Talk on Derived Categories

November 5, 2014

An overview on triangulated categories

Let $\mathcal{D}$ be an additive category, a translation functor on $\mathcal{D}$ is an autoequivalence T : $\mathcal{D} \to \mathcal{D}$, for short $\mathcal{T}(\mathcal{D}) = \mathcal{D}[1]$; a triangle is the datum of 3 objects and 3 morphisms of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

Definition 0.1. *A triangulated category is an additive category $\mathcal{T}$ equipped with a transition functor and a class of* distinguished *triangles satisfying the following axioms*

TR1 *For any object X the triangle*

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished.

For any morphism $u : X \to Y$ there is an object Z (called the mapping cone*) fitting into a distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

Any triangle isomorphic to a distinguished triangle, i.e. where objects are isomorphic and morphisms are given by composition with the objects isomorphisms, are distinguished.

TR2 *If*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle, then so are the two rotated triangles

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z\ .$$

TR3 *Given a map between two morphisms u and u', there is a morphism between their mapping cones that makes everything commute. This means that in the following diagram there is a morphism $h : Z \to Z'$ making all the squares commute.*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f\downarrow & & g\downarrow & & h\downarrow & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

TR4 *This the octahedral axiom. If you truly want to see it check*

`http://en.wikipedia.org/wiki/Triangulated_category`

We are interested in triangulated categories on one hand because derived categories of abelian ones are some examples of them in a very natural way, and on the other hand because the fact that the homotopy category $\mathrm{K}(\mathcal{A})$ of an additive category $\mathcal{A}$ is triangulated, with mapping cone being the mapping cone(!), will be central in our study of $\mathcal{D}(\mathcal{A})$.

Definition and existence of $\mathcal{D}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category, and $\mathrm{Kom}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. We start defining immediately the notion of derived category.

Theorem 0.2. *There exists a category $\mathcal{D}(\mathcal{A})$ and a functor $\mathrm{Q} : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ with the following properties*

1. $\mathrm{Q}(f)$ *is an isomorphism for any quasi-isomorphism f.*

2. *Any functor* $\mathrm{F} : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ *transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through* $\mathcal{D}(\mathcal{A})$*; i.e., there exists a unique functor* $\mathrm{G} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}$ *with* $\mathrm{F} = \mathrm{G} \circ \mathrm{Q}$.

Before proving the theorem let us observe that if such a category exists the it is unique up to unique equivalence, therefore the following definition makes sense.

Definition 0.3. *The category* $\mathcal{D}(\mathcal{A})$ *is called the derived category of* $\mathcal{A}$.

The proof of Theorem 0.2 follows by the following construction of localization of a category.

Localization of a Category Let $\mathcal{B}$ be an arbitrary category and S an arbitrary class of morphisms in $\mathcal{B}$; then there exists a universal functor "localization by S" $\mathrm{Q} : \mathcal{B} \to \mathcal{B}[\mathrm{S}^{-1}]$ transforming morphisms in S into isomorphisms.

The objects in the category $\mathcal{B}[S^{-1}]$ are the objects in $\mathcal{B}$, and Q is the identity on objects.

To construct morphisms in $\mathcal{B}[\mathrm{S}^{-1}]$ we introduce formal symbols s^{-1} for any $s \in \mathrm{S}$, and we construct an oriented graph Γ as follows; vertices of Γ are objects of $\mathcal{B}$, edges between two vertices X, Y are either morphisms $f : X \to Y$ oriented from X to Y, or morphisms $g : Y \to X$ oriented from Y to X or formal symbols s^{-1} with the opposite orientation of s. Now a path on Γ is a finite sequence of edges such that the end of any edge coincides with the beginning of the next one.

Finally a morphism in $\mathcal{B}[\mathrm{S}^{-1}]$ is an equivalence class of paths in Γ with common beginning and common end, where the equivalences are generated by the following elementary ones: two consecutive arrows are equivalent to their composition; the composition of s with s^{-1} (resp. of s^{-1} with s) is equivalent to the identity morphism. Morphisms are composed patching paths, and Q maps a morphism into its equivalence class.

Moreover the pair $(\mathrm{Q}, \mathcal{B}[\mathrm{S}^{-1}])$ is universal in the sense of Theorem 0.2; the verification is easy.

