Exact functors

Rostislav Devyatov

03.12.2014

1 Reminder: derived functors

Let $\mathscr{A}$ be an abelian category. A category $D(\mathscr{A})$ with a functor $Q_{\mathscr{A}}\colon \operatorname{Com}(\mathscr{A}) \to D(\mathscr{A})$ is called *the derived category of* $\mathscr{A}$ if:

1. If f is a quasiisomorphism between two complexes, then $Q_{\mathscr{A}}(f)$ is an isomorphism.
2. (Universal property) Suppose that D' is another category, and $Q'\colon \operatorname{Com}(\mathscr{A}) \to D'$ is another functor such that for each quasiisomorphism f of complexes, $Q'(f)$ is an isomorphism in D'. Then there exists a unique up to isomorphism functor $F\colon D(\mathscr{A}) \to D'$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Com}(\mathscr{A}) & & \\ {\scriptstyle Q_{\mathscr{A}}}\downarrow & \searrow^{Q'} & \\ D(\mathscr{A}) & \xrightarrow{F} & D' \end{array}$$

Let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor between two abelian categories. A functor $D(F)$ (if it exists) is called the *derived functor of* F if $D(F) \circ Q_{\mathscr{A}} \cong Q_{\mathscr{B}} \circ F$. In fact, a derived functor exists if and only if F is exact. If it does not exist, it may still be possible to construct left and right derived functors.

A functor $RF\colon D(\mathscr{A}) \to D(\mathscr{B})$ is called the *right derived functor of* F if there exists a morphism of functors $\rho_F\colon Q_{\mathscr{B}} \circ F \to RF \circ Q_{\mathscr{A}}$ such that the following universal property is satisfied:

> If $G\colon D(\mathscr{A}) \to D(\mathscr{B})$ is a functor that preserves distinguished triangles, and $\varphi\colon G \circ Q_{\mathscr{A}} \to Q_{\mathscr{B}} \circ F$ is a morphism of functors, then there exists a unique morphism of functors $\varphi'\colon RF \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} Q_{\mathscr{B}} \circ F & \xrightarrow{\rho_F} & RF \circ Q_{\mathscr{A}} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi'} \\ & & G \circ Q_{\mathscr{A}} \end{array}$$

Similarly, a functor $LF\colon D(\mathscr{A}) \to D(\mathscr{B})$ is called the *left derived functor of* F if there exists a morphism of functors $\lambda_F\colon LF \circ Q_{\mathscr{A}} \to Q_{\mathscr{B}} \circ F$ that satisfies the following universal property:

> If a functor $G\colon D(\mathscr{A}) \to D(\mathscr{B})$ preserves distinguished triangles, and $\varphi\colon G \circ Q_{\mathscr{A}} \to Q_{\mathscr{B}} \circ F$ is a morphism of functors, then there exists a unique morphism of functors $\varphi'\colon G \to LF$ such that the following diagram commutes:

$$\begin{array}{ccc} G \circ Q_{\mathscr{A}} & & \\ {\scriptstyle \varphi'}\downarrow & \searrow^{\varphi} & \\ LF \circ Q_{\mathscr{A}} & \xrightarrow[\lambda_F]{} & Q_{\mathscr{B}} \circ F \end{array}$$

One can easily check that a right (resp. left) derived functor is unique if it exists, and that if a functor F has a derived functor $D(F)$, then $D(F)$ is both the left and the right derived functor of F. The same definitions work for $D^+(\mathscr{A})$, for $D^-(\mathscr{A})$, and for $D^b(\mathscr{A})$ instead of $D(\mathscr{A})$.

Theorem 1.1. *If $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor, and $\mathscr{A}$ has enough injectives (resp. each object in $\mathscr{A}$ has a bounded injective resolution), then the right derived functor of F from $D^+(\mathscr{A})$ to $D^+(\mathscr{B})$, (resp. from $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$) exists and can be constructed as follows. Given a complex $A^\bullet \in D^+(\mathscr{A})$ (resp. $A^\bullet \in D^b(\mathscr{A})$), let $A'^\bullet$ be a injective resolution (resp. a bounded injective resolution) of $A^\bullet$. Then the formula $RF(A^\bullet) = F(A'^\bullet)$ defines the right derived functor of F.*

Theorem 1.2. *If $F\colon \mathscr{A} \to \mathscr{B}$ is a right exact functor, and $\mathscr{A}$ has enough projectives (resp. each object in $\mathscr{A}$ has a bounded projective resolution), then the left derived functor of F from $D^-(\mathscr{A})$ to $D^-(\mathscr{B})$, (resp. from $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$) exists and can be constructed as follows. Given a complex $A^\bullet \in D^-(\mathscr{A})$ (resp. $A^\bullet \in D^b(\mathscr{A})$), let $A'^\bullet$ be a projective resolution (resp. a bounded projective resolution) of $A^\bullet$. Then the formula $LF(A^\bullet) = F(A'^\bullet)$ defines the left derived functor of F.*

Not that one could try to use the same construction without the assumption that F is a left exact functor, but the resulting "right derived functor" can turn out to be badly defined, i. e. it may depend on the choice of the resolution $A'^\bullet$.

