1. Projectivity of Abelian varieties

Proposition 1. *For any $L \in \mathrm{Pic}(X)$ the set $K(L) = \ker(\phi_L)$ is a Zariski closed subset of X.*

Proof. Apply the Seesaw theorem ([Mu] p.54 Corollary 6) to the line bundle $m^*L \otimes p_2^*L^{-1}$ on $X \times X$, where m is the addition map and p_2 is the second projection. This way we obtain that the set

$$S = \{x \in X \mid m^*L \otimes p_2^*L^{-1}|_{\{x\}\times X} \text{ is trivial}\}$$

is Zariski closed. Now we will show that $S = K(L)$.

To see this, first note that if i denotes the inclusion map then the compositions $\{x\} \times X \xrightarrow{i} X \times X \xrightarrow{m} X$ and $\{x\} \times X \xrightarrow{i} X \times X \xrightarrow{p_2} X$ are equal to t_x and 1_X, respectively (if we make the obvious identification $\{x\} \times X \cong X$). Hence

$$m^*L \otimes p_2^*L^{-1}|_{\{x\}\times X} = i^*(m^*L \otimes p_2^*L^{-1}) \cong (m \circ i)^*L \otimes (p_2 \circ i)^*L^{-1} \cong t_x^*L \otimes L^{-1}.$$

Therefore $S = K(L)$. □

Proposition 2. *Let D be an effective divisor on an abelian variety X and $L = \mathcal{O}_X(D)$. Then the following conditions are equivalent.*

i) The subgroup $H = \{x \in X \mid t_x^*D = D\}$ of X is finite (Here equality means really the equality of divisors, not just linear equivalence);
ii) $K(L)$ is finite;
iii) The linear system $|2D|$ is base point free and defines a finite morphism $\varphi: X \to \mathbb{P}^r$;
iv) L is an ample line bundle.

Proof. **iii) ⇒ iv)** : By the definition of φ, we have $\varphi^*\mathcal{O}(1) \cong L^{\otimes 2}$. We will prove that it is ample, then clearly it follows that L is ample as well. So we need to show that given a coherent sheaf $\mathcal{F}$ on X there exists $n \in \mathbb{Z}$ such that $\mathcal{F} \otimes \varphi^*\mathcal{O}(1)^{\otimes n}$ is globally generated. Since φ is finite $\varphi_*\mathcal{F}$ is a coherent sheaf (Ex 5.5 in [Ha]) and since $\mathcal{O}(1)$ is ample on $\mathbb{P}^r$ there exists $n \in \mathbb{Z}$ such that we have a surjection

$$\mathcal{O}^{\oplus I} \to \varphi_*\mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}.$$

Pulling this morphism back, we obtain the surjection

$$\mathcal{O}_X^{\oplus I} \to \varphi^*\varphi_*\mathcal{F} \otimes \varphi^*\mathcal{O}(1)^{\otimes n}.$$

Finally since φ is affine, the natural map $\varphi^*\varphi_*\mathcal{F} \to \mathcal{F}$ is a surjection (reduce to the affine case and use that $\mathcal{F}$ is coherent) we obtain that $\mathcal{F} \otimes \varphi^*\mathcal{O}(1)^{\otimes n}$ is globally generated.

iv) ⇒ ii) : Assume that $K(L)$ is infinite. Let Y be the connected component of 0. Then Y is an abelian subvariety of X of positive dimension (One can show using connectedness that Y is closed under the group operation. Y is also irreducible, since the translation map of closed points induces isomorphisms of local rings at closed points).

The restriction L_Y of L to Y is again ample. (In the proof of *iii) ⇒ iv)* we essentially showed that the pullback of an ample bundle along a finite morphism is again ample. The restriction is pulling back along a closed immersion and closed immersions are clearly finite morphisms.)

Consider now the line bundle $M = m^*L_Y \otimes p_1^*L_Y^{-1} \otimes p_2^*L_Y^{-1}$, where $m: Y \times Y \to Y$ is the addition map and p_1, p_2 are projections. Since for any $y \in Y$,the pullback

$t_y^* L_Y \cong L_Y$, the restriction $M|_{\{y\}\times Y}$ is trivial. Then by the Seesaw principle, $M \cong p_1^* R$ for some $R \in \mathrm{Pic}(Y)$. By the same argument, the restriction $p_1^* R|_{Y\times\{y\}}$ is also trivial, but this is nothing but the pullback along the composition map

$$Y \times \{y\} \xrightarrow{i} Y \times Y \xrightarrow{p_1} Y,$$

which is an isomorphism. Therefore R and hence M is trivial.