Example 0.4. *Let* R *be a commutative ring with unity, and* $\mathrm{S} \subset \mathrm{R}$ *a multiplicatively closed subset; then we construct the category* $\mathcal{C}_{\mathrm{R}}$ *which has only* *one object and* R *as morphisms. The localized category* $\mathcal{C}_{\mathrm{R}}[\mathrm{S}^{-1}]$ *is equivalent to the category* $\mathcal{C}_{\mathrm{R}[\mathrm{S}^{-1}]}$ *associated in the same fashion to the localized ring* $\mathrm{R}[\mathrm{S}^{-1}]$.

Problem 0.5. *This existence result doesn't give us any grasp on the category* $\mathcal{D}(\mathcal{A})$*; we give an example of it in the next construction.*

Let us define $\mathrm{Kom}^{+}(\mathcal{A})$ to be the category of complexes on $\mathcal{A}$ bounded on the left, i.e. for any $\mathrm{K} \in \mathrm{Kom}^{+}(\mathcal{A})$ there exists $i(\mathrm{K})$ such that for any $i \leq i(\mathrm{K})$ we have $\mathrm{K}^{i} = 0$. This is a full subcategory of $\mathrm{Kom}(\mathcal{A})$ and we can form the corresponding derived category $\mathcal{D}^{+}(\mathcal{A})$ in two different ways

- $\mathcal{D}^{+}(\mathcal{A})$ is the localization of $\mathrm{Kom}^{+}(\mathcal{A})$ by quasi-isomorphisms.
- $\mathcal{D}^{+}(\mathcal{A})$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of complexes K' with $\mathrm{H}^{i}(\mathrm{K}') = 0$ for $i < i(\mathrm{K}')$.

We would like those two constructions to produce the same objects, but so far we have no clue on how to do it. The problem is that morphisms in $\mathcal{B}[\mathrm{S}^{-1}]$ as constructed above are just formal (rigid) expressions of the form

$$f_1 \circ s_1^{-1} \circ f_2 \circ ... \circ s_k^{-1} \circ f_k, \qquad f_i \in \mathrm{Hom}(\mathcal{B}),\ s_i \in \mathrm{S}; \tag{1}$$

and we cannot "find the common denominator", or manipulate these expressions with other algebraic identities.

Localizing classes of morphisms

Now we define a nice setting where algebraic identities to manipulate morphisms in $\mathcal{B}[\mathrm{S}^{-1}]$ do exist.

Definition 0.6. *A class of morphisms* S *on an arbitrary category* $\mathcal{B}$ *is said to be* localizing *if the following conditions are satisfied:*

1. S *is closed under composition, in specific* $\mathrm{id}_X \in \mathrm{S}$ *for any object* X.
2. *Extension conditions. For any morphisms* f *in* $\mathcal{B}$ *and* $s \in \mathrm{S}$ *there exists* g *morphisms in* $\mathcal{B}$ *and* $t \in \mathrm{S}$ *such that the following square:*

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \tag{2}$$

commute. And the same condition reversing all the arrows.

3. *Let $f, g \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$; then $s \in \mathrm{S}$ such that $sf = sg$ exists if and only if $t \in \mathrm{S}$ with $ft = gt$ does.*

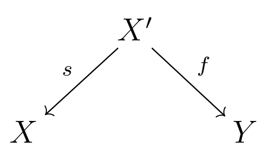
Observe that the paths $s^{-1} f$ and $g\, t^{-1}$ represent the same morphism $X \to Z$ in $\mathcal{B}[\mathrm{S}^{-1}]$. Indeed by commutativity we have $f\, t = s\, g$ in $\mathrm{Hom}_{\mathcal{B}}$, therefore the paths $s^{-1} f\, t\, t^{-1}$ and $s^{-1} s\, g\, t^{-1}$ are equivalent and so also $s^{-1} f$ and $g\, t^{-1}$ are, therefore we have equality in $\mathcal{B}[\mathrm{S}^{-1}]$. In specific if S is a localizing class of morphisms (we actually use for the moment only the first two properties) then (1) reduces to an expression of the form $f \circ s^{-1}$ for $f \in \mathrm{Hom}_{\mathcal{B}}$ and $s \in S$.