The following theorem shows connections between left and right derived functors and classical derived functors.

Theorem 1.3. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor such that there exists the right derived functor $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ (resp. the left derived functor $LF\colon D^-(\mathscr{A}) \to D^-(\mathscr{B})$). Let $A \in \mathscr{A}$. Consider the complex $A^\bullet$ defined by $A^0 = A$, $A^i = 0$ for $i \neq 0$. Then the formula $R^iF(A) = H^i(RF(A^\bullet))$ (resp. $L_iF(A) = H^{-i}(LF(A^\bullet))$) define the classical right (resp. left) derived functors of F.*

So, from now on, if the right derived functor $RF\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ exists, and $A^\bullet \in \mathrm{Com}^+(\mathscr{A})$, we will shortly write $R^iF(A^\bullet)$ instead if $H^i(RF(A^\bullet))$, and if $A \in \mathscr{A}$ is a single object, we will write $RF(A)$ and $R^iF(A)$ instead of $RF(A^\bullet)$ and $R^iF(A^\bullet)$, respectively, where $A^\bullet$ is the complex defined by $A^0 = A$, $A^i = 0$ for $i \neq 0$. We will use symmetric notations for left derived functors.

To check the existence of compositions of derived functors and compute them, one can use the following definition and theorems.

Definition 1. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor. A class of objects $\mathscr{K}$ in $\mathscr{A}$ is called *adapted* (resp. a positive, negative, bounded adapted class) if:

1. If $A^\bullet \in \mathrm{Com}(\mathscr{A})$ is an acyclic complex (i. e. $H^i(A^\bullet) = 0$ for all i) and $A^i \in \mathscr{K}$ for all i, then the complex $F(A^\bullet)$ is also acyclic.

2. For each $A^\bullet \in \mathrm{Com}(\mathscr{A})$ (resp. $A^\bullet \in \mathrm{Com}^+(\mathscr{A})$, $A^\bullet \in \mathrm{Com}^-(\mathscr{A})$, $A^\bullet \in \mathrm{Com}^b(\mathscr{A})$) there exists a complex $A'^\bullet \in \mathrm{Com}(\mathscr{A})$ (resp. $A'^\bullet \in \mathrm{Com}^+(\mathscr{A})$, $A'^\bullet \in \mathrm{Com}^-(\mathscr{A})$, $A'^\bullet \in \mathrm{Com}^b(\mathscr{A})$) quasiisomorphic to $A^\bullet$ and such that $A'^i \in \mathscr{K}$ for all i. In this case, $A'^\bullet$ is called a *$\mathscr{K}$-resolution* (resp. a *positive $\mathscr{K}$-resolution, negative $\mathscr{K}$-resolution, bounded $\mathscr{K}$-resolution*) of $A^\bullet$.

If $A \in \mathscr{A}$, we will shortly say "a $\mathscr{K}$-resolution of A" instead of "a $\mathscr{K}$-resolution of the complex $A^\bullet$ that has A at degree 0 and the zero object at all other degrees". The same is for positive, negative, and bounded resolutions.

Theorem 1.4. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor, and suppose that there exists a positive (resp. bounded) adapted class $\mathscr{K}$ for F. Then the right derived functor of F from $D^+(\mathscr{A})$ to $D^+(\mathscr{B})$ (resp. from $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$) exists and $\mathscr{K}$-resolutions can be used instead of injective resolutions to compute it as in Theorem 1.1.*

Theorem 1.5. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be a right exact functor, and suppose that there exists a negative (resp. bounded) adapted class $\mathscr{K}$ for F. Then the left derived functor of F from $D^-(\mathscr{A})$ to $D^-(\mathscr{B})$ (resp. from $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$) exists and $\mathscr{K}$-resolutions can be used instead of injective resolutions to compute it as in Theorem 1.2.*

The following two propositions provide sufficient conditions for existence of derived functors between *bounded* derived categories.

Proposition 1.6. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor, and suppose that there exists a positive adapted class for F. Suppose that there exists $n \in \mathbb{N}$ such that for each $A \in \mathscr{A}$ for classical right derived functors one has $R^iF(A) = 0$ if $i > n$. Then RF actually maps $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$.*

Proposition 1.7. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be a right exact functor, and suppose that there exists a negative adapted class for F. Suppose that there exists $n \in \mathbb{N}$ such that for each $A \in \mathscr{A}$ for classical left derived functors one has $L^iF(A) = 0$ if $i > n$. Then LF actually maps $D^b(\mathscr{A})$ to $D^b(\mathscr{B})$.*

The category of quasicoherent sheaves has enough injectives, but it does not have enough projectives. So, the latter of these two theorems will be helpful to compute the left derived functor of tensor product.

Finally, we are ready to formulate a theorem about the composition of two derived functors.