Pulling M back along the map

$$\begin{aligned} Y &\to Y \times Y \\ y &\mapsto (y, -y), \end{aligned}$$

we obtain that $L_Y \otimes (-1_Y)^* L_Y$ is trivial on Y. Since -1_Y is an automorphism of Y, the pullback $(-1_Y)^* L_Y$ and hence $L_Y \otimes (-1_Y)^* L_Y$ is also ample. However, the trivial bundle is ample only if the dimension is zero. Since $\dim Y > 0$, this is a contradiction.

ii) $\Rightarrow$ **i)** : Trivial since $H \subseteq K(L)$.

i) $\Rightarrow$ **iii)** : By the theorem of the square, $t_x^*(D) + t_{-x}^*(D) \in |2D|$. For any $u \in X$, we can find an $x \in X$ such that $u \pm x \notin \mathrm{Supp}(D)$ (because the complement of the set of such x's lie in $t_u^*(D) \cup t_{-u}^*((-1_X)^* D)$, which is of codimension one), equivalently $u \notin \mathrm{Supp}(t_x^*(D) + t_{-x}^*(D))$. Therefore, the linear system $|2D|$ is base point free and we have a morphism $\varphi: X \to \mathbb{P}^r$.

Now assume that φ is not finite. Since for φ, being finite or quasi-finite are equivalent ([GW] p.358 Cor.12.89), that means that there exists a fiber $\varphi^{-1}(q)$ for some $q \in \mathbb{P}^r$ such that $\dim \varphi^{-1}(q) \geq 1$. Pick an irreducible curve $C \subseteq \varphi^{-1}(q)$. Since the map φ restricted to C is constant, that means that the linear system $|2D|$ restricts trivially to C. This in turn means that for all $E \in |2D|$, either $C \cap E = \varnothing$ or $C \subseteq E$.

We need a technical lemma to finish the proof of the theorem.

Lemma 3. *Let E be an effective irreducible divisor on X such that $C \cap E = \varnothing$. Then $t_{x_i - x_j}^*(E) = E$ for all $x_i, x_j \in C$.*

Proof. Let $L = \mathcal{O}_X(E)$ and consider the bundle $m^* L$ on $X \times C$, m being the restriction of the addition map $X \times X \to X$. The first projection $X \times C \xrightarrow{p_1} X$ is obviously flat. Therefore the degree of the bundle $m^* L\,|_{\{x\}\times C}$ is the same for all $x \in X$. Clearly, $m^* L\,|_{\{x\}\times C} \cong t_x^* L\,|_C$ and since $t_0^* L\,|_C \cong L\,|_C$ is trivial (as $C \cap E = \varnothing$), we conclude that for all $x \in X$, $t_x^* L\,|_C$ has degree zero. Since $t_x^* L\,|_C$ is effective (as E is effective), that means $t_x^* L\,|_C$ is trivial for all $x \in X$.

It follows that for all $x \in X$, either $t_x(C) \cap E = \varnothing$ or $t_x(C) \subseteq E$. Let now $x_1, x_2 \in C$ and $y \in E$. Then $t_{y-x_1}(C) \cap E \neq \varnothing$ as they meet at y, therefore $t_{y-x_1}(C) \subseteq E$ and hence $y - x_1 + x_2 \in E$. This proves the lemma. □

Now let $x \in X$ such that $C \cap t_x^* D + t_{-x}^* D = \varnothing$ (To see that such an element exists, let $u \in C$ and then find $x \in X$ such that $u \notin t_x^*(D) + t_{-x}^*(D)$. Since C cannot be contained in $t_x^*(D) + t_{-x}^*(D)$, they should be disjoint). So we have that $t_x(C) \cap D = \varnothing$. In particular, we have $t_x(C) \cap D_i = \varnothing$, where $D = \sum n_i D_i$ is the decomposition into irreducible divisors. By lemma, we conclude that the set H is infinite, contradicting the assumption. □

Corollary 4. *Abelian varieties are projective.*

Proof. Let U be an open affine subset of X containing the point 0. Then it is a general fact that the complement $X \smallsetminus U$ has pure codimension 1 (Check the lecture notes of Bryden Cais [Ca] for this fact). Let $D_1, \dots, D_t$ be the irreducible components of $X \smallsetminus U$ and let $D = \sum D_i$. We will show that D satisfies $i)$ of the above proposition.

