest $n \in \mathbb{Z}^+$ such that $w - nv \geq 0$. Necessarily $v > w - nv$ as well. By minimality of v this forces $w = nv$.

Now for the general case $\dim V > 1$. Choose $0 \neq v \in \Lambda$ and consider the quotient W of V defined by

$$0 \to \mathbb{R}\langle v\rangle \to V \to W \to 0\,.$$

The quotient map is open, thus the image of Λ in W, say Λ_W, is discrete. Using the induction hypothesis we may conclude Λ_W is generated by $\dim V - 1$ elements. Let $\Lambda_v = \mathbb{R}\langle v\rangle \cap \Lambda$, this is generated by a single element by our base case. Then

$$0 \to \Lambda_v \to \Lambda \to \Lambda_W \to 0$$

implies that Λ can be generated by $\dim V$ elements. But the $\mathbb{R}$-span of Λ equals V, consequently these generators are $\mathbb{R}$-linearly independent. $\square$

4 Cohomology

4.1 Group Cohomology

Let G be a group and Mod_G be the category of G-modules. To $M \in \mathrm{Mod}_G$ we associate $M^G := \{m \in M \mid \forall g \in G,\, gm = m\}$. This gives a functor $F : M \mapsto M^G \in \mathrm{Ab}$ which is left exact and the right derived functors RF^n $n \geq 0$ are called the cohomology functors of G and are often denoted by $H^n(G, \cdot)$. Sometimes we will refer to it as $H^n_{gp}(G, \cdot)$ to avoid possible confusion.

For a given M we will work with an explicit chain complex that calculates the cohomology groups $H^n(G, M)$. Let $C^n(G, M) := \{f : G^n \to M \mid f \text{ is a function}\}$, where $G^n = G \times \cdots \times G$ is the n-fold product. These have a natural G-module structure, where G acts on the image of a function.

The coboundary maps $\delta : C^{n-1}(G, M) \to C^n(G, M)$ are defined as follows, $\forall u_1, \ldots, u_n \in G^n$:

$$\delta(f)(u_1, \ldots, u_n) = u_1 f(u_2, \ldots, u_n) + \sum_{i=1}^{n-1} (-1)^i f(u_1, u_2, \ldots, u_i u_{i+1}, \ldots, u_n) + (-1)^n f(u_1, \ldots, u_{n-1})$$

Note the empty product is a singleton set, containing the empty set. So $C^0(G, M) \simeq M$. For $m \in M$ and $u \in G$ we have $\delta(m)(u) = um - m$. Notice that the minus sign denotes the inversion in M and in the cases we consider in the following sections, this inversion will be division rather than subtraction.

We will often refer to cocycles. The elements of the module

$$Z^n(G, M) := \ker(\delta : C^n \to C^{n+1})$$

are called n-cocycles. When $n = 1$ let us be explicit. Let $e \in C^1$, then e is a 1-cocycle if and only if

$$\delta(e)(u_1, u_2) = u_1 e(u_2) - e(u_1 u_2) + e(u_1) = 0, \quad \forall u_1, u_2 \in G.$$

In other words, a function $e : G \to M$ is a 1-cocycle if and only if it satisfies the following transformation rule:

$$e(u_1 u_2) = u_1 e(u_2) + e(u_1).$$

Recall that we will be using the multiplicative notation in the future for $M = H^*$, e will take values from the group as a subscript and moreover because our group $G = U$ will be abelian the group operation will be denoted by $+$. So the 1-cocycle condition will in later sections look like this

$$e_{u_1+u_2} = {}^{u_1}e_{u_2} \cdot e_{u_1}.$$

We then have the coboundaries. These are the images of the coboundary map,

$$B^n(G, M) = \mathrm{Im}(\delta : C^{n-1} \to C^n).$$

Then the cohomology groups are defined as

$$H^n_{gp}(G, M) = Z^n(G, M)/B^n(G, M).$$

There is a cup product here and to every short exact sequence of G-modules the associated long exact sequence of cohomology. This gives us the whole of cohomology theory. But we will not develop these here.