Non-example 0.7. *The class of quasi-isomorphisms in* $\mathrm{Kom}(\mathcal{A})$ *in general is not a localizing class of morphisms. The "pathology" is the same as for* $\mathrm{Kom}(\mathcal{A})$ *not being a triangulated category.*

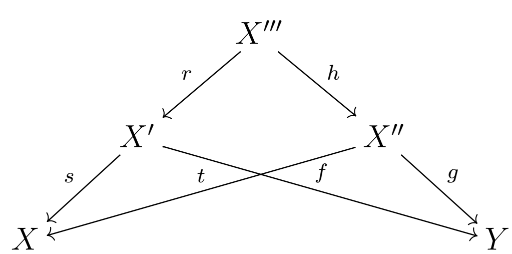
The last property required in the definition of localizing class of morphisms is used in the proof of the next Lemma, which gives a practical representation for morphisms in $\mathcal{B}[\mathrm{S}^{-1}]$.

Lemma 0.8. *Let* S *be a localizing class of morphisms in a category* $\mathcal{B}$*. Then*

1. *A morphisms $X \to Y$ in $\mathcal{B}[\mathrm{S}^{-1}]$ is a class of "roofs", i.e. diagrams $(s, f) \in \mathrm{S} \times \mathrm{Hom}_{\mathcal{B}}$ of the form*

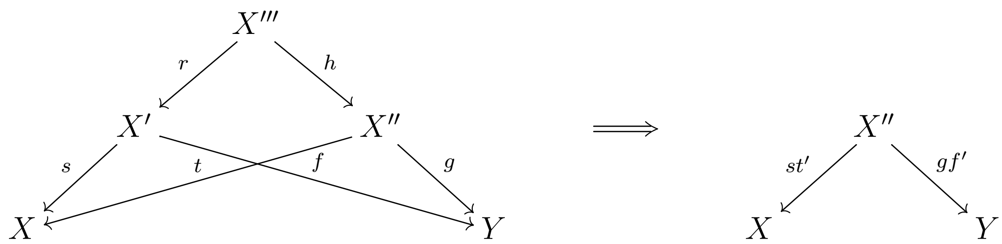


where two roofs (s, f) and (t, g) are equivalent if there exists a third roof forming a commutative diagram of the form



And the identity morphisms for X is the class of the roof $(\mathrm{id}_X, \mathrm{id}_X)$.

2. *The composition of the morphisms represented by roofs* (s, f) *and* (t, g) *is represented by the roof* $(s\,t', g\,f')$ *obtained using part* 2 *of the definition of a localizing class of morphisms:*



We skip the proof of this Lemma. But we observe that from it follows a criterion to decide when $\mathcal{B}[(\mathrm{S} \cap \mathrm{Hom}_{\mathcal{B}})^{-1}]$ is a full subcategory of $\mathcal{C}[\mathrm{S}^{-1}]$ in case $\mathcal{B} \subset \mathcal{C}$ is. In turns this proves that the two ways we presented to construct $\mathcal{D}^{+}(\mathcal{A})$ actually give rise to the same object.

Construction of $\mathcal{D}(\mathcal{A})$

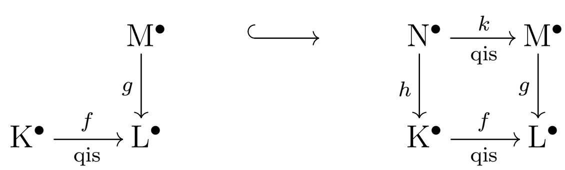
The aim of this paragraph is to construct $\mathcal{D}(\mathcal{A})$ as a localization through a localizing class of morphisms of a well behaved category. The well behaved category in consideration is the homotopic category $\mathrm{K}(\mathcal{A})$ already introduced by Nicholas; its objects are the same as $\mathrm{Kom}(\mathcal{A})$, and morphisms are the morphisms of $\mathrm{Kom}(\mathcal{A})$ modulo homotopic equivalence. Moreover $\mathrm{K}(\mathcal{A})$ is an additive category on which the homology functors H^i are well defined, specifically the definition of quasi-isomorphism makes sense for morphisms in $\mathrm{K}(\mathcal{A})$.

The main technical result of the section is the following Theorem.

Theorem 0.9. *The class of quasi-isomorphisms in the category* $\mathrm{K}(\mathcal{A})$ *is localizing.*

Sketch of the Proof. We have to verify the three properties of the definition; the first one is obvious.