Theorem 1.8. *Let $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be two left exact functors. Suppose that $\mathscr{A}$ has enough injectives (resp. each bounded complex in $\mathscr{A}$ has a bounded injective resolution). Suppose that there exists a positive adapted class $\mathscr{K}$ for G in $\mathscr{B}$, and F maps each injective object into an object from $\mathscr{K}$, then the right derived functor $R(G \circ F)\colon D^+(\mathscr{A}) \to D^+(\mathscr{C})$ (resp. $R(G \circ F)\colon D^b(\mathscr{A}) \to D^b(\mathscr{C})$) exists, and $R(G \circ F) = RG \circ RF$. Moreover, if $A^\bullet \in Com^+(\mathscr{A})$ (resp. $A^\bullet \in \mathrm{Com}^b(\mathscr{A})$ is a complex, then there exists a spectral sequence with the second sheet*

$$E_2^{p,q} = RG^p(RF^q(A^\bullet))$$

converging to $R^{p+q}(G \circ F)(A^\bullet)$.

A similar theorem holds for the left derived functors of right exact functors.

2 Examples of left and right derived functors for coherent sheaves

We are going to consider quasicoherent sheaves on smooth projective varieties. All varieties are over $\mathbb{C}$.

2.1 Cohomology

Let X be a smooth projective variety. The global sections functor is left exact, and the category of quasicoherent sheaves has enough injectives, so it has a right derived functor and classical right derived functors. The classical right derived functors (which can be applied to a single quasicoherent sheaf $\mathscr{F}$) are called sheaf cohomology and are denoted by $H^i(X, \mathscr{F})$. The right derived functor can be applied to a complex $\mathscr{F}^\bullet$ of quasicoherent sheaves, and the cohomology groups of the resulting complex $R\Gamma(\mathscr{F}^\bullet)$ are called the *hypercohomology* groups of $\mathscr{F}^\bullet$, and are denoted by $\mathbb{H}^i(\mathscr{F}^\bullet)$.

Theorem 2.1. *Let $n = \dim X$. Then, if $\mathscr{F}$ is a quasicoherent sheaf on X, $H^i(X, \mathscr{F}) = 0$ for $i > n$. Moreover, if $\mathscr{F}^\bullet \in D^b(\mathrm{QCoh}(X))$, then $R\Gamma(\mathscr{F}^\bullet) \in D^b(\mathrm{Vec}(\mathbb{C}))$.*

For coherent sheaves there is the following Serre theorem:

Theorem 2.2. *Let $\mathscr{F}$ be a coherent sheaf on X. Then all spaces $H^i(X, \mathscr{F})$ are finite-dimensional.*

Corollary 2.3. *Let $\mathscr{F}^\bullet \in D^b(X)$. Then all spaces $\mathbb{H}^i(\mathscr{F}^\bullet)$ are finite dimensional.*

Proof. Apply Theorem 1.8 to the composition $\Gamma = \Gamma \circ \mathrm{rk}_{\mathrm{Coh}(X)}$. We get a spectral sequence with the second sheet

$$E_2^{p,q} = R\Gamma^p(H^q(\mathscr{F}^\bullet)) = H^p(X, H^q(\mathscr{F}^\bullet)),$$

where $H^p(X, -)$ denotes taking sheaf cohomology, this way we get a vector space out of a single sheaf, and H^q denotes taking the cohomology of a complex, in this particular case we get a sheaf out of a complex of sheaves.

So, we have a spectral sequence with only finitely many (by Theorem 2.1 and since $\mathscr{F}^\bullet \in D^b(X)$) nonzero terms, and these terms are finite-dimensional vector spaces by Serre's theorem. Hence, all spaces $\mathbb{H}^i(\mathscr{F})$ are finite-dimensional. If a bounded complex of (not necessarily finite-dimensional) vector spaces has finite dimensional cohomology groups, it is quasiisomorphic to the complex consisting of these groups with zero differentials. □

2.2 Pushforward

Let X and Y be two smooth projective varieties over $\mathbb{C}$, and let $f\colon X \to Y$ be a morphism. Let $\mathscr{F} \in \mathrm{QCoh}(X)$. Then the sheaf $f_*\mathscr{F}$, which is called the *pushforward* of $\mathscr{F}$ is defined as follows: for each open subset $U \subseteq Y$, $\Gamma(U, f_*\mathscr{F}) = \Gamma(f^{-1}(U), \mathscr{F})$. The functor of taking pushforward is left exact since the global sections functor is left exact. Therefore, it has a right derived functor $Rf_*\colon D^+(\mathrm{QCoh}(X)) \to D^+(\mathrm{QCoh}(Y))$.

The following theorem, which we formulate without a proof, together with Proposition 1.6 gives a right derived functor $Rf_*\colon D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(Y))$.

Theorem 2.4. *Let X and Y be two smooth projective varieties over $\mathbb{C}$, and let $f\colon X \to Y$ be a morphism. Let $\mathscr{F} \in \mathrm{QCoh}(X)$. Then $R^i f_*\mathscr{F} = 0$ if $i > \dim X$.*

The following theorem and proposition, which we also formulate without proofs, imply that if f is a *projective* morphism, then Rf_* maps $D^b(X)$ to $D^b(Y)$. For an algebraic variety X, denote by $D^b_{\mathrm{Coh}}(\mathrm{QCoh}(X))$ the subcategory of $D^b(\mathrm{QCoh}(X))$ generated by complexes of quasicoherent sheaves with coherent cohomology.

