The dual abelian variety

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These are notes for a talk given in the IRTG College Seminar on abelian varieties and Fourier-Mukai transforms in Berlin on December 10, 2014. The outline mainly follows Chapter 9 of Polishchuk's book [5], whereas most proof details are taken from the classic textbook of Mumford [4]. The lecture notes of Milne [3] and van der Geer/Moonen [1] provided additional inspiration.

1 The group Pic^0 of an abelian variety

1.1 Set-up

Any line bundle $\mathcal{L}$ on an abelian variety X over some algebraically closed field k defines a map

$$\phi_{\mathcal{L}} : X(k) \to \mathrm{Pic}(X), \quad x \mapsto [t_x^*\mathcal{L} \otimes \mathcal{L}^\vee].$$

Remark. Since $t_0^*\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$ and by the theorem of the square,

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L} \quad \rightsquigarrow \quad t_{x+y}^*\mathcal{L} \otimes \mathcal{L}^\vee \cong (t_x^*\mathcal{L} \otimes \mathcal{L}^\vee) \otimes (t_y^*\mathcal{L} \otimes \mathcal{L}^\vee),$$

the map $\phi_{\mathcal{L}}$ is a homomorphism of abelian groups.

Recall that Irfan introduced the set

$$K(\mathcal{L})(k) := \ker \phi_{\mathcal{L}} = \{x \in X(k) \mid t_x^*\mathcal{L} \cong \mathcal{L}\}.$$

Definition. $\mathrm{Pic}^0(X) := \{[\mathcal{L}] \in \mathrm{Pic}(X) \mid \phi_{\mathcal{L}} \equiv 0\} = \{[\mathcal{L}] \in \mathrm{Pic}(X) \mid \forall x \in X(k) : t_x^*\mathcal{L} \cong \mathcal{L}\}.$

From now on, simplify notation by writing $\mathcal{L}$ for the isomorphism class of a line bundle $\mathcal{L}$ in $\mathrm{Pic}(X)$ or $\mathrm{Pic}^0(X)$.

1.2 Properties of $\mathrm{Pic}^0(X)$

Lemma 1. *Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ and denote by $m, p_1, p_2 : X \times X \to X$ the addition and the two natural projections. Then*

(a) $m^*\mathcal{L} \cong p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$,

(b) If Y is a scheme and $f, g : Y \to X$ are morphisms, then $(f+g)^*\mathcal{L} \cong f^*\mathcal{L} \otimes g^*\mathcal{L}$.

(c) If $n \in \mathbb{Z}$, then $n_X^*\mathcal{L} \cong \mathcal{L}^{\otimes n}$.

Proof. (a) Let $x \in X(k)$. Then

$$(m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee)|_{X\times\{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X \quad \text{and} \quad (m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee)|_{\{0\}\times X} \cong \mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X,$$

since the maps in question are given explicitly by

$$\begin{aligned} m, p_1, p_2 &: X \cong \{x\} \times X \hookrightarrow X \times X \to X, \quad m(y) = x+y, \quad p_1(y) = x, \quad p_2(y) = y, \\ m, p_1, p_2 &: X \cong X \times \{x\} \hookrightarrow X \times X \to X, \quad m(y) = x+y, \quad p_1(y) = y, \quad p_2(y) = x. \end{aligned}$$

By the seesaw principle (as in Irfan's talk), this implies $m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee \cong \mathcal{O}_{X\times X}$.

(b) Consider the compositions $Y \xrightarrow{f\times g} X \times X \xrightarrow{m,p_1,p_2} X$. Then by (a),

$$(f+g)^*\mathcal{L} = (f\times g)^* m^*\mathcal{L} \cong (f\times g)^* p_1^*\mathcal{L} \otimes (f \times g)^* p_2^*\mathcal{L} = f^*\mathcal{L} \otimes g^*\mathcal{L}.$$

(c) Induction. □

Lemma 2. *Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ be non-trivial. Then $H^i(X, \mathcal{L}) = 0$ for all $i \geq 0$.*

Proof.
- If $H^0(\mathcal{L}) \neq 0$, then $\mathcal{L}$ has a non-trivial section s, so $(-1_X)^*\mathcal{L}$ has the non-trivial section $(-1_X)^*s$. But by Lemma 1, $(-1_X)^*\mathcal{L} \cong \mathcal{L}^\vee$, so both $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non-trivial section. Hence $\mathcal{L} \cong \mathcal{O}_X$, contradiction.
- Let $i > 0$ be the smallest positive integer such that $H^i(X, \mathcal{L}) \neq 0$. The maps

$$X \xrightarrow{\mathrm{id}\times 0} X \times X \xrightarrow{m} X, \quad x \mapsto (x, 0) \mapsto x$$

give in cohomology

$$H^i(X, \mathcal{L}) \to H^i(X \times X, m^*\mathcal{L}) \to H^i(X, \mathcal{L}),$$

the composition being the identity. Using the first statement of Lemma 1, the Künneth formula tells

$$H^i(X \times X, m^*\mathcal{L}) \cong H^i(X \times X, p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) \cong \bigoplus_{j=0}^{i} H^j(X, \mathcal{L}) \otimes H^{i-j}(X, \mathcal{L}).$$

Since $H^0(\mathcal{L}) = 0$ by the first bullet and $H^{i-j}(X, \mathcal{L}) = 0$ for $j \geq 1$ by the choice of i, this yields $H^i(X \times X, m^*\mathcal{L}) = 0$. So the identity of $H^i(X, \mathcal{L})$ factors through 0. □

Proposition 3. *Let $\mathcal{L}$ be a line bundle on the abelian variety X. Then:*

(a) $\mathrm{im}\,\phi_{\mathcal{L}} \subseteq \mathrm{Pic}^0(X)$.