For the second property we need an embedding of the form



observe that this would be trivial, by using the fiber product, if we were not asking k to be a quasi-isomorphism, using the mapping cone we embed the diagram on the left into

$$\begin{array}{ccccccc}
C(\pi\, g)[-1] & \xrightarrow{k} & \mathrm{M}^\bullet & \xrightarrow{\pi\, g} & C(f) & \longrightarrow & C(\pi\, g) \\
{\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \downarrow{\scriptstyle h[1]} \\
\mathrm{K}^\bullet & \underset{\mathrm{qis}}{\xrightarrow{f}} & \mathrm{L}^\bullet & \xrightarrow{\pi} & C(f) & \longrightarrow & \mathrm{K}^\bullet[1]
\end{array}$$

where h is given by composition of the natural map $C(\pi\, g) \rightarrow C(\pi)$ with the isomorphism (in $\mathrm{K}(\mathcal{A})$, not in $\mathrm{Kom}(\mathcal{A})$!) $C(\pi) \simeq \mathrm{K}[1]$ implied by the properties of the mapping cone. Using the properties of long exact sequences in cohomology one proves that $C(\pi\, g)[-1] \rightarrow \mathrm{M}^\bullet$ is a quasi-isomorphism.

We skip the proof of the third property, which again is just a computation based on the properties of the mapping cone. □

The result we were looking for is then given by the next easy proposition.

Proposition 0.10. *The localization of* $\mathrm{K}(\mathcal{A})$ *by quasi-isomorphisms is equivalent to the derived category* $\mathcal{D}(\mathcal{A})$.

Sketch of the Proof. Let $\tilde{\mathcal{D}}(\mathcal{A})$ denote the localization of $\mathrm{K}(\mathcal{A})$ by quasi-isomorphisms, then the composition $\mathrm{Kom}(\mathcal{A}) \rightarrow \mathrm{K}(\mathcal{A}) \rightarrow \tilde{\mathcal{D}}(\mathcal{A})$ is a bijection on objects and maps quasi-isomorphisms into isomorphisms; because the notion of quasi-isomorphism is invariant by homotopic equivalence. Therefore, by the universal property of $\mathcal{D}(\mathcal{A})$ it factors through a unique functor $\mathrm{G} : \mathcal{D}(\mathcal{A}) \rightarrow \tilde{\mathcal{D}}(\mathcal{A})$, moreover, by construction, G is a bijection on objects; therefore to verify that it is an equivalence of categories we only need to prove that it is fully faithful.

Since morphisms in $\tilde{\mathcal{D}}(\mathcal{A})$ are classes of roofs, those can be lifted to $\mathrm{K}(\mathcal{A})$ and, choosing a representative, to $\mathrm{Kom}(\mathcal{A})$. The image of this last morphism in $\mathcal{D}(\mathcal{A})$ is a roof where the left morphism is an isomorphism, therefore it maps to the original class through G; proving it is surjective.

The injectivity follows from the following, quite involved, fact; if f, g in $\mathrm{Hom}_{\mathrm{Kom}}$ are homotopic to each other, then $\mathrm{Q}(f) = \mathrm{Q}(g)$. The idea behind the proof is to write $f = g + \mathrm{d}\, h + h\, \mathrm{d}$ and verify that Q has to ignore the latter parts. □

$\mathcal{D}(\mathcal{A})$ as homotopy category of injectives

Let $\mathcal{A}$ be our abelian category, and A_0 an object of it. Then the contravariant functor

$$\mathrm{Hom}(\cdot, A_0) : \mathcal{A} \longrightarrow \mathbf{Ab}$$

is left exact.

Definition 0.11. *An object* $I \in \mathcal{A}$ *is called injective if* $\mathrm{Hom}(\cdot, I)$ *is right exact.*

Definition 0.12. $\mathcal{A}$ *contains enough injective objects if for any object* $A \in \mathcal{A}$ *there exists an injective morphisms* $A \to I$ *with* $I \in \mathcal{A}$ *injective. An injective resolution of an object* $A \in \mathcal{A}$ *is an exact sequence*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \ldots$$

with all I^i *injective.*

We observe that if $\mathcal{A}$ has enough injectives then any objects of $\mathcal{A}$ admits an injective resolution; moreover the subcategory $\mathcal{I} \subset \mathcal{A}$ of injective objects is full.