Proposition 2.5. *Let X be an algebraic variety. The natural functor $D^b(X) \to D^b(\mathrm{QCoh}(X))$ defines an equivalence between $D^b(X)$ and $D^b_{\mathrm{Coh}}(\mathrm{QCoh}(X))$.*

Theorem 2.6. *Let X and Y be two smooth projective varieties over $\mathbb{C}$, and let $f\colon X \to Y$ be a* projective *morphism. Let $\mathscr{F}$ be a coherent sheaf on X. Then all classical right derived pushforwards $R^i f_*\mathscr{F}$ are coherent sheaves.*

A quasicoherent sheaf $\mathscr{F}$ on X is called flabby if for any open subset $U \subseteq X$, the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F})$ is surjective. Injective sheaves are flabby, and flabby sheaves form an adapted class for the pushforward functor, and the pushforward functor maps flabby sheaves to flabby sheaves.

Applying Theorem 1.8 yields the following *Leray spectral sequences*:

Theorem 2.7. *Let X, Y, and Z be three projective varieties, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms. Let $\mathscr{F}^\bullet \in D^+(\mathrm{QCoh}(X))$. Then there is a spectral sequence with the second page*

$$E_2^{p,q} = R^p g_*(R^q f_*(\mathscr{F}^\bullet))$$

converging to $R^{p+q}(g \circ f)_*(\mathscr{F}^\bullet)$.

Theorem 2.8. *Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Let $\mathscr{F}^\bullet \in D^+(\mathrm{QCoh}(X))$. Then there is a spectral sequence with the second page*

$$E_2^{p,q} = R^p f_*(H^q(\mathscr{F}^\bullet))$$

converging to $R^{p+q} f_*(\mathscr{F}^\bullet)$. Here $H^q(\mathscr{F}^\bullet)$ is the qth cohomology object of the complex $\mathscr{F}^\bullet$, which is a sheaf, it is not a (coherent) cohomology group of a particular sheaf or a hypercohomology group of the complex.

Theorem 2.9. *Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Let $\mathscr{F}^\bullet \in D^+(\mathrm{QCoh}(X))$. Then there is a spectral sequence with the second page*

$$E_2^{p,q} = H^p(Y, R^q f_*\mathscr{F}^\bullet)$$

converging to $\mathbb{H}^{p+q} f_*(\mathscr{F}^\bullet)$.

Note also that the category of quasicoherent (resp. coherent) sheaves on a point is equivalent to the category of vector spaces (resp. finite-dimensional vector spaces). In terms of this equivalence, the functor of taking global sections (on any variety) is equivalent to the pushforward to a point. Therefore, flabby sheaves also form an adapted class for Γ.

2.3 Pullback

Again, let $f\colon X \to Y$ be a morphism of smooth projective varieties. Let $\mathscr{F}$ be a quasicoherent sheaf on Y. We are going to define the pullback of f. First, let us define the *inverse image* of $\mathscr{F}$, which will a priori be only a sheaf of vector spaces, namely, the inverse image of $\mathscr{F}$ is the following sheaf $f^{-1}\mathscr{F}$:

$$\Gamma(U, f^{-1}\mathscr{F}) = \lim_{V \supset f(U) \text{ open in } Y} \mathscr{F}(V)$$

for all open subsets U in X. Then $f^{-1}\mathscr{O}_Y$ is a sheaf of algebras, and $f^{-1}\mathscr{F}$ is a sheaf of $f^{-1}\mathscr{O}_Y$-modules. It is easy to see that $\mathscr{O}_X$ is also a sheaf of $f^{-1}\mathscr{O}_Y$-modules. Now we define the *pullback* of $\mathscr{F}$ as the following sheaf $f^*\mathscr{F}$:

$$\Gamma(U, f^*\mathscr{F}) = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{F}.$$

Clearly, $f^*\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules.

Proposition 2.10. *1. $f^*\mathscr{F}$ is a quasicoherent sheaf, and it is even coherent if $\mathscr{F}$ was coherent.*

2. If $\mathscr{F}$ is a vector bundle, then $f^*\mathscr{F}$ coincides with the pullback of $\mathscr{F}$ as of a vector bundle.

3. The functor f^* is right exact.

Corollary 2.11. *The class of vector bundles (i. e. locally free sheaves) is a negative adapted class for the functor f^* applied to coherent sheaves.*

Proof. This follows directly from the fact that the pullback functor is exact for vector bundles and that every coherent sheaf can be presented as a (sheaf) quotient of a vector bundle. □

In fact, each coherent sheaf has a resolution of finite length consisting of vector bundles. Moreover, each bounded complex of coherent sheaf has a resolution of finite length consisting of vector bundles, and the class of coherent sheaves is also a bounded adapted class for the functor f^*.