(b) If $K(\mathcal{L})(k)$ is finite, then $\mathrm{im}\,\phi_{\mathcal{L}} = \mathrm{Pic}^0(X)$.

Proof. (a) Let $x \in X(k)$ and $y \in X(k)$. Using the theorem of the square,

$$t_y^*(t_x^*\mathcal{L} \otimes \mathcal{L}^\vee) = t_{x+y}^*\mathcal{L} \otimes t_y^*\mathcal{L}^\vee \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes t_y^*\mathcal{L}^\vee \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee.$$

(b) Pick $\mathcal{M} \in \mathrm{Pic}^0(X)$ and suppose $\mathcal{M} \notin \mathrm{im}\,\phi_{\mathcal{L}}$.

A useful line bundle. On $X \times X$, define the Mumford line bundle

$$\Lambda(\mathcal{L}) := m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee,$$

where m denotes the addition and p_1, p_2 are the natural projections. Put $\mathcal{N} := \Lambda(\mathcal{L}) \otimes p_1^*\mathcal{M}^\vee$ and let $x \in X(k)$. Then, similar to the proof of Lemma 1,

$$\mathcal{N}|_{\{x\}\times X} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \quad \text{and} \quad \mathcal{N}|_{X\times\{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee,$$

The first projection. Since $\mathcal{M} \notin \operatorname{im}\phi_{\mathcal{L}}$, the line bundle $t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee$ is not trivial for all $x \in X(k)$, because otherwise $\mathcal{M} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee$. Now Lemma 2 states that

$$H^i(X \times \{x\}, \mathcal{N}|_{X\times\{x\}}) \cong H^i(X, t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee) = 0, \quad i \geq 0.$$

By Grauert's theorem [2, Corollary III.12.9], for any $i \geq 0$ and $x \in X(k)$, the sheaf $\mathrm{R}^i p_{2*}\mathcal{N}$ is locally free and

$$\mathrm{R}^i p_{2*}\mathcal{N} \otimes \kappa(x) \cong H^i(X \times \{x\}, \mathcal{N}|_{X\times\{x\}}).$$

So $\mathrm{R}^i p_{2*}\mathcal{N} = 0$ for any $i \geq 0$. Applying the Leray spectral sequence

$$H^j(X, \mathrm{R}^i p_{2*}\mathcal{N}) \Rightarrow H^{i+j}(X \times X, \mathcal{N})$$

yields $H^i(X \times X, \mathcal{N}) = 0$, $i \geq 0$.

The second projection. If $x \notin K(\mathcal{L})(k)$, then $\mathcal{N}|_{\{x\}\times X} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ is not isomorphic to $\mathcal{O}_X$. In particular, by Lemma 2,

$$H^i(\{x\} \times X, \mathcal{N}|_{\{x\}\times X}) \cong H^i(X, t_x^*\mathcal{L} \otimes \mathcal{L}^\vee) = 0, \quad i \geq 0.$$

Similar to the situation above, Grauert's theorem states now that for any $i \geq 0$ and $x \in X(k)$, $\mathrm{R}^i p_{1*}\mathcal{N}$ is locally free and

$$\mathrm{R}^i p_{1*}\mathcal{N} \otimes \kappa(x) \cong H^i(\{x\} \times X, \mathcal{N}|_{\{x\}\times X}).$$

This vanishes for $x \notin K(\mathcal{L})(k)$, so the support of $\mathrm{R}^i p_{1*}\mathcal{N}$ is contained in $K(\mathcal{L})(k)$, which is a finite set by hypothesis. This forces the Leray spectral sequence

$$H^j(X, \mathrm{R}^i p_{1*}\mathcal{N}) \Rightarrow H^{i+j}(X \times X, \mathcal{N})$$

to degenerate to

$$\bigoplus_{x \in K(\mathcal{L})(k)} (\mathrm{R}^i p_{1*}\mathcal{N})_x \cong H^i(X \times X, \mathcal{N}) = 0, \quad i \geq 0.$$