Let us make two basic observations. First, $H^0(G, M) = M^G$ as it should. Secondly, if G acts trivially on M then $B^1 = 0$ and the 1-cocycle condition translates to being a homomorphism of groups. That is, if G acts trivially on M then $H^1(G, M) = \mathrm{Hom}(G, M)$.

4.2 Cohomology of X

We make a list of results that we will use about the cohomologies of X and the lattice U. Here, we treat X as a complex manifold and U as a group, this is how the cohomologies are to be interpreted. We will refer to the cohomologies as (graded) algebras together with their cup product. The entire graded algebra will be denoted by $H^*(\cdot, \cdot)$ as is usual. The exterior algebra on an abelian group M is denoted by $\bigwedge^* M$, multiplication is just the wedge product.

I) We have an isomorphism of algebras $H^*(X, \mathbb{Z}) \xrightarrow{\sim} \bigwedge^* H^1(X, \mathbb{Z})$.

II) The first $\mathbb{Z}$ cohomology of X is just the $\mathbb{Z}$ dual of its first homology group, but this latter is the abelianization of $\pi_1(X)$. Therefore, $H^1(X, \mathbb{Z}) = H_1(X, \mathbb{Z})^\vee \simeq \mathrm{Hom}(\pi_1(X), \mathbb{Z})$. Since X is a torus, $\pi_1(X, e)$ is naturally identified with the lattice U. But using the fact $H^1(U, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(U, \mathbb{Z})$ we get $H^1(X, \mathbb{Z}) \xrightarrow{\sim} H^1(U, \mathbb{Z})$.

III) Hence there is an isomorphism of algebras $H^*(X, \mathbb{Z}) \xrightarrow{\sim} \bigwedge^* \mathrm{Hom}(U, \mathbb{Z})$

IV) Moreover, there exists an isomorphism of algebras $H^*(X, \mathbb{Z}) \xrightarrow{\sim} H^*(U, \mathbb{Z})$.

V) We can explicitly state the resulting isomorphism $H^*(U, \mathbb{Z}) \xrightarrow{\sim} \bigwedge^* \mathrm{Hom}(U, \mathbb{Z})$. Let $f : U^n \to \mathbb{Z}$ be a n-cocycle, denote the cohomology class of f by $[f]$. Then to f we can associate an alternating form

$$A(f)(u_1, \ldots, u_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}).$$

It turns out that the map $H^*(U, \mathbb{Z}) \xrightarrow{\sim} \bigwedge^* \mathrm{Hom}(U, \mathbb{Z})$ is simply $[f] \mapsto A(f)$.

Let $T = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{T} = \mathrm{Hom}_{\mathbb{C}-\mathbf{antilinear}}(V, \mathbb{C})$. Let $\imath : V \to V$ be the $\mathbb{R}$-linear anti-involution which corresponds to the multiplication by i. It is easy to check that $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is isomorphic to the direct sum $T \bigoplus \overline{T}$, with the isomorphism given by $\mu \mapsto \frac{1}{2i}(i\mu + \mu \circ \imath, i\mu - \mu \circ \imath)$.

VI) There is an isomorphism of algebras $H^*(X, \mathbb{C}) \xrightarrow{\sim} \bigwedge^*(T \oplus \overline{T})$.

VII) There is an isomorphism of algebras $H^*(X, \mathcal{O}_X) \xrightarrow{\sim} \bigwedge^* \overline{T}$.

VIII) The morphism between the cohomology rings $H^*(X, \mathbb{C}) \to H^*(X, \mathcal{O}_X)$ corresponding to the inclusion $\mathbb{C} \to \mathcal{O}_X$ commutes, through the isomorphisms, with the projection $\bigwedge^*(T \oplus \overline{T}) \xrightarrow{\mathrm{pr}} \bigwedge^* \overline{T}$.