Lemma 2.12. *Let X be a smooth projective variety, let $\mathscr{F}$ be a finite-dimensional vector bundle on X, and let $\mathscr{G}$ be a coherent sheaf on X. Let $\{U_i\}$ be a covering of X by n not necessarily affine open sets such that $\mathscr{F}$ is trivial on each of them. Then an element of* $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ *is determined by its values on all sets U_i, i. e. by n morphisms $\Gamma(U_i, \mathscr{F}) \to \Gamma(U_i, \mathscr{G})$ that agree well on the intersections.*

Proof. By definition, an element of $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is determined by morphisms $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$ defined for all open subsets $U \subseteq X$, but it follows from the axioms of sheaves that it is sufficient to define these morphisms for "small enough" sets U, where "small enough" means that a subset of a "small" set is "small" again, and that each point of X is contained in at least one "small" subset. Since $\mathscr{F}$ and $\mathscr{G}$ are coherent sheaves, instead of defining morphisms $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$ on all "small" sets U, we can define them only on sets V_i forming an affine covering of X. Finally, let us choose an affine covering $\{V_i\}$ so that each set V_i is contained in some U_j. Then, since $\mathscr{F}$ is already a trivial vector bundle on U_j, each section of $\mathscr{F}$ on V_i can be written as $\sum f_k s_k$, where f_k are regular functions on V_j, and $s_k \in \Gamma(U_i, \mathscr{F})$. Therefore, it is possible to recover the morphisms $\Gamma(V_j, \mathscr{F}) \to \Gamma(V_j, \mathscr{G})$ out of the morphisms $\Gamma(U_i, \mathscr{F}) \to \Gamma(U_i, \mathscr{G})$, and they will agree well on the intersections if the morphisms $\Gamma(U_i, \mathscr{F}) \to \Gamma(U_i, \mathscr{G})$ agree well on the intersections. □

Lemma 2.13. *Let $f\colon X \to Y$ be a morphism of smooth projective varieties. The functor f^* is left adjoint to the functor f_* for coherent sheaves.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on X, and let $\mathscr{G}$ be a coherent sheaf on Y. We have to prove that $\mathrm{Hom}(f^*\mathscr{G}, \mathscr{F}) = \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})$. First, let us consider the case when $\mathscr{G}$ is a vector bundle. Let U_i be an affine covering of Y such that $\mathscr{G}$ is trivial above each U_i. Then $f^*\mathscr{G}$ is trivial above each of the sets $f^{-1}(U_i)$. Let $m = \mathrm{rk}\,\mathscr{G}$. Choose m basis sections of each of these sets U_i. Denote the transition matrix for the line bundle $\mathscr{G}$ between U_i and U_j by $M_{i,j}$. The entries of this matrix are regular functions on $U_i \cap U_j$. By the previous lemma, an element of $\mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})$ is determined by the following data: for each $U_i \subseteq X$ we have a $\Gamma(U_i, \mathscr{O}_Y)$-linear map from $\Gamma(U_i, \mathscr{G})$ to $\Gamma(U_i, f_*\mathscr{F}) = \Gamma(f^{-1}(U_i), \mathscr{F})$. Such a $\Gamma(U_i, \mathscr{O}_Y)$-linear map is determined by m images of the basis sections of $\mathscr{G}$. Denote these images by

$s_{i,1}, \ldots, s_{i,m} \in \Gamma(U_i, f_*\mathscr{F})$. The fact that these images for U_i and for U_j agree well on $U_i \cap U_j$ means that $(s_{j,1}, \ldots, s_{j,m})M_{i,j} = (s_{i,1}, \ldots, s_{i,m})$ in $\Gamma(U_i, f_*\mathscr{G})^{\oplus m}$.

On the other hand, the pullbacks of the basis sections of $\mathscr{G}$ on U_i form a basis for the sections on $f^{-1}(U_i)$. The transition matrices for these bases are $f^*M_{i,j}$. By the previous lemma again, an element of $\mathrm{Hom}(f^*\mathscr{G}, \mathscr{F})$ is determined by $\Gamma(f^{-1}(U_i), \mathscr{O}_X)$-linear maps from $\Gamma(f^{-1}(U_i), f^*\mathscr{G})$ to $\Gamma(f^{-1}(U_i), \mathscr{F})$ for each U_i, in other words, for each U_i we again have to choose m elements of $\Gamma(f^{-1}(U_i), \mathscr{F}) = \Gamma(U_i, f_*\mathscr{F})$. If we again denote these elements (understood as sections on $f^{-1}(U_i)$ this time) by $s_{i,1}, \ldots, s_{i,m}$, then the equation verifying that they agree with $s_{j,1}, \ldots, s_{j,m}$ on $f^{-1}(U_i) \cap f^{-1}(U_j) = f^{-1}(U_i \cap U_j)$ will look like $(s_{j,1}, \ldots, s_{j,m})f^*M_{i,j} = (s_{i,1}, \ldots, s_{i,m})$. But the definition of $\mathscr{O}_Y$-module structure on $f_*\mathscr{F}$ means that this is the same equation as we had before. So, we have proved that $\mathrm{Hom}(f^*\mathscr{G}, \mathscr{F}) = \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})$ is $\mathscr{G}$ is a vector bundle.