Thus $\mathrm{R}^i p_{1*}\mathcal{N}$ vanishes everywhere for all $i \geq 0$. Using the Grauert isomorphism, this implies for $x \in K(\mathcal{L})(k)$

$$0 = p_{1*}\mathcal{N} \otimes \kappa(x) \cong H^0(\{x\} \times X, \mathcal{N}|_{\{x\}\times X}) \cong H^0(X, t_x^*\mathcal{L} \otimes \mathcal{L}^\vee) = H^0(X, \mathcal{O}_X),$$

contradiction. □

On the level of abelian groups, the second part of Propostion 3 constructs an isomorphism

$$X(k)/K(\mathcal{L})(k) \cong \mathrm{Pic}^0(X),$$

provided that $K(\mathcal{L})(k)$ is finite (this is e. g. satisfied if $\mathcal{L}$ is ample). The aim is now to construct a *scheme* $K(\mathcal{L})$ so that the quotient $X/K(\mathcal{L})$ becomes an abelian variety, namely the dual abelian variety of X. In this case, $\phi_{\mathcal{L}}$ becomes an isogeny with (finite) kernel $K(\mathcal{L})$.

2 Construction of the dual abelian variety

Lemma 4. *Let X be a complete variety, Y an arbitrary scheme, $\mathcal{L}$ a line bundle on $X \times Y$ and $p : X \times Y \to Y$ the projection onto the second factor. Then $\mathcal{L}$ is isomorphic to the pullback of a line bundle on Y via p if and only if $p_*\mathcal{L}$ is a line bundle and the natural map $p^*p_*\mathcal{L} \to \mathcal{L}$ is an isomorphism.*

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) If $\mathcal{L} \cong p^*\mathcal{M}$ for some line bundle $\mathcal{M}$ on Y, then by the projection formula, $p_*\mathcal{L} \cong p_*p^*\mathcal{M} \cong \mathcal{M} \otimes p_*\mathcal{O}_{X\times Y}$. Since X is complete, $p_*\mathcal{O}_{X\times Y} \cong \mathcal{O}_Y$, giving an isomorphism of $p_*\mathcal{L}$ with the line bundle $\mathcal{M}$. Applying p^* gives the natural map $p^*p_*\mathcal{L} \to \mathcal{L}$. □

Proposition 5. *Let X be a complete variety, Y an arbitrary scheme, $\mathcal{L}$ a line bundle on $X \times Y$. Then there exists a unique closed subscheme $Y_1 \subseteq Y$ such that for every scheme Z, a morphism $f : Z \to Y$ factors through Y_1 if and only if the line bundle $(\mathrm{id} \times f)^*\mathcal{L}$ on $X \times Z$ is the pullback of some line bundle on Z via the projection onto the second factor.*

Remark. Y_1 is the maximal closed subscheme of Y over which $\mathcal{L}$ is trivial: For each $y \in Y_1$, the restricted line bundle $\mathcal{L}|_{X\times\{y\}}$ is trivial, and Y_1 is minimal among the closed subschemes with this property.

Proof. Uniqueness. If Y_1, Y_2 are two such closed subschemes, then they factor through each other.

Localizing. Let $\{U_i\}$ be an open covering of Y. If the proposition holds for $X \times U_i \to U_i$ and $\mathcal{L}|_{X\times U_i}$, then we can glue the obtained closed subschemes together, as they have to be equal on the intersections $V_i \cap V_j$. Hence the statement is local on Y.

Shrinking Y. The set $S := \{y \in Y \mid \mathcal{L}|_{X\times\{y\}} \text{ is trivial}\}$ is closed. Moreover, $\mathcal{L}|_{X\times S}$ is the pullback of some line bundle on S (Angela's talk). Fix $y \in S$. By Lemma 4, the natural map $p^*p_*\mathcal{L} \to \mathcal{L}$ is thus an isomorphism over $X \times \{y\}$, where p denotes the projection onto the second factor. On the open subset $Y \setminus S$, the empty scheme satisfies the conditions of the proposition. So Y may be shrinked to some $\operatorname{Spec} A$, which is a neighborhood of some point in $y \in S$ so that $p^*p_*\mathcal{L} \to \mathcal{L}$ is an isomorphism on $X \times \operatorname{Spec} A$.

Applying proper base change. We want to equip S with a scheme structure. By proper base change (again Angela's talk), there is a finite complex

$$\mathcal{P} : 0 \to P_0 \to P_1 \to \cdots \to P_n$$

of finitely generated projective A-modules and an isomorphism of functors

$$H^i(X \times_{\operatorname{Spec} A} \operatorname{Spec} -, \mathcal{L} \otimes_A -) \cong H^i(\mathcal{P} \otimes_A -)$$

for all i on the category of A-algebras. Let $M := \operatorname{coker}(P_1^\vee \to P_0^\vee)$. Then for any A-algebra B,

$$(P_1^\vee \otimes_A B) \to (P_0^\vee \otimes_A B) \to M \otimes_A B \to 0$$

is exact, and thus is

$$0 \to \operatorname{Hom}_B(M \otimes_A B, B) \to P_0 \otimes_A B \to P_1 \otimes_A B.$$

