DEFINITION OF ABELIAN VARIETIES AND THE THEOREM OF THE CUBE

ANGELA ORTEGA
(NOTES BY B. BAKKER)

Throughout k is a field (not necessarily closed), and all varieties are over k. For a variety X/k, by a basepoint we'll mean a k-rational point $x_0 \in X(k)$.

1. Definition and examples

Definition 1. An *abelian variety* X/k is a complete variety X with the structure of a group (in the category of varieties). Thus there exists:

(1) an identity basepoint $e \in X(k)$;
(2) a multiplication map $m : X \times X \to X$ satisfying the associative property;
(3) an inverse map $i : X \to X$ interacting with m in the usual way

A homomorphism of abelian varieties is a morphism of varieties respecting the group structure.

Remark 2. A more precise definition is that the functor of points of X is given a factorization through the forgetful functor **Groups** $\to$ **Sets**. A homomorphism of abelian varieties is a natural transformation of the corresponding **Groups**-valued functors of points.

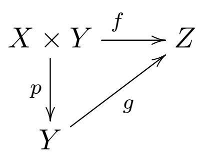
Example 3. (1) An elliptic curve E/k is an abelian variety. E can be realized as a plane cubic $E \subset \mathbb{P}^2$, and addition is given by the usual condition that $x + y + z = 0$ if they are colinear.
(2) If $\mathbb{Z}^{2g} \cong \Lambda \subset \mathbb{C}^g$ is a lattice, then the complex torus $\mathbb{C}^g/\Lambda$ is a complex (in fact Kähler) manifold with the structure of a group; when it happens to be the $\mathbb{C}$-points of a variety, that variety is an abelian variety.
(3) We saw last time that in fact *every* compact complex group manifold is a torus.
(4) If C/k is a curve, the Jacobian $\mathrm{Jac}(C)$ is a projective abelian variety, defined over k. In fact, we'll see later that all abelian varieties are projective.
(5) For $X/\mathbb{C}$ a smooth complete variety, there is a natural map $H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)$ given by integration along a cycle, and the albanese is defined as

$$\mathrm{Alb}(X) := H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})$$

For $X = C$ a complex curve, this is just the Jacobian.

Our first aim is to prove an algebraic analog of Example 3.(3) above.

Lemma 4. *For X, Y, Z varieties with X complete, x_0, y_0, z_0 basepoints, and $f : X \times Y \to Z$ a morphism such that $f(X \times y_0) = z_0$, there is a morphism g so that*



commutes, where p is projection onto the second factor.

Date: December 11, 2014.

Proof. Define $g(y) = f(x_0, y)$. The condition that two morphisms be equal is a closed condition on the source[1], so we need only show $f = g \circ p$ on a nonempty open set. Let $z_0 \in U \subset Z$ be an open affine, and let $F = Z \smallsetminus U$, $G = p(f^{-1}(F)) \subset Y$. X is complete so $G \subset Y$ is the closed set of Y-coordinates that don't arise among the points of $X \times Y$ that get sent to U. In particular, $y_0 \in Y \smallsetminus G =: V$ which is an open set. Moreover, for all closed points $y \in V$, $X \times y$ gets mapped to U under f by construction. As X is complete, it must be sent to a point, *i.e.* $g(y)$. ☐

Corollary 5. *If X, Y are abelian varieties and $f : X \to Y$ is any morphism, then $f(x) = h(x) \cdot f(e)$ for a homomorphism $h : X \to Y$.*

Proof. We may as well assume $f(e) = e$. Define $F : X \times X \to Y$ by $F(x, y) = f(xy) \cdot f(y)^{-1} \cdot f(x)^{-1}$. This sends $X \times e$ to e, and now apply the lemma. ☐

Corollary 6. *An abelian variety X is commutative.*

Proof. Apply Corollary 5 to the inversion map $i : X \to X$. ☐

From now on, we'll therefore write the group law additively, and denote by 0 the identity. Note that we're really using the completeness of X here; there are many noncommutative connected group schemes (like $\mathrm{GL}_n = \operatorname{Spec} k[X][\det(X)^{-1}]$), but they have to be non-complete.