Let A and B be the last two terms in a resolution of $\mathscr{G}$ by vector bundles, i. e. there exists an exact sequence of the form $A \to B \to \mathscr{G} \to 0$ of coherent sheaves on Y. Since the functor $\mathrm{Hom}(-, f_*\mathscr{F})$ is right exact, we also have the following exact sequence: $0 \to \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}(B, f_*\mathscr{F}) \to \mathrm{Hom}(A, f_*\mathscr{F})$. Since the functors f^* and $\mathrm{Hom}(-, \mathscr{F})$ are right exact, the following sequence is also exact: $0 \to \mathrm{Hom}(f^*\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}(f^*B, \mathscr{F}) \to \mathrm{Hom}(A, \mathscr{F})$. But we already have constructed isomorphisms $\mathrm{Hom}(f^*A, \mathscr{F}) = \mathrm{Hom}(A, f_*\mathscr{F})$ and $\mathrm{Hom}(f^*B, \mathscr{F}) = \mathrm{Hom}(B, f_*\mathscr{F})$, and they are functorial, so $\mathrm{Hom}(f^*\mathscr{G}, \mathscr{F}) = \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F})$. □

The functors Rf_* and Lf^* are also adjoint, but we will not prove this fact.

2.4 Tensor product

Let X be a smooth projective variety. The tensor product functor for quasicoherent sheaves on X is right exact with respect to each of the arguments, and for the coherent sheaves on X, vector bundles of finite rank form a (negative and bounded) adapted class for the functor of tensoring by a single object. So, the functor of tensoring coherent sheaves by a single (quasi)coherent sheaf has the left derived functor, which can be computed using resolutions by vector bundles. So, we get functors $\mathrm{Coh}(X) \times D^b(X) \to D^b(X)$, $\mathrm{Coh}(X) \times D^-(X) \to D^-(X)$, $\mathrm{QCoh}(X) \times D^b(X) \to D^b(X)$, and $\mathrm{QCoh}(X) \times D^-(X) \to D^-(X)$. They can be extended to functors $D^b(X) \times D^b(X) \to D^b(X)$ and $D^-(X) \times D^-(X) \to D^-(X)$ as follows. Given two complexes $\mathscr{E}^\bullet, \mathscr{F}^\bullet \in \mathrm{Com}^-(\mathrm{Coh}(X))$, or two complexes $\mathscr{E}^\bullet \in \mathrm{Com}^-(\mathrm{QCoh}(X)), \mathscr{F}^\bullet \in \mathrm{Com}^-(\mathrm{Coh}(X))$ define their *tensor product* by

$$(\mathscr{E}^\bullet \otimes \mathscr{F}^\bullet)^i = \bigoplus_{p+q=i} \mathscr{F}^p \otimes \mathscr{E}^q.$$

The differential in this complex is $d = d_{\mathscr{F}} \otimes 1 + (-1)^i 1 \otimes d_{\mathscr{E}}$. This is again a complex of quasicoherent sheaves bounded above, it is in fact a complex of coherent sheaves if $\mathscr{E}^\bullet, \mathscr{F}^\bullet \in \mathrm{Com}^-(\mathrm{Coh}(X))$, and it is also bounded below if both $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ are bounded below.

To define the derived tensor product of two objects in $D^b(X)$ (or in $D^-(X)$), we replace one of them by a vector bundle resolution, and compute the tensor products of two complexes as explained above. One can check that this operation is well-defined, that it does not depend on the choice of the complex we replace by a resolution and that this functor is exact (i. e. maps distinguished triangles to distinguished triangles) in both arguments.

Using the same definition, we can define tensor product functors $D^b(\mathrm{QCoh}(X)) \times D^b(X) \to D^b(\mathrm{QCoh}(X))$ and $D^-(\mathrm{QCoh}(X)) \times D^-(X) \to D^-(\mathrm{QCoh}(X))$. Here we only replace the complex at the right by a vector bundle resolution. This functor is also exact in both arguments.

2.5 Projection formula

Before we state and prove the projection formula, let us make the following remark about sections of a tensor product by a finite-dimensional vector bundle, similar to Lemma 2.12.

Remark 2.14. *Let X be a smooth projective variety, let $\mathscr{F}$ be a quasicoherent sheaf on X, and let $\mathscr{G}$ be a finite-dimensional vector bundle on X. Let $\{U_i\}$ be a covering of X by n not necessarily affine open sets such that $\mathscr{G}$ is trivial on each of them. Let $V \subseteq X$ be another open set. Then a element of $\Gamma(V, \mathscr{F} \otimes \mathscr{G})$ can be understood as follows: for each U_i, one has to choose a section from $\Gamma(U_i \cap V, \mathscr{F} \otimes \mathscr{G})$, and these* *sections must agree well on $U_i \cap U_j$ for all pairs (i,j). Each element of $\Gamma(U_i \cap V, \mathscr{F} \otimes \mathscr{G})$ can be written as a finite sum of the form $\sum s_k \otimes t_k$, where $s_k \in \Gamma(U_i \cap V, \mathscr{F})$ and $t_k \in \Gamma(U_i \cap V, \mathscr{G})$*