Before we state the final item on our list, observe that we have a series of inclusions

$$\mathrm{Hom}(U, \mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}).$$

The first inclusion follows from the fact that U contains a basis for V and so we may $\mathbb{R}$-linearly extend a homomorphism $U \to \mathbb{Z}$ to get an $\mathbb{R}$-linear homomorphism $V \to \mathbb{R}$. Composing, we have a natural inclusion

$$\mathrm{Hom}(U, \mathbb{Z}) \hookrightarrow T \oplus \overline{T}.$$

IX) The morphism between the cohomology rings $H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{C})$ corresponding to the inclusion $\mathbb{Z} \to \mathbb{C}$ commutes, through the isomorphisms, with the inclusion $\bigwedge^* H^1(U, \mathbb{Z}) \hookrightarrow \bigwedge^*(T \oplus \overline{T})$.

5 Line bundles on complex tori

Lemma 5.1 ($\bar{\partial}$-Poincaré lemma)**.**
Using the Euclidean topology on $\mathbb{C}^n$ the higher cohomology of the sheaf of holomorphic functions $\mathcal{O}_{\mathbb{C}^n}$ vanishes. That is:

$$H^p(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0 \textit{ for } p > 0.$$

Sketch. Resolve $\mathcal{O}$ using the differential forms, this is an acyclic resolution, thus capable of giving the cohomology. Now the generalized Poincaré lemma as is usually stated, as in here [2, p.46-47], simply proves the exactness of this resolution. □

Corollary 5.2. *All holomorphic line bundles on $\mathbb{C}^n$ are trivial.*

Proof. We will use the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}^*_{\mathbb{C}^n} \to 0$$

to calculate $H^p(\mathbb{C}^n, \mathcal{O}^*_{\mathbb{C}^n})$ for $p > 0$. Since the affine space is contractible, all the higher $\mathbb{Z}$ cohomologies vanish, that is $H^p(\mathbb{C}^n, \mathbb{Z}) = 0$ for $p > 0$. Combined with the Poincaré Lemma above, the long exact sequence corresponding to the exponential sequence yields $H^p(\mathbb{C}^n, \mathcal{O}^*_{\mathbb{C}^n}) = 0$ for $p > 0$. Since $H^1(\mathbb{C}^n, \mathcal{O}^*_{\mathbb{C}^n}) \simeq \mathrm{Pic}(\mathbb{C}^n)$ as groups, we are done. □

Notation. *Following the lead of Mumford, from now on we will denote the exponential map* $\exp : V \to X$ *with* π *instead. Things would get ugly otherwise.*

Recall we had $0 \to U \to V \xrightarrow{\pi} X \to 0$ where $U \simeq \mathbb{Z}^{2g}$ is a lattice in V and $g = \dim_{\mathbb{C}} X$. In order to study the line bundles of X we will pull them back to V where they will be isomorphic to a trivial bundle with a U action on it.

We will first describe how a group action on the line bundle could be realized. Let $L \to X$ be a line bundle. For $u \in U$ let $\phi_u : V \to V : v \mapsto u + v$. Now consider the following diagram:

$$\begin{array}{ccccc} \phi_u^*(\pi^* L) & \to & \pi^* L & \to & L \\ \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{\phi_u} & V & \xrightarrow{\pi} & X \end{array}$$

By definition of pull back, the two small squares are cartesian. It follows that the larger rectangle is also cartesian. This means there is a canonical isomorphism $\pi^* L = (\pi \circ \phi_u)^* L \simeq \phi_u^*(\pi^* L)$, respecting the horizontal and vertical arrows. Therefore we could replace the diagram above with the following:

$$\begin{array}{ccccc} \pi^* L & \xrightarrow{a_u} & \pi^* L & \to & L \\ \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{\phi_u} & V & \xrightarrow{\pi} & X \end{array}$$

The map a_u gives a group homomorphism $a : U \to \mathrm{Aut}(\pi^* L) : u \mapsto a_u$.