Consider the set $H = \{x \in X \mid t_x^* D = D\}$. Clearly, for any $x \in H$ we have that $t_x(U) = U$. Since $0 \in U$, it follows that $H \subseteq U$. On the other hand, H is a closed set as it can be seen as $f^{-1}(D)$ where $f\colon K(L) \to |D|$ which is defined as $f(x) = t_x^* D$. H is clearly proper and being a closed subset of an affine scheme U, it is affine. Therefore H is finite (by [GW] p.358 Cor.12.89). □

2. Isogenies

Definition 5. A homomorphism of abelian varieties $\alpha\colon X \to Y$ is called an isogeny if it is surjective and has zero dimensional kernel.

Proposition 6. *For a homomorphism $\alpha : X \to Y$ of abelian varieties the following are equivalent:*

i) α *is an isogeny;*
ii) $\dim X = \dim Y$ *and* $\ker(\alpha)$ *is finite;*
iii) $\dim X = \dim Y$ *and* α *is surjective;*
iv) α *is finite, flat and surjective.*

Proof. First observe that for any $y \in Y$, choosing an element $x \in \alpha^{-1}(y)$ we obtain an isomorphism of the fibers $t_x|_{\alpha^{-1}(0)} : \alpha^{-1}(0) \to \alpha^{-1}(y)$. In particular all fibers of the map $\alpha : X \to \alpha(X)$ have the same dimension. Moreover, for any $y \in Y$ we have that $\dim \alpha^{-1}(y) \geq \dim X - \dim \alpha(X)$ and that equality holds on an open subset of Y (p.95 Ex 3.22 [Ha]). It follows that for any $y \in Y$ we have the equality $\dim \alpha^{-1}(y) = \dim X - \dim \alpha(X)$. This proves the equality of i),ii) and iii).

Now we prove the equality of i) and iv). It is obvious that iv) implies i), so assume i) now. Since $\alpha^{-1}(0)$ is finite and every fiber has the same dimension, we conclude that α is quasi finite. Moreover, the composition $X \xrightarrow{\alpha} Y \to \operatorname{Spec}(k)$ is proper and $Y \to \operatorname{Spec}(k)$ is separated, therefore α is proper (p.102 Corollary 4.8 [Ha]). Finally α is finite, since it is quasi finite and proper. Finally, since every fiber is finite, $|\alpha^{-1}(y)| = h^0(X_y, \mathcal{O}_{X_y})$ for any $y \in Y$. Since each fiber is isomorphic, they have the same cardinality, which in turn means that the map $y \mapsto h^0(X_y, \mathcal{O}_{X_y})$ is constant. By ([Ha] p.125 Ex 5.8), we conclude that $\alpha_* \mathcal{O}_X$ is locally free. Hence α is flat. □

Proposition 7. *Let X be an abelian variety of dimension g. Then $n_X\colon X \to X$ is an isogeny of degree n^{2g}.*

Proof. Let L be a symmetric (i.e. $(-1_X)^* L \cong L$) very ample line bundle on X (To see that this choice can be made, pick an ample line bundle M. Then $(-1_X)^* M \otimes M$ is also ample. Take a sufficiently high power of it to ensure that it is very ample. The resulting bundle is clearly symmetric). First observe that $n_X^* L \cong L^{n^2}$ (by Theorem 17 from Angela's talk). Let $Z = \ker(n_X)$. Then $n_X^* L|_Z$ is trivial since the map $Z \hookrightarrow X \xrightarrow{n_X} X$ is a constant map. It is clearly ample as well. Hence it follows that $\dim Z = 0$. By ii) of the above proposition, we see that n_X is an isogeny.