Using the above isomorphism of functors,

$$\operatorname{Hom}_A(M, B) \cong \operatorname{Hom}_B(M \otimes_A B, B) \cong H^0(\mathcal{P} \otimes_A B) \cong H^0(X \times_Y \operatorname{Spec} B, \mathcal{L} \otimes_A B).$$

Let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to the point $y \in S$. Then, as $A/\mathfrak{m}$-vector spaces,

$$\begin{aligned}\dim M/\mathfrak{m}M &= \dim \operatorname{Hom}_{A/\mathfrak{m}}(M \otimes_A A/\mathfrak{m}, A/\mathfrak{m}) \\ &= \dim \operatorname{Hom}_A(M, A/\mathfrak{m}) \\ &= \dim H^0(X \times \{y\}, \mathcal{L}|_{X\times\{y\}}) \\ &= \dim H^0(X \times \{y\}, \mathcal{O}_{X\times\{y\}}) \\ &= 1.\end{aligned}$$

Consequently, Nakayama's lemma implies that M is generated by a single element in an open neighborhood of y. Shrinking Y further, we can assume $M = A/I$ for some ideal I in A. Define Y_1 to be the closed subscheme of Y corresponding to I.

Checking the "only if" part. Denote by $\mathcal{L}_1$ the restriction of $\mathcal{L}$ to $X \times Y_1$. Then $p_*\mathcal{L}_1$ is the sheafification of $\operatorname{Hom}_A(A/I, A/I) \cong A/I$ on $Y_1 = \operatorname{Spec} A/I$, hence it is a line bundle. In view of Lemma 4, consider the natural map $\lambda : p^*p_*\mathcal{L}_1 \to \mathcal{L}_1$. Both sides are invertible sheaves, so the stalks at $x \in X \times Y_1$ are isomorphic if and only if $(p^*p_*\mathcal{L}_1)_x \otimes \kappa(x) \to (\mathcal{L}_1)_x \otimes \kappa(x)$ is surjective. Now

$$\operatorname{Hom}_A(A/I, A/I) \to \operatorname{Hom}_A(M, A/\mathfrak{m}) = H^0(X \times \{y\}, \mathcal{L}|_{X\times\{y\}}) \cong H^0(X \times \{y\}, \mathcal{O}_{X\times\{y\}}).$$

is surjective, so λ is an isomorphism at all $x \in X \times \{y\}$. Let V denote the projection onto Y of the union of the supports of $\ker \lambda$ and $\operatorname{coker} \lambda$. Then V is a closed subset of Y not containing y. We can shrink Y even further so that V is actually empty. Now Lemma 4 states that $\mathcal{L}_1$ is the pullback of some line bundle on Y_1. This shows the "only if" direction of the proposition: If $f : Z \to Y$ factors as $Z \xrightarrow{g} Y_1 \hookrightarrow Y$, then $(\operatorname{id} \times f)^*\mathcal{L} = (\operatorname{id} \times g)^*\mathcal{L}_1$.

Universal property. Assume that $(\operatorname{id} \times f)^*\mathcal{L} \cong p^*\mathcal{M}$ for some line bundle $\mathcal{M}$ on Z. The statement is local on Z, thus suppose $Z = \operatorname{Spec} C$, where C becomes an A-algebra via f. We can shrink Z further in order to assume that $\mathcal{M}$ is trivial. As X is complete, $p_*(\operatorname{id} \times f)^*\mathcal{L} \cong p_*\mathcal{O}_{X\times Z} \cong \mathcal{O}_Z$. Translated into algebra, this is an isomorphism of C-modules $\operatorname{Hom}_A(A/I, C) \cong C$, so $A \to C$ factors throguh A/I. □

Let X be an abelian variety over k, and let $\mathcal{L}$ be a line bundle on X. Apply Proposition 5 to the Mumford line bundle $\Lambda(\mathcal{L})$ on $X \times X$. This yields a closed subscheme $X_1 \subseteq X$ with the universal property as described above. For each $x \in X(k)$, by definition of the Mumford bundle, $\Lambda(\mathcal{L})|_{X\times\{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee$. Thus

$$K(\mathcal{L})(k) = \{x \in X(k) \mid \Lambda(\mathcal{L})|_{X\times\{x\}} \text{ is trivial}\} = X_1(k),$$

and we can view $K(\mathcal{L})$ as a scheme whose k-rational points are $K(\mathcal{L})(k)$.

Proposition 6. *$K(\mathcal{L})$ is a subgroup scheme of X.*

Proof. Let $f' : Z \to K(\mathcal{L})$ be a morphism of schemes. Composing with $K(\mathcal{L}) \hookrightarrow X$ gives a morphism $f : Z \to X$. By Proposition 5, $(\operatorname{id} \times f)^*\Lambda(\mathcal{L}) = q_2^*\mathcal{M}$, where $q_2 : X \times Z \to Z$ is the natural projection onto Z. Let $\mathcal{L}_Z := q_1^*\mathcal{L}$, where $q_1 : X \times Z \to X$. Let further

$$t_f : X \times Z \to X \times Z, \quad (x, z) \mapsto (x + f(z), z)$$

be the translation by f. Then

$$\begin{aligned}t_f^*\mathcal{L}_Z = (\operatorname{id} \times f)^*m^*\mathcal{L} &= (\operatorname{id} \times f)^*\Lambda(\mathcal{L}) \otimes (\operatorname{id} \times f)^*p_1^*\mathcal{L} \otimes (\operatorname{id} \times f)^*p_2^*\mathcal{L} \\ &\cong q_2^*\mathcal{M} \otimes q_1^*\mathcal{L} \otimes q_2^*f^*\mathcal{L}, \\ &= q_2^*(\mathcal{M} \otimes f^*\mathcal{L}) \otimes \mathcal{L}_Z.\end{aligned}$$