2. Cohomology and base-change

This is a really important theorem, and its application to proving the theorem of the cube is as good a time as any to learn it. The seemingly technical heart of the result is the following theorem:

Theorem 7. *Let $f : X \to Y$ be a morphism of noetherian schemes with $Y = \operatorname{Spec} A$ affine and F a coherent sheaf on X flat over Y. Then there is a finite complex*

$$K^\bullet = [0 \to K^0 \to K^1 \to \cdots \to K^{n-1} \to K^n \to 0]$$

of finitely-generated locally free A-modules such that there is an isomorphism of functors

$$H^p(X \times_Y \operatorname{Spec} B, F \otimes_A B) \cong H^p(K^\bullet \otimes_A B)$$

on the category of A-algebras for each $p \geq 0$.

The key part of this is the finiteness and the finite-generation. Indeed, just taking the Ĉech complex associated to some affine cover of X would give us a complex of flat A-modules universally computing the cohomology, but typically this won't be finite or finitely-generated.

Let's see the consequences of this theorem. As a matter of notation, for a point $y \in Y$, denote by $X(y) = X \times_Y \operatorname{Spec} k(y)$ and $F(y) = F \otimes_{\mathcal{O}_Y} k(y)$ the fibers over y. For a scheme X/k we define $h^p(X, F) = \dim_k H^p(X, F)$, so for example $h^p(X(y), F(y)) = \dim_{k(y)} H^p(X(y), F(y))$.

Corollary 8. *In the above situation, for all $p \geq 0$,*

1. *$y \mapsto h^p(X(y), F(y))$ is upper semicontinuous;*
2. *$y \mapsto \chi(X(y), F(y))$ is locally constant.*

No one can ever remember which semicontinuity means which thing, so the above says that cohomology can jump up on special fibers.

[1] The set where they're equal is the base-change of the diagonal (and has a natural scheme structure).

Proof. Let $d^p : K^p \to K^{p+1}$ be the differential in the complex guaranteed by the theorem. The key idea is that

$$\begin{aligned} h^p(X(y), F(y)) &= \dim_{k(y)} \ker(d^p \otimes k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes k(y)) \\ &= \dim_{k(y)} K^p \otimes k(y) - \dim_{k(y)} \operatorname{im}(d^p \otimes k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes k(y)) \end{aligned}$$

and the first term on the right in the last line is locally constant, while the last two are semicontinuous. Indeed, for any map of sheaves $\varphi : E \to F$ on Y, the set

$$\{y \in Y \mid \operatorname{rk}_{k(y)}(\varphi \otimes k(y)) < r\}$$

is the zero set of the map $\bigwedge^r \varphi : \bigwedge^r E \to \bigwedge^r F$ and is closed. In fact, this even gives a natural scheme structure to this set.

The euler characteristic is the alternating sum of $\dim_{k(y)} K^p \otimes k(y)$, which is clearly locally constant as the K^p are locally free. □

Corollary 9. *Now assume Y is reduced and connected. The following are equivalent:*

1. *$y \mapsto h^p(X(y), F(y))$ is constant;*
2. *$R^p f_* F$ is locally free and the natural map*

$$(R^p f_* F)(y) \xrightarrow{\cong} H^p(X(y), F(y))$$

is an isomorphism.

Proof. The backward implication is clear. For the forward direction, we need to know that for E a coherent sheaf on Y, if $\operatorname{rk} E(y)$ is constant then it is locally free (this of course uses the reducedness!). By the proof of Corollary 9, if $h^p(X(y), F(y))$ is constant, then

$$\dim_{k(y)} \operatorname{im}(d^p \otimes k(y)) \text{ and } \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes k(y))$$

are both constant, which implies $\operatorname{im} d^p$, $\operatorname{im} d^{p-1}$, $\ker d^p$, and $\ker d^{p-1}$ are all locally free. This gives a splitting of our complex at the pth place:

$$\begin{array}{ccccc} K^{p-1} & \longrightarrow & K^p & \longrightarrow & K^{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ \ker d^{p-1} \oplus K'^{p-1} & \xrightarrow[\begin{pmatrix} 0 & \cong \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]{} & \operatorname{im} d^{p-1} \oplus H \oplus K'^p & \xrightarrow[\begin{pmatrix} 0 & 0 & \cong \\ 0 & 0 & 0 \end{pmatrix}]{} & \operatorname{im} d^p \oplus K'^{p+1} \end{array}$$

Now the theorem says H universally computes the pth cohomology. Right off the bat that means $H \cong R^p f_* F$, and base-changing to y,

$$H^p(X(y), F(y)) \cong H \otimes k(y) \cong (R^p f_* F)(y)$$

by the canonical maps. □

Corollary 10 (Seesaw theorem). *For X a complete variety and L a line bundle on $X \times T$, the set*

$$T_1 = \{t \in T \mid L|_{X \times t} \text{ is trivial}\}$$

is closed in T and $L_{X \times T_1} \cong p_2^* M$ for a line bundle M on T_1 (with the reduced scheme structure).