Proposition 0.13. *Suppose* $\mathcal{A}$ *to have enough injectives, then for any* $A^\bullet \in K^+(\mathcal{A})$ *there exists a complex* $I^\bullet \in K^+(\mathcal{A})$ *with* $I^i \in \mathcal{A}$ *injective objects and a quasi-isomorphism* $A^\bullet \to I^\bullet$.

Before sketching the proof we observe that this Proposition is not so hard to believe, having enough injectives already means that it holds for any 1-term complex.

Sketch of the Proof. Assume $A^\bullet = 0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \ldots$. By assumption on $\mathcal{A}$ there exists an injective object I^0 and a monomorphism $A^0 \to I^0$; the induced morphism $f_0 : A^\bullet \to (I^0 \longrightarrow 0 \longrightarrow \ldots)$ has the property that $H^i(f_0)$ is an isomorphism for $i < 0$ and injective for $i = 0$.

Now we consider the object $(I^0 \otimes A^1)/A^0$ and an embedding of it into an injective object I^1; the natural maps $I^0 \to I^1$ and $A^1 \to I^1$ induce a morphism of complexes $f_1 : A^\bullet \to (I^0 \longrightarrow I^1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots)$ with the property that $H^0(f_1) : H^0(A^\bullet) \to H^0(I^\bullet)$ is an isomorphism and $H^1(f_1) : H^1(A^\bullet) \to H^1(I^\bullet)$ is an injection.

Proceeding in this fashion, and overcoming some technical issues, one can conclude the argument by induction. □

Lemma 0.14. *Let* $\mathrm{A}^\bullet, \mathrm{I}^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ *such that all the* I^i *are injective objects. Then*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet).$$

Sketch of Proof. There is a natural map $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet) \rightarrow \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet)$, we have to show that for any morphism

$$\mathrm{A}^\bullet \xleftarrow{\text{qis}} \mathrm{B}^\bullet \longrightarrow \mathrm{I}^\bullet$$

in $\mathcal{D}(\mathcal{A})$ there exists a unique morphism of complexes $\mathrm{A}^\bullet \rightarrow \mathrm{I}^\bullet$ making the whole diagram commutative up to homotopy. In other words we have to show that if $\mathrm{A}^\bullet \rightarrow \mathrm{B}^\bullet$ is a quasi-isomorphism in $\mathrm{Kom}^+(\mathcal{A})$, then the induced map $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet) \rightarrow \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\mathrm{A}^\bullet, \mathrm{I}^\bullet)$ is bijective. The proof of this fact is technical and we skip it. □

Since $\mathcal{I} \subset \mathcal{A}$ is full, the construction of $\mathrm{K}(\mathcal{I})^+ \subset \mathrm{K}(\mathcal{A})^+$ makes sense and is again triangulated; we can compose this inclusion with the natural exact functor $\mathrm{Q} : \mathrm{K}^+(\mathcal{A}) \rightarrow \mathcal{D}^+(\mathcal{A})$ to get the exact functor $i : \mathrm{K}^+(\mathcal{I}) \rightarrow \mathcal{D}^+(\mathcal{A})$. Those are exact by construction, because the notion of exactness for functors between triangulated categories is to map distinguished triangles into distinguished triangles; but the one on $\mathcal{D}^+(\mathcal{A})$ are defined to be the ones coming from $\mathrm{K}^+(\mathcal{A})$.

Theorem 0.15. *Suppose that the category* $\mathcal{A}$ *contains enough injectives; then the natural functor*

$$i : \mathrm{K}^+(\mathcal{I}) \rightarrow \mathcal{D}^+(\mathcal{A})$$

is an equivalence of categories.

Sketch of Proof. Even without the hypothesis on $\mathcal{A}$ the functor i is fully faithful; this statement is the content of Lemma 0.14.

The fact that i is essentially surjective, i.e. that any object in $\mathcal{D}^+(\mathcal{A})$ is isomorphic to the image of an object in $\mathrm{K}^+(\mathcal{I})$, is the content of Proposition 0.13. □

Exercise 0.16. *Let* k *be a field, and* $\mathcal{A}$ *the abelian category of finite vector spaces over* k*. Prove that for any* $\mathrm{A}^\bullet \in \mathrm{Kom}(\mathcal{A})$ *the class of* A *in the derived category is*

$$\bigotimes \mathrm{H}^i(\mathrm{A}^\bullet)[-i].$$