Conversely, if $f : Z \to X$ is any morphism such that $t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee$ is the pullback of a line bundle on Z via q_2, Proposition 5 states that f factors through $K(\mathcal{L})$.

Now let $f, g : Z \to X$ be morphisms of schemes such that $t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee$ and $t_g^*\mathcal{L}_Z \otimes \mathcal{L}^\vee$ are pullbacks of line bundles on Z via q_2. That is, f, g are points of $X(Z)$ that lie in $K(\mathcal{L})(Z)$. By a slightly enhanced version of the theorem of the square (which follows from Lemma 1),

$$t_{f+g}^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee \cong t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee \otimes t_g^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee,$$

so $f + g$ lies in $K(\mathcal{L})(Z)$ as well. As a consequence, $K(\mathcal{L})(Z)$ is a subgroup of $X(Z)$. □

Remark. If $\mathcal{L}$ is ample, then $K(\mathcal{L})(k)$ is finite (Irfan). In this case, $K(\mathcal{L})$ is a finite group scheme of X.

Theorem 7. *Let X be an abelian variety over a field k, $\mathcal{L}$ an ample line bundle on X. Then the quotient scheme $X/K(\mathcal{L})$ exists and is an abelian variety over k with the same dimension as X.*

Idea of proof. X is an (abelian) group scheme, $K(\mathcal{L})$ a finite subgroup scheme. By [4, p. 118], this implies that $X/K(\mathcal{L})$ is a group scheme of the same dimension as X. If $\operatorname{char} k = 0$, then X is automatically a variety, as group schemes in $\operatorname{char} 0$ are smooth [1, Theorem 3.20]. For positive characteristic, the proof requires more work (see [1, Theorem 6.18]). □

Definition. This quotient is the dual abelian variety $X^\vee$ of X.

Remark. By construction, $X^\vee(k) = \operatorname{Pic}^0(X)$ and the quotient morphism $X(k) \to X^\vee(k)$ is $\phi_\mathcal{L}$. In this way, $\phi_\mathcal{L}$ may be thought of as an isogeny from X to $X^\vee$, whose restriction to k-rational points is the "old" $\phi_\mathcal{L}$.

3 Properties of the dual abelian variety

3.1 Functoriality and the Poincaré bundle

Theorem 8 (Universal property of the dual abelian variety)**.** *Let X be an abelian variety over k. Then there is a uniquely determined line bundle $\mathcal{P}$ on $X \times X^\vee$, called the Poincaré bundle, such that*

(a) $\mathcal{P}|_{X\times\{y\}} \in \operatorname{Pic}^0(X \times \{y\})$ for all $y \in X^\vee$,

(b) $\mathcal{P}|_{\{0\}\times X^\vee}$ is trivial,

and if Z is a scheme with a line bundle $\mathcal{R}$ on $X \times Z$ such that $\mathcal{R}|_{X\times\{z\}} \in \operatorname{Pic}^0(X \times \{z\})$ for all $z \in Z$ and $\mathcal{R}|_{\{0\}\times Z}$ is trivial, then there is a unique morphism $f : Z \to X^\vee$ such that $(\operatorname{id} \times f)^*\mathcal{P} = \mathcal{R}$.

In other words, $(X^\vee, \mathcal{P})$ represents the functor

$$Z \mapsto \{\mathcal{L} \in \operatorname{Pic}(X \times Z) \mid \mathcal{L}|_{X\times\{z\}} \in \operatorname{Pic}^0(X \times \{z\}) \text{ for all } z \in Z \text{ and } \mathcal{L}|_{\{0\}\times Z} \text{ is trivial}\},$$

and the Poincaré bundle $\mathcal{P}$ corresponds to $\operatorname{id}_{X^\vee}$.

Remark. This shows the uniqueness of $X^\vee$ as well.

Chunks of the proof. Let $\mathcal{L}$ be an ample line bundle on X.

Strategy. If $K(\mathcal{L})$ acts on the second factor of $X \times X$, then there is a quotient map $\pi : X \times X \to X \times X^\vee$, which is given on k-rational points by $\mathrm{id} \times \phi_{\mathcal{L}}$. By [4, p. 112], there is an equivalence of categories

$$\{K(\mathcal{L})\text{-line bundles on } X \times X\} \leftrightarrow \{\text{line bundles on } X \times X^\vee\}, \quad \mathcal{M} \mapsto \pi^*\mathcal{M}$$

The strategy is now to show that the Mumford line bundle $\Lambda(\mathcal{L})$ is a $K(\mathcal{L})$-bundle, hence corresponds to a line bundle $\mathcal{P}$ on $X \times X^\vee$ such that $\pi^*\mathcal{P} = \Lambda(\mathcal{L})$.