Proof. First, observe

Lemma 11. *A line bundle M on a complete variety X is trivial if and only if $h^0(M) > 0$ and $h^0(M^{-1}) > 0$.*

Proof. The forward direction is clear. If there is a section $\mathcal{O}_X \to M$ and a section $\mathcal{O}_X \to M^{-1}$, then their tensor product is a map $\mathcal{O}_X \to \mathcal{O}_X$, which must be a constant since neither section was the zero section. But then neither section is ever zero, and hence they are isomorphisms. □

Now by the lemma,

$$T_1 = \{t \in Y \mid h^0(L|_{X\times t}) > 0\} \cap \{t \in Y \mid h^0(L^{-1}|_{X\times t}) > 0\}$$

which by Corollary 8 is closed. Replace T by T_1 with the reduced scheme structure, so we can assume L is trivial on every fiber of the projection $p : X \times T \to T$. Then by Corollary 9, $M = p_*L$ is a line bundle. Now, it follows from the fact that the natural map $p^*M \to L$ is an isomorphism on every fiber that in fact its an isomorphism. □

Theorem 12 (Theorem of the cube)**.** *Let X, Y be complete varieties, Z any variety, and x_0, y_0, z_0 basepoints. Any line bundle L on $X \times Y \times Z$ whose restriction to each of $X \times Y \times z_0$, $X \times y_0 \times Z$, $x_0 \times Y \times Z$ is trivial is itself trivial.*

Let's first give a proof over $\mathbb{C}$ using the exponential sequence, at least in the case Z is also compete. For simplicity, assume none of X, Y, Z have torsion in their cohomology, though it won't matter. Then the Künneth theorem tells us that the natural map

$$H^*(X,\mathbb{Z}) \otimes H^*(Y,\mathbb{Z}) \otimes H^*(Z,\mathbb{Z}) \xrightarrow{p_1^*\cup p_2^*\cup p_3^*} H^*(X\times Y\times Z,\mathbb{Z}) \tag{1}$$

is an isomorphism, where p_i is the projection to the ith factor (we let p_{ij} be the projection to the i and j factors). Let ι_i be the inclusion of the ith factor and ι_{ij} the inclusion of the i and j factors using the basepoints. Concretely,

$$\iota_1 : X \times y_0 \times z_0 \to X \times Y \times Z, \qquad \iota_{12} : X \times Y \times z_0 \to X \times Y \times Z$$

If you think about the isomorphism (1) in degree 2, it means that for any class $\alpha \in H^2(X \times Y \times Z,\mathbb{Z})$,

$$\alpha = \alpha_{12} + \alpha_{13} + \alpha_{23} - \alpha_1 - \alpha_2 - \alpha_3$$

where α_{ij} (respectively α_i) is α inserted in the i and jth slots (respectively $p_i^*\alpha$ inserted in the ith slot) via the Künneth formula, so $\alpha_{12} = \iota_{12}^*\alpha \otimes 1$, $\alpha_1 = \iota_1^*\alpha \otimes 1 \otimes 1$. In particular, this means if $\iota_{ij}^*\alpha = 0$ for all i, j, then $\alpha = 0$ (this is what it means to be "quadratic").

The long exact sequence associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives us an exact sequence

$$H^1(X\times Y\times Z,\mathcal{O}) \xrightarrow{\exp} H^1(X\times Y\times Z,\mathcal{O}^*) \xrightarrow{c_1} H^2(X\times Y\times Z,\mathbb{Z})$$

Given a line bundle L on $X \times Y \times Z$ thought of as an element of the middle group, $\iota_{ij}^*c_1(L) = c_1(\iota_{ij}^*L) = 0$ for all i, j, so $c_1(L) = 0$ and $L = \exp(A)$ for some $A \in H^1(X \times Y \times Z,\mathcal{O})$. But

$$H^1(X,\mathcal{O}) \oplus H^1(Y,\mathcal{O}) \oplus H^1(Z,\mathcal{O}) \xrightarrow{p_1^*+p_2^*+p_3^*} H^1(X\times Y\times Z,\mathcal{O})$$

is an isomorphism, and the hypotheses imply that $\iota_i^*\exp(A) = \exp(\iota_i^*A) = 0$ for each i. Thus,

$$L = \exp\left(\sum_i \iota_i^*A\right) = 0$$

Here's a sketch of the algebraic proof:

Sketch of proof. By the seesaw theorem, its enough to show that L is trivial when restricted to $x \times Y \times z$ for all $x \times z \in X \times Z$, for then L is a pullback from $X \times Z$, but it is trivial on $X \times y_0 \times Z$. We can prove this using the theorem in the case that X is a curve using the following:

Lemma 13. *For any two $x, x' \in X$, there is an irreducible curve on X passing through x, x'.*

Proof. This is obvious by Bertini for X projective, but by Chow's lemma this is enough. □

So now assume X is a curve; after normalizing we can assume X is smooth. Secretly we can conclude because considering L as a family of line bundles on X parametrized by $Y \times Z$, we have

$$Y \times Z \to \mathrm{Jac}(X)$$

but by the hypotheses and Lemma 4, this factors through the projection to Y, *i.e.* L is a pullback from $X \times Y$, and therefore must be trivial, again by the hypotheses.

If you don't want to use the existence of the Jacobian, there is a work-around, but I'm afraid you'll have to look in Mumford for that. □

3. Consequences for abelian varieties

First note the following immediate corollary of the theorem of the cube:

Corollary 14. *For X, Y, Z as in Theorem 12, let L be any line bundle on $X \times Y \times Z$. Then*

$$L \cong p_{12}^*L \otimes p_{13}^*L \otimes p_{23}^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \otimes p_3^*L^{-1}$$

Proof. Both sides have the same restriction to $X \times Y \times z_0$ etc. □

For X an abelian variety, denote by $m_{ij} : X \times X \times X \to X$ the sum of the i and jth coordinates (*i.e.* $m_{ij} = m \circ p_{ij}$ where $m : X \times X \to X$ is the addition map), by m_{123} the sum of all three coordinates, and for consistency $m_i = p_i$.

Corollary 15. *Let X be an abelian variety and L a line bundle on X. Then*

$$m_{123}^*L \cong m_{12}^*L \otimes m_{13}^*L \otimes m_{23}^*L \otimes m_1^*L^{-1} \otimes m_2^*L^{-1} \otimes m_3^*L^{-1}$$

Proof. Apply the last corollary to m_{123}^*L. □

Corollary 16. *Let X be any variety, Y an abelian variety, $f, g, h : X \to Y$ morphisms, and L a line bundle on Y. Then*

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}$$

Proof. Pull back the previous corollary along $f \times g \times h : X \to Y \times Y \times Y$. □

For X an abelian variety, let $n : X \to X$ be the multiplication by n map, which can be inductively defined by $n+1 = m \circ (n \times \mathrm{id})$. We also denote by $-1 = i$ the inversion. The following two results are the most memorable results of this section:

Theorem 17. *For X an abelian variety and L a line bundle on X,*

$$n^*L \cong L^{\frac{n^2+n}{2}} \otimes (-1)^*L^{\frac{n^2-n}{2}}$$

Proof. The theorem is true for $n = 0, 1, -1$, and if its true for n its true for $-n$ as well. By Corollary 16 applied to $f = n, g = 1, h = -1$, we have

$$n^*L \cong (n+1)^*L \otimes (n-1)^*L \otimes \mathcal{O} \otimes n^*L^{-1} \otimes L^{-1} \otimes (-1)^*L^{-1}$$

Computing this out using the theorem for all the terms except $(n+1)^*L$, we conclude by induction that the result is true for $n > 0$. □

For any k-point $x \in X(k)$, there is a translation map $t_x : X \to X$ given by $t_x(y) = y + x$.

Theorem 18 (Theorem of the square)**.** *For any line bundle L on an abelian variety X and any two k-points $x, y \in X(k)$,*

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L$$

Proof. Apply Corollary 16 to $f = \mathrm{id}$, and g, h the constant maps with images x, y, respectively. □

This theorem says that for any line bundle L, the map

$$\varphi_L : X \to \mathrm{Pic}(X), \qquad x \mapsto t_x^* L \otimes L^{-1}$$

is a homomorphism. Technically at the moment $\mathrm{Pic}(X)$ is just the set of line bundles on X defined over k, and the map is only defined set theoretically on the k-points of X, but we'll see soon that this is actually a homomorphism of abelian varieties.