There is another way to see these automorphisms of the pull-back bundle. As a set we simply have $\pi^* L = \{(l, v) \mid l \in L_{\pi(v)} \text{ and } v \in V\}$. Then the action of a_u looks innocent, taking $(l, v) \mapsto (l, u + v)$. However, there is some subtlety involved in this action as we will see.

5.1 $\mathrm{Pic}(X)$ and $H^1_{gp}(U, H^*)$

Observe that the quotient of $\pi^* L \to V$ with respect to U results in $L \to X$ again, this is immediate from the last description of the action a. Although $\pi^* L$ is isomorphic to the trivial bundle, the quotient L need not be! To better see what this action of U is doing, we fix a trivialization $\rho : \pi^* L \to \mathbb{C} \times V$.

Letting $b_u := \rho \circ a_u \circ \rho^{-1} : \mathbb{C} \times V \to \mathbb{C} \times V$ transfers the action of U on L to an action of the trivial bundle. (*Warning*: These are not actions of a fiber bundle in the usual sense, since the base is not preserved.)

The action of U through b can be described as follows: For each $u \in U$ there is a holomorphic function $e_u : V \to \mathbb{C}^*$ such that $b_u : (t, v) \mapsto (e_u(v)t, u + v)$. Here is how to see this. Afterall, we know that b is just translation on the base. Moreover, b_u maps individual fibers isomorphically, hence the we need a multiplicative factor $e_u(v)$ which must be holomorphic.

What remains to understand is how $e_{u+u'}$ relates to e_u and $e_{u'}$. This can be calculated immediately, but we will introduce some notation for future convenience.

Let $H = H^0(V, \mathcal{O}_V)$ and $H^* = H^0(V, \mathcal{O}_V^*)$. There is the standard action of U on these spaces, acting by translating the domain, that is by $u : f \mapsto uf := f \circ \phi_u$. Then $e : U \to H^*$ defined above has the following transformation rule:

$$e_{u+u'} = ue_{u'} \cdot e_u$$

Throughout the rest of this subsection we will show that this approach establishes an isomorphism of groups $\mathrm{Pic}(X) \xrightarrow{\sim} H^1_{gp}(U, H^*)$.

Let Z^1 be the 1-cocycles of U in H^*, that is functions $f : U \to H^*$ such that $\delta(f) = 0$. The transformation rule of e above translates to $\delta(e) = 0$. Therefore we have established a map

$$\{(L, \rho) \mid L \in \mathrm{Pic}(X),\ \rho : \pi^* L \xrightarrow{\sim} \mathbb{C} \times V\} \quad \xrightarrow{\Psi} \quad Z^1, \quad \Psi : (L, \rho) \mapsto e.$$

A few words must be said to describe the set on the left hand side because we are abusing notation a bit. $L \in \mathrm{Pic}(X)$ is not a line bundle but an isomorphism class of line bundles. We should similarly treat ρ to be an equivalence class of trivializations. Here is how: Suppose ρ and ρ' are trivializations of L and L', if there exists an isomorphism $\nu : L \xrightarrow{\sim} L'$ such that $\rho = \rho' \circ \pi^*(\nu)$ then we consider (L, ρ) and (L', ρ') to be equivalent. Now, the set on the left hand side must be considered as the equivalence classes of pairs for it to even make sense.

We will now show that Ψ is a bijection. That is, given $e \in Z^1$ we can construct a line bundle on X and a trivialization of its pullback. The first of these statements requires nothing new. Tracing our steps back, it's immediate to see that such an e induces a U action on the trivial bundle $\mathbb{C} \times V$. The quotient will be a line bundle $L \in \mathrm{Pic}(X)$. As for the trivialization, the argument is not difficult but cumbersome, so we will merely sketch the idea.