$\mathcal{P}$ satisfies (a) and (b). This line bundle $\mathcal{P}$ satisfies the following: If $y = \phi_{\mathcal{L}}(x) \in X^\vee$ for some $x \in X(k)$, then

$$\mathcal{P}|_{X\times\{y\}} = (\mathrm{id} \times \phi_{\mathcal{L}})^*\mathcal{P}|_{X\times\{x\}} = \Lambda(\mathcal{L})|_{X\times\{x\}} = t_x^*\mathcal{L} \otimes \mathcal{L}^\vee = \phi_{\mathcal{L}}(x) \in \mathrm{Pic}^0(X)$$

by Proposition 3. Since $\Lambda(\mathcal{L})|_{\{0\}\times X} \cong \mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$, the line bundle $\mathcal{P}|_{\{0\}\times X^\vee}$ is trivial as well.

The universal property. Assume that Z is a normal variety. Consider the line bundle $\mathcal{M} := p_{12}^*\mathcal{R} \otimes p_{13}^*\mathcal{P}^\vee$ on $X \times Z \times X^\vee$, where $p_{12} : X \times Z \times X^\vee \to X \times Z$ and $p_{13} : X \times Z \times X^\vee \to X \times X^\vee$ are the natural projections. If $z \in Z$, $y \in X^\vee$, then $\mathcal{M}|_{X\times\{z\}\times\{y\}} = \mathcal{R}|_{X\times\{z\}} \otimes \mathcal{P}^\vee|_{X\times\{y\}}$. The subset

$$\Gamma := \{(z, y) \in Z \times X^\vee \mid \mathcal{M}|_{X\times\{z\}\times\{y\}} \text{ is trivial}\} = \{(z, y) \in Z \times X^\vee \mid \mathcal{R}|_{X\times\{z\}} \cong \mathcal{P}|_{X\times\{y\}}\}$$

is Zariski-closed in $Z \times X^\vee$. Moreover, it is the graph of a set-theoretic map $Z \to X^\vee$, because the map $X^\vee \to \mathrm{Pic}^0(X), y \mapsto \mathcal{P}|_{X\times\{y\}}$ is a bijection. In particular, the natural projection $\Gamma \to Z$ is bijective on points. In characteristic zero, this means that it must be birational of degree one, and hence an isomorphism by Zariski's Main Theorem (see e.g. [2, Corollary III.11.4]). Hence we get a unique morphism $Z \cong \Gamma \to X^\vee$, where the last arrow is given by projection.

Uniqueness of $\mathcal{P}$. A priori, $\mathcal{P}$ is only unique up to tensoring with pullbacks of line bundles on $X^\vee$ via the projection $p_2 : X \times X^\vee \to X^\vee$. But since $X^\vee \cong \{0\} \times X^\vee \to X \times X^\vee \xrightarrow{p_2} X^\vee$ is the identity, one obtains $(\mathcal{P} \otimes p_2^*\mathcal{L})|_{\{0\}\times X^\vee} \cong \mathcal{P}|_{\{0\}\times X^\vee} \otimes \mathcal{L}$. This implies that $\mathcal{L}$ is trivial by condition (b). □

Remark. Due to the uniqueness of $\mathcal{P}$, it is normalized in the sense that by construction, both $\mathcal{P}|_{X\times\{0\}}$ and $\mathcal{P}|_{\{0\}\times X^\vee}$ are trivial.

3.2 Dual morphisms and double-duals

Let $f : X \to Y$ be a homomorphism of abelian varieties. Denote by $\mathcal{P}_X$ and $\mathcal{P}_Y$ the Poincaré bundles on $X \times X^\vee$ and $Y \times Y^\vee$, respectively. Consider the line bundle $\mathcal{M} := (f \times \mathrm{id}_{Y^\vee})^*\mathcal{P}_Y$ on $X \times Y^\vee$. By the properties of the Poincaré bundle, $\mathcal{M}|_{X\times\{y\}} \in \mathrm{Pic}^0(X \times \{y\})$ and $\mathcal{M}|_{\{0\}\times Y^\vee}$ is trivial. Hence by Theorem 8, $\mathcal{M}$ defines a unique morphism $f^\vee : Y^\vee \to X^\vee$ with the propery that $(\mathrm{id}_X \times f^\vee)^*\mathcal{P}_X \cong (f \times \mathrm{id}_{Y^\vee})^*\mathcal{P}_Y$.

Definition. If $f : X \to Y$ is a homomorphism of abelian varieties, then $f^\vee : Y^\vee \to X^\vee$ is called the dual morphism of f.

Remark. If a point in $Y^\vee$ corresponds to a line bundle $\mathcal{L} \in \mathrm{Pic}^0(Y)$, then its image under $f^\vee$ is given by the pullback $f^*\mathcal{L}$.