Now we will show that the degree is n^{2g}. Let $L \cong \mathcal{O}_X(D)$ for some effective divisor D. It is a classical fact that the g-fold intersection product

$$n_X^* D \dots n_X^* D = \deg(n_X).D \dots D$$

On the other hand, by our observation above, $n_X^* D$ is linearly equivalent to $n^2 D$. So we obtain

$$\deg(n_X).D \dots D = n^{2g}.D \dots D$$

So we will be done if we can show that $D \dots D \neq 0$. But this is obvious, since $|D|$ is very ample. In fact $D \dots D = \deg(X)$, where by the degree I mean the degree of X under the projective embedding given by $|D|$.

□

Remark 8. The above proposition tells us that X is divisible as an abstract group.

Theorem 9. *Let X be an algebraic variety, G a finite group of automorphisms of X. Suppose that for any $x \in X$ the orbit Gx is contained in an open affine subset of X. Then there is a pair (Y, π), where Y is a variety and $\pi : X \to Y$ a morphism, satisfying the following conditions:*

i) as a topological space, (Y, π) is the quotient space for the G-action;
ii) if $(\pi_* \mathcal{O}_X)^G$ denotes the subsheaf of invariants of $\pi_* \mathcal{O}_X$ for the action of G on $\pi_* \mathcal{O}_X$ deduced from i), the natural homomorphism $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism.

The pair (Y, π) is determined up to an isomorphism by these conditions. The morphism π is finite, surjective and separable. Y is affine if X is affine.

If further G acts freely on X then π is an etale morphism.

Proof. See [Mu] p.66. □

Theorem 10. *Let X be an abelian variety. Then there is a 1-1 correspondence between the two sets of objects:*

i) finite subgroups $K \subseteq X$;
ii) separable isogenies $f : X \to Y$, where two isogenies $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are considered equal if there is an isomorphism $h : Y_1 \to Y_2$ such that $h \circ f_1 = f_2$.

Proof. First let $K \subseteq X$ be a finite subgroup. By the above theorem, X/K is a variety and the morphism $X \xrightarrow{f} X/K$ is finite, surjective and separable. We want to show that X/K is an abelian variety. X/K is the image of a complete variety X and therefore it is complete. X/K has clearly the structure of an abstract group. To see that the multiplication map on X/K is a morphism consider the following diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{m} & X \\ \Big\downarrow{\scriptstyle f \times f} & & \Big\downarrow{\scriptstyle f} \\ X/K \times X/K & \xrightarrow[n]{} & X/K \end{array}$$

$X/K \times X/K$ can be identified with $(X \times X)/(K \times K)$ such that the map $f \times f$ becomes the quotient map $X \times X \to (X \times X)/(K \times K)$. Now observe that the map $f \circ m$ is clearly $K \times K$ invariant. Since these quotients are good quotients, they are also categorical, i.e. they enjoy the universal property that any G−invariant map factors through the quotient variety. Therefore there is a unique morphism $(X \times X)/(K \times K) \xrightarrow{g} X/K$ such that $f \circ m = g \circ (f \times f)$. Clearly the morphism g is equal to n on closed points and thus is the multiplication map on X/K.

To see that the inversion map is a morphism, we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X/K & \xrightarrow[i_2]{} & X/K \end{array}$$

where i_1, i_2 are the inversion maps. Now observe that $f \circ i_1$ is K−invariant. Therefore using the universal property, there exists a unique morphism $g: X/K \to X/K$ such that $f \circ i_1 = g \circ f$. Again, the morphism g is equal to i_2 on closed points and thus is the inversion map of X/K. Obviously, $\ker(f) = K$ and f is an isogeny.

For the converse, let $f: X \to Y$ be a separable isogeny. $\ker(f)$ is a finite subgroup of X, so we can consider the isogeny $X \xrightarrow{\pi} X/\ker(f)$. Again since f is obviously $\ker(f)$−invariant, there is a unique morphism $g: X/\ker(f) \to Y$ such that $g \circ \pi = f$. All we need to show now is that g is an isomorphism. Since f is separable, so is g. A separable morphism of varieties, which is a bijection is a birational isomorphism (p.35 [Hu]). Then by Zariski's main theorem (p.152 Corollary 4.6 [Li]) it follows that g is an isomorphism. □

Corollary 11. *A separable isogeny is an etale morphism.*

Proof. By the theorem any separable isogeny is of the form $X \to X/K$ for some finite subgroup $K \subseteq X$ and this quotient is etale since any subgroup of X acts freely on X. □

References

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