Around any $v \in V$ choose a small neighbourhood W such that U translates of W are disjoint. Each point of L corresponds to a U orbit of $\mathbb{C} \times V$. By our choice of W there exists precisely one representative on $\mathbb{C} \times W$ for each point of $L|_{\pi(W)}$. Mapping each orbit to its equivalence class gives a trivialization of $\pi^* L|_W$, say ρ_W. These trivializations glue, without calculation simply due to the nature of the construction, to give a global trivialization $\rho : \pi^* L \xrightarrow{\sim} \mathbb{C} \times V$. This trivialization has been uniquely determined by e, hence we establish a bijection:

$$\{(L, \rho) \mid L \in \mathrm{Pic}(X),\ \rho : \pi^* L \xrightarrow{\sim} \mathbb{C} \times V\} \longleftrightarrow Z^1$$

There is a natural group structure on Z^1 and through this bijection we may inherit this structure on to the left hand side. Then one might ask how this group structure looks like on the pairs (L, ρ). Let us fix an isomorphism $\mu : (\mathbb{C} \times V) \otimes (\mathbb{C} \times V) \xrightarrow{\sim} \mathbb{C} \times V$ taking the constant $1 \otimes 1$ section to constant 1 section. Then the group action looks like this: $(L, \rho) \cdot (L', \rho') = (L \otimes L', \mu \circ (\rho \otimes \rho'))$. In particular, the first component respects the group structure of $\mathrm{Pic}(X)$.

The next question to ask is this: How do the 1-cocylces corresponding to the same line bundle with different trivializations differ? Let $B^1 \subset Z^1$ be the images of the 0-cycles, that is the 1-coboundaries.

Fact 5.3. *The difference between $\Psi(L, \rho)$ and $\Psi(L', \rho')$ lies in B^1 if and only if $L \simeq L'$.*

Proof.

($\Longleftarrow$) Identify L and L'. Let $\gamma = \rho' \circ \rho^{-1}$. This is a base preserving automorphism of $\mathbb{C} \times V$, therefore it corresponds to scaling each fiber by a nonzero value. In other words, there is an $m \in H^*$ such that $\gamma(t, v) = (m(v)t, v)$.

Let $b = \rho \circ a \circ \rho^{-1}$ and $b' = \rho' \circ a \circ {\rho'}^{-1}$ be the induced actions on the trivial bundle, recall $a : U \to \hom(\pi^* L)$ is the action on the pull back bundle. Therefore, $b = \gamma^{-1} \circ b' \circ \gamma$. Then, by simply expanding out this expression, one can calculate that $e = e' \cdot \delta(m)$.

($\Longrightarrow$) Conversely, if $e = e' \cdot \delta(m)$ then we define $\gamma(t, v) = (m(v)t, v)$ and deduce $b = \gamma \circ b' \circ \gamma^{-1}$. But γ is a base preserving automorphism of the trivial bundle, hence the U actions induced by b and b' will be compatible, yielding isomorphic quotients. That is $L \simeq L'$. $\square$

We may summarize the results of this section with the following

Fact 5.4. *We have a commutative diagram, with the lower horizontal arrow a group isomorphism:*

$$\begin{array}{ccc} \{(L,\rho)\} & \overset{\Psi}{\to} & Z^1 \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \overset{\sim}{\to} & H^1_{gp}(U, H^*)\,, \end{array}$$

where the left vertical arrow forgets the trivialization of the pullback.

5.2 Chern classes

Recall that there are natural isomorphisms $H^2(X, \mathbb{Z}) \simeq H^2_{gp}(U, \mathbb{Z}) \simeq \bigwedge^2 \mathrm{Hom}(U, \mathbb{Z})$. Moreover, we may bi-linearly extend any element $\bigwedge^2 \mathrm{Hom}(U, \mathbb{Z})$ to give an alternating 2-form in $\bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \subset \bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. The image of $\bigwedge^2 \mathrm{Hom}(U, \mathbb{Z}) \hookrightarrow \bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is the set of alternating two forms taking integer values on $U \times U$.