Example. The dual morphism of the multiplication-by-n map $n_X : X \to X$ is $(n_X)^\vee = n_{X^\vee}$. This is basically due to Lemma 1 (c).

Lemma 9. *Let $f : X \to Y$ be a homomorphism of abelian varieties, $\mathcal{L}$ a line bundle on Y. Then $\phi_{f^*\mathcal{L}} = f^\vee \circ \phi_{\mathcal{L}} \circ f$.*

Proof. For $x \in X(k)$ holds $f^\vee(\phi_{\mathcal{L}}(f(x))) = f^* t^*_{f(x)}\mathcal{L} \otimes f^*\mathcal{L}^\vee = t^*_x f^*\mathcal{L} \otimes f^*\mathcal{L}^\vee$. □

Proposition 10. *If $f : X \to Y$ is an isogeny of abelian varieties, then so is $f^\vee$. Moreover,* $\ker f^\vee$ *is the kernel of the pullback map $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$, and this is the Cartier dual of* $\ker f$.

Proof. See [4, p. 143]. □

Definition. Let X be an abelian variety.

- The morphism $X \to X^\vee$ corresponding to the Poincaré bundle on $X \times X^\vee$ seen as a family of line bundles on $X^\vee$ parametrized by X, is called the canonical identification of X with its double-dual $X^{\vee\vee}$.
- A morphism $f : X \to X^\vee$ is called symmetric if $f = f^\vee \circ \mathrm{can}_X$.
- A polarization of X is a symmetric isogeny $f : X \to X^\vee$ such that $f = \phi_{\mathcal{L}}$ on X for some ample line bundle $\mathcal{L}$.
- A prinicipal polarization of X is a polarization which is an isomorphism, i. e. the isogeny is of degree one.

Remarks.

- $\mathrm{can}_X : X \to X^{\vee\vee}$ is an isomorphism of abelian varieties, see for instance [4, p.132].
- If $\mathcal{L}$ is a line bundle on X, then $\phi_{\mathcal{L}}$ is symmetric. Conversely, any symmetric morphism is of the form $\phi_{\mathcal{L}}$ for some line bundle $\mathcal{L}$ on X [1, Proposition 11.2].
- An ample line bundle defines a principal polarization if and only if $K(\mathcal{L}) = 0$.

4 Further topics

4.1 Elliptic curves

Let E be an elliptic curve over k with origin ∞. Then ∞ is a divisor on E. If $x \in E(k)$, then $t_x^{-1}(\infty) = x$, so $t_x^*\mathcal{O}_E(\infty) \otimes \mathcal{O}_E(-\infty) \cong \mathcal{O}_E(x - \infty)$. But the divisor ∞ is of degree one and hence ample, thus $\mathrm{Pic}^0(E) = \mathrm{im}\,\phi_\infty$ by Proposition 3. This proves:

Lemma 11. $\mathrm{Pic}^0(E) = \{[\mathcal{O}_E(x - \infty)] \mid x \in E(k)\}$.

Remark. Another characterization is the following: Let $D = \sum_i n_i P_i$ be a divisor on E. Then $\mathcal{O}_E(D - (\sum_i n_i)\infty) = \mathcal{O}_E(D - \deg(D)\infty) \in \mathrm{Pic}^0(E)$, so $\mathrm{Pic}^0(E) = \{[\mathcal{O}_E(D)] \mid \deg D = 0\}$, i. e. $\mathrm{Pic}^0(E)$ consists of isomorphism classes of line bundles of degree zero.

Proposition 12. *Let (E, ∞) be an elliptic curve over k. Then $K(\mathcal{O}_E(\infty)) = \{\infty\}$. In particular, $\phi_{\mathcal{O}_E(\infty)} : E \to E^\vee$ is an isomorphism sending $x \in X(k)$ to $\mathcal{O}_E(x - \infty)$.*

Proof. Let $f : Z \to E$ for some scheme Z and put $\mathcal{L} := \mathcal{O}_E(\infty)$. Suppose that $(\mathrm{id} \times f)^*\Lambda(\mathcal{L})$ is the pullback of some line bundle on Z via the projection p_2 onto the second factor. Performing the same computations as in the proof of Proposition 6, $t_f^* p_1^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee$ is the pullback of some line bundle $\mathcal{M}$ on Z via the projection $p_2 : E \times Z \to Z$. This means that

$$\mathcal{O}_{E\times Z}((p_1 + f \circ p_2)^*\infty - p_1^*\infty) \cong p_2^*\mathcal{M}.$$