The connecting homomorphism of the exponential map $\mathrm{Pic}(X) \overset{ch}{\to} H^2(X, \mathbb{Z})$ associates to a line bundle its Chern class (this is one way to define the first Chern class of a line bundle). If we make the canonical identifications $H^2(X, \mathbb{Z}) \simeq \bigwedge^2 \mathrm{Hom}(U, \mathbb{Z}) \subset \bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we ask which 2-forms are realized as the Chern classes of line bundles.

We will use $\mathrm{Im}\, ch = \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X))$. Recall that for the sequence of injections $\mathbb{Z} \to \mathbb{C} \to \mathcal{O}_X$ the cohomology yields:

$$\begin{array}{ccccc} H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathbb{C}) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge^2 \mathrm{Hom}(U,\mathbb{Z}) & \overset{\mathbb{R}-\text{linear extension}}{\longrightarrow} & \bigwedge^2 (T \bigoplus \bar{T}) & \overset{\text{projection}}{\longrightarrow} & \bigwedge^2 \bar{T} \end{array}$$

Here the vertical arrows are isomorphisms and the diagram is commutative. Let $E \in \bigwedge^2 \mathrm{Hom}(U, \mathbb{Z})$ be a 2-form, which is necessarily real valued. Then, one could check by hand that E is killed by the projection map to $\bigwedge^2 \bar{T}$ if and only if $\forall x, y \in V$ the form E satisfies $E(x, y) = E(ix, iy)$. This gives a complete description of the 2-forms that are Chern classes. We put together these observations in the following:

Fact 5.5. *Any two form $E \in \bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is a Chern class if and only if*

1. *E is integral valued on the lattice, that is $E(U \times U) \subset \mathbb{Z}$. This characterizes the image of $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})$.*

2. $\forall x, y \in V$ *we get* $E(x,y) = E(ix, iy)$*. This ensures* E *is in the kernel of the map*

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\,.$$

The two forms satisfying $E(x,y) = E(ix,iy)$ are special in the sense that they correspond to Hermitian forms on V.

Definition 5.6. *A function* $h : V \times V \to \mathbb{C}$ *that is* $\mathbb{C}$*-linear in the first entry and that satisfies* $h(x,y) = \overline{h(y,x)}$ *is called a* Hermitian form *on* V*.*

Given a Hermitian form H on V, let g and ω be real valued functions such that $H = g + i\omega$. Then, ω is a 2-form satisfying $\omega(x,y) = \omega(ix,iy)$ and $g(x,y) = \omega(ix,y)$. So we may recover the Hermitian form H from its imaginary part ω. Therefore we may say that the Chern classes of line bundles on X are in one to one correspondance with Hermitian forms on V whose imaginary part on $U \times U$ is integer valued.

5.3 Appel-Humbert Theorem

Having determined the Chern classes of line bundles, we are inclined to ask; If we fix a Chern class E then what extra data do we need to specify a line bundle on X with Chern class E?

After a couple of intricate but explicit cohomology calculations, chasing around various connecting homomorphisms and identifications one arrives at the following result. (*Warning*: When in doubt use the following rule of thumb. Any e that is exponentiated refers to Euler's number. But an e appearing with a subscript is a cocycle.)

Lemma 5.7 (Mumford, p. 19)**.** *Let* H *be a Hermitian form whose imaginary part, which we will denote by* E*, takes integral values on* $U \times U$*. Let* $\alpha : U \to \mathbb{C}_1^* := \{z \in \mathbb{C} \mid \|z\| = 1\}$ *be a function with the following transformation rule:*

$$\alpha(u + u') = e^{i\pi E(u,u')}\alpha(u)\alpha(u') \quad \forall u, u' \in U.$$