Pushing the line bundle on the left forward via p_2 gives the trivial line bundle. Thus in view of Lemma 4, $\mathcal{M}$ is trivial. This gives an isomorphism

$$\mathcal{O}_{E\times Z}((p_1 + f\circ p_2)^*\infty) \cong \mathcal{O}_{E\times Z}(p_1^*\infty),$$

yielding two sections s_1, s_2 vanishing on the divisors $(p_1 + fp_2)^*\infty$ and $p_1^*\infty$, respectively. Pushing these sections forward via p_2 gives to sections of $p_{2*}\mathcal{M} \cong \mathcal{O}_Z$ that do not vanish anywhere on Z. Hence s_1 and s_2 differ only by an invertible function on Z. In particular, the vanishing loci coincide. That is, f is constant with value ∞, so f factors through $\{\infty\}$. ☐

Proposition 13. *Let (E, ∞) be an elliptic curve. The Poincaré bundle $\mathcal{P}$ on $E \times E$ is given by $\mathcal{P} = \mathcal{O}_{E\times E}(\Delta - p_1^*\infty - p_2^*\infty)$, where Δ is the diagonal divisor and $p_1, p_2 : E\times E \to E$ are the canonical projections.*

Proof. Omitted, see [5, Section 9.4]. ☐

Remark. This Poincaré bundle differs from $\Lambda(\mathcal{O}_E(\infty))$ by the automorphism $\mathrm{id}\times(-1)_E$.

4.2 Quotients by abelian subvarieties

Proposition 14. *Let X be an abelian variety, $Y \subsetneq X$ an abelian subvariety. Then there is an abelian variety Z and a surjective homomorphism $f : X \to Z$ such that $Y = \ker f$.*

Proof. Let $i : Y \hookrightarrow X$ denote the embedding. This gives a dual morphism $i^\vee : X^\vee \to Y^\vee$. If $\mathcal{L}$ is an ample line bundle on X, then $\phi_{\mathcal{L}|_Y} = i^\vee \circ \phi_{\mathcal{L}} \circ i$ by Lemma 9. Since $\phi_{\mathcal{L}|_Y} : Y \to Y^\vee$ is surjective, this implies that $i^\vee$ is surjective.

Let W be the abelian variety given by the connected component of 0 in $\ker i^\vee \subseteq X^\vee$. Dualizing the embedding $W \hookrightarrow X^\vee$ gives a morphism $X^{\vee\vee} \to W^\vee$, composing this with the canonical identification can_X yields a morphism $g : X \to W^\vee$. Since the composition $W \hookrightarrow X^\vee \xrightarrow{i^\vee} Y^\vee$ is the zero map, $Y \subseteq \ker g$.

Let $X \xrightarrow{f} Z \to W^\vee$ be the Stein factorization of g, where $Z \to W^\vee$ is finite and f has connected fibers. Now Z is an abelian variety, f is a homomorphism and $\ker f$ is an abelian subvariety of X containing Y. But as $\dim Y = \dim X - \dim W = \dim \ker f$, necessarily $Y = \ker f$. ☐

Proposition 15. *Let $f : Y \to X$ be a finite morphism of abelian varieties. Then there is a homomorphism $g : X \to Y$ such that $g \circ f$ is the multiplication-by-n map on Y for some $n \in \mathbb{N}$.*

Proof. If $\mathcal{L}$ is an ample line bundle on X, then $f^*\mathcal{L}$ is an ample line bundle on Y, as f is finite. By Irfan's results, $K(f^*\mathcal{L})$ is finite, so it is annihilated by some positive integer n, i. e. $K(f^*\mathcal{L}) \subseteq Y[n]$. Thus the map $\pi : Y^\vee \to Y, (\phi_{\mathcal{L}}|_Y)(y) \mapsto ny$ is well-defined and by Lemma 9, $n_Y = \pi \circ \phi_{\mathcal{L}|_Y} = \pi \circ f^\vee \circ \phi_{\mathcal{L}} \circ f$. ☐

Corollary 16 (Poincaré's complete reducibility theorem)**.** *Let X be an abelian variety and $Y \subseteq X$ an abelian subvariety. Then there is an abelian subvariety $Z \subseteq X$ such that $Y + Z = X$ and $Y \cap Z$ is finite.*

Proof. According to 15, there is a homomorphism $p : X \to Y$ such that $p|_Y = n_Y$ for some $n \in \mathbb{N}$. Define Z to be the connected component of 0 in $\ker p$. Then $Y \cap Z \subseteq \ker p|_Y$, which is finite as $p|_Y$ is an isogeny. So $Y \times Z \to X, (y, z) \mapsto y + z$ has finite kernel as well and since $\dim Y \times Z \geq \dim X$, it is an isogeny too and hence surjective. ☐

Corollary 17. *Let $f : Y \to X$ be an isogeny of abelian varieties of degree n. Then there is an isogeny $g : X \to Y$ such that $g \circ f = n_Y$ and $f \circ g = n_X$.*

Proof. The kernel $\ker f$ is a finite group scheme of order n, hence we can proceed as in the proof of Proposition 15 to obtain a map $g : X \to Y$ such that $g \circ f = n_Y$. Since n_Y is surjective and $\dim X = \dim Y$, g is an isogeny as well. Moreover, $f \circ g \circ f = f \circ n_Y = n_X \circ f$, therefore $g \circ f = n_X$, as f is flat and surjective and hence an epimorphism of schemes. □

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