There exists such an α*. Moreover, if we define* $e : U \to H^*$ *in the following way,*

$$e_u(z) = \alpha(u)e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}$$

we get a 1-cocycle. If we denote the line bundle associated to this 1-cocycle with $L(H,\alpha)$ *then the Chern class of* $L(H,\alpha)$ *is* E*.*

Note that if e and e' are 1-cocycles obtained from the data (H,α) and (H',α') respectively, then their product $e \cdot e'$ will be obtained from $(H + H', \alpha \cdot \alpha')$. In particular, this implies that $L(H,\alpha) \otimes L(H',\alpha') \simeq L(H + H', \alpha \cdot \alpha')$ because the main result of Section 5.1 was that tensor product and multiplication of 1-cocycles are compatible.

When $H = 0$ the corresponding α is just a group homomorphism $\alpha : U \to \mathbb{C}_1^*$. From now on H will always denote a Hermitian form corresponding to a Chern class, and pairs (H,α) will denote pairs compatible as in the lemma above. Then we have an exact sequence of groups:

$$0 \to \mathrm{Hom}(U, \mathbb{C}_1^*) \to \{(H,\alpha)\} \to \{H\} \to 0$$

Let us denote by μ the map $\{(H,\alpha)\} \to \mathrm{Pic}(X) : (H,\alpha) \mapsto L(H,\alpha)$. If we denote by $\mathrm{Pic}^0(X)$ the group of line bundles with Chern class 0, then composing μ with the inclusion of $\mathrm{Hom}(U, \mathbb{C}_1^*)$ gives a map $\lambda : \mathrm{Hom}(U, \mathbb{C}_1^*) \to \mathrm{Pic}^0(X)$. Finally, we have the good old isomorphism between $\{H\} \xrightarrow{\sim}$ Chern Classes $= \ker(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X))$, let us denote this map by ν. We have the following important theorem, whose proof we will omit.

Theorem 5.8 (Appel-Humbert)**.** *The following diagram with exact rows is commutative with isomorphic vertical arrows:*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(U,\mathbb{C}_1^*) & \to & \{(H,\alpha)\} & \to & \{H\} & \to & 0 \\ & & \lambda\downarrow & & \mu\downarrow & & \nu\downarrow & & \\ 0 & \to & \mathrm{Pic}^0(X) & \to & \mathrm{Pic}(X) & \to & \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)) & \to & 0 \end{array}$$

6 Consequences

Fix a compatible pair (H,α) as before. Let $N = \{v \in V \mid H(v,\cdot) \equiv 0\}$ be the degeneracy locus of H. Let $U' = U \cap N$ be the sublattice of U lying in the degeneracy loci and denote by $X' = N/U'$ the corresponding subtorus of $X = V/U$.

Here is a brief selection of results that follow from the infrastructure we have established so far. Their proofs still require quite a lot of work, but now the statements are understandable.

1. If $L(H,\alpha)$ induces a morphism from X then it has to factor through X/X'. In particular, if H is degenerate then the line bundle $L(H,\alpha)$ can not be ample.
2. If H is not positive definite, then $L(H,\alpha)$ has no non-zero sections.
3. Assume H is positive definitive. Let $[E]$ be a matrix expressing the 2-form E associated to H with respect to a minimal set of generators of the lattice U. Then:
$$h^0(X, L(H,\alpha)) = \sqrt{\det[E]}$$
4. H is positive definitive if and only if $L(H,\alpha)^{\otimes 3}$ is very ample.
5. X 'is' an algebraic variety if and only if it 'is' a projective variety.
6. If $\dim_{\mathbb{C}} V \geq 2$ then for almost all lattices $U \in V$, there are no non-zero Hermitian forms whose imaginary part is integral valued on $U \times U$. Hence, in light of the previous results, almost all complex tori $X = V/U$ of dimension ≥ 2 are not algebraic.

References

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[2] Daniel Huybrechts, *Complex Geometry, An Introduction.* Springer-Verlag, Berlin Heidelberg, 2005.