Lemma 2.15. *(Non-derived version of projection formula) Let $f\colon X \to Y$ be a morphism of smooth projective varieties, let $\mathscr{F}$ be a quasicoherent sheaf on X, and let $\mathscr{G}$ be a vector bundle of finite rank on Y. Then $f_*\mathscr{F} \otimes \mathscr{G} = f_*(\mathscr{F} \otimes f^*\mathscr{G})$.*

Proof. The proof is similar to the proof of Lemma 2.13. Namely, let $\{U_i\}$ be an affine open covering of Y such that $\mathscr{G}$ is trivial on each of the sets U_i. Then, by the definition of th pullback of a vector bundle, $f^*\mathscr{G}$ is trivial on each of the sets $f^{-1}(U_i)$, and these sets form a (not necessarily affine) open covering of X. Choose basis sections $e_{i,k}$ of $\mathscr{G}$ on U_i, then they also define basis sections $f^*e_{i,k}$ of $f^*\mathscr{G}$ on $f^{-1}(U_i)$.

Let $V \subseteq Y$ be an open subset. By the axioms of sheaves, an element of $\Gamma(V, f_*\mathscr{F} \otimes \mathscr{G})$ is determined by its restrictions to the sets $V \cap U_i$ that agree on the intersections. Since $\mathscr{G}$ is trivial on each U_i, and each of these restrictions can be written as $\sum s_{i,k} \otimes e_{i,k}$, where $s_{i,k} \in \Gamma(U_i \cap V, f_*\mathscr{F}) = \Gamma(f^{-1}(U_i \cap V), \mathscr{F})$. If we understand $s_{i,k}$ as sections of $\mathscr{F}$ on $f^{-1}(U_i \cap V) \subset X$, then each $\sum s_{i,k} \otimes f^*e_{i,k}$ becomes a section of $\mathscr{F} \otimes f^*\mathscr{G}$ on $f^{-1}(U_i \cap V)$, and these sections agree well on the intersections $V \cap U_i \cap U_j$, one can check this directly using a transition matrix argument similar to the one in the proof of Lemma 2.13.

On the other hand, an element of $\Gamma(V, f_*(\mathscr{F} \otimes f^*\mathscr{G})) = \Gamma(f^{-1}(V), \mathscr{F} \otimes f^*\mathscr{G})$ is also determined by its restrictions to the sets $f^{-1}(V) \cap f^{-1}(U_i) = f^{-1}(V \cap U_i)$, and, since $f^*\mathscr{G}$ is trivial on each $f^{-1}(U_i)$ and $f^*e_{i,k}$ are basis sections of $f^*\mathscr{G}$ on $f^{-1}(U_i)$, these restrictions can be written as $\sum s_{i,k} \otimes f^*e_{i,k}$, and the coincidence on the intersections is verified by the same equations involving transition matrices for $\mathscr{G}$. $\square$

There is also a derived version of projection formula, which we will not prove.

Proposition 2.16. *(Derived projection formula) Let $f\colon X \to Y$ be a* projective *morphism of smooth projective varieties, let $\mathscr{F}^\bullet \in D^b(X)$ and $\mathscr{G}^\bullet \in D^b(Y)$. Then $Rf_*\mathscr{F}^\bullet \otimes_L \mathscr{G}^\bullet = Rf_*(\mathscr{F}^\bullet \otimes_L Lf^*\mathscr{G}^\bullet)$.*

2.6 Hom and Inner (sheafy) Hom

Given a smooth projective variety X and two quasicoherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on it, we can define the sheaf $\mathscr{H}om(\mathscr{F},\mathscr{G})$ on X by $\Gamma(U, \mathscr{H}om(\mathscr{F},\mathscr{G})) = \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathscr{F}|_U, \mathscr{G}|_U)$ (one needs to check that these formulas indeed define a quasicoherent sheaf). This sheaf is coherent if both $\mathscr{F}$ and $\mathscr{G}$ are coherent.

This functor is right exact with respect to the first argument and is left exact with respect to the second argument. In particular, injective sheaves form an adapted class for the second argument. Moreover, vector bundles form a (negative and bounded) adapted class for the first argument. So, we can define the left derived functor with respect to the first argument and the right derived functor for the second argument. Both of them can be generalized to the same bifunctor $R\mathscr{H}om\colon (D^-(\mathrm{QCoh}(X)))^{\mathrm{op}} \times D^+(\mathrm{QCoh}(X)) \to D^+(X)$ (or $R\mathscr{H}om\colon (D^-(X))^{\mathrm{op}} \times D^+(X) \to D^+(X)$, or $R\mathscr{H}om\colon (D^b(X))^{\mathrm{op}} \times D^b(X) \to D^b(X)$) constructed as follows:

If $\mathscr{F}^\bullet \in \mathrm{Com}^-(\mathrm{QCoh}(X))$ and $\mathscr{G}^\bullet \in \mathrm{Com}^+(\mathrm{QCoh}(X))$, we define the complex $\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet)^\bullet$ by

$$\mathscr{H}om(\mathscr{F},\mathscr{G})^i = \prod_p \mathscr{H}om(\mathscr{F}^p, \mathscr{G}^{i+p}),$$

and the differential is defined by $d = d_{\mathscr{G}} - (-1)^i d_{\mathscr{F}}$. Note that $\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet)^\bullet \in \mathrm{Com}^b(\mathrm{QCoh}(X))$ if $\mathscr{F}^\bullet \in \mathrm{Com}^b(\mathrm{QCoh}(X))$ and $\mathscr{G}^\bullet \in \mathrm{Com}^b(\mathrm{QCoh}(X))$, and that the sheaves $\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet)^i$ are coherent if both $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are complexes of coherent sheaves. Now, to compute $R\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet)$, we either replace $\mathscr{G}^\bullet$ by an injective resolution, or, if $\mathscr{F}^\bullet$ is a complex of coherent sheaves, replace $\mathscr{F}^\bullet$ by a vector bundle resolution. One can prove that the result in $D^+(\mathrm{QCoh}(X))$ (or in $D^b(\mathrm{QCoh}(X))$) does not depend on which complex we replaced by a resolution and what this resolution was. It already follows from these claims that if both $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ consist of coherent sheaves, then $R\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet) \in D^+(X)$ (or even $R\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{G}^\bullet) \in D^b(X)$ if both $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are, in addition, bounded).

Now let us consider the usual Hom functor in the category of (quasi)coherent sheaves. It is (contravariant) right exact in the first argument and left exact in the second argument. It cannot be derived in the first argument using the technics explained above, because the category of (quasi)coherent sheaves does not have enough projectives, but in the second argument it can be derived. Again, it turns out (but we will not prove this fact) that this functor can be generalized to the following functor $R\operatorname{Hom}\colon (D^-(X))^{\mathrm{op}} \times D^+(X) \to D^+(\mathrm{Vec}(\mathbb{C}))$: Given $\mathscr{F}^\bullet \in D^-(X)$ and $\mathscr{G}^\bullet \in D^+(X)$, we say that the ith term of $R\operatorname{Hom}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ is

$$R\operatorname{Hom}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)^i = \operatorname*{Hom}_{D(X)}(\mathscr{F}^\bullet, \mathscr{G}^\bullet[i]).$$

The differential in this complex is zero. Clearly, if both complexes are bounded (from both sides), then the result is also bounded from both sides.

The following proposition (which we will not prove) relates $R\operatorname{Hom}$ and $R\mathscr{H}om$.

Proposition 2.17. *If X is a smooth projective variety, we have the following equality of functors $(D^b(X))^{\mathrm{op}} \times D^b(X) \to D^b(\mathrm{Vec}(\mathbb{C}))$: $R\operatorname{Hom} = R\Gamma \circ R\mathscr{H}om$.*

An important particular case of $\mathscr{H}om$ functor (with a single object on the right-hand side) is the *dual* functor, that is, the contravariant functor $\mathscr{F} \mapsto \mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$. It has a left derived functor, which we call the derived dual and denote by $\mathscr{F}^\bullet \mapsto \mathscr{F}^{\bullet L\vee}$ (where $\mathscr{F}^\bullet \in D^-(\mathrm{QCoh}(X))$ Note that we do need to replace $\mathscr{F}^\bullet$ by a vector bundle resolution despite $\mathscr{O}_X$ is already a vector bundle, since $\mathscr{H}om$ is right exact with respect to the second argument, not left exact, and we need an injective resolution for it. An alternative way to compute $\mathscr{F}^{\bullet L\vee}$ is, indeed, to replace $\mathscr{O}_X$ by an injective resolution, but this is usually much harder.

For the bounded derived category of coherent sheaves, the functors ${}^{L\vee}$, $R\mathscr{H}om$, and $\otimes_L$ behave very similarly to finite dimensional vector spaces, as expressed in the following proposition.

Proposition 2.18. *Let X be a smooth projective variety, and let $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b(X)$.*

1. *$R\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet) = \mathscr{F}^{\bullet L\vee} \otimes_L \mathscr{G}^\bullet$.*
2. *$\mathscr{F}^{\bullet L\vee L\vee} = \mathscr{F}^\bullet$.*
3. *$(\mathscr{F}^\bullet \otimes_L \mathscr{G}^\bullet)^{L\vee} = \mathscr{F}^{\bullet L\vee} \otimes_L \mathscr{G}^{\bullet L\vee}$.*

Proof. To compute the derived $\mathscr{H}om$ and the derived tensor product, we should replace $\mathscr{F}^\bullet$ by a vector bundle resolution, but we can also replace $\mathscr{G}^\bullet$ by any quasiisomorphic complex, so let us also replace it by a vector bundle resolution. Now the claim follows from the same facts for vector bundles, which can be verified locally. □

We will need one more lemma relating the left derived pullback with the left derived tensor product and the left derived dual.

Lemma 2.19. *Let $f\colon X \to Y$ be a morphism of smooth projective varieties, and let $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b(Y)$. Then $Lf^*(\mathscr{F}^\bullet \otimes_L \mathscr{G}^\bullet) = (Lf^*\mathscr{F}^\bullet) \otimes (Lf^*\mathscr{G}^\bullet)$ and $(Lf^*\mathscr{F}^\bullet)^{L\vee} = Lf^*(\mathscr{F}^{\bullet L\vee})$.*

Proof. Again, to compute each side of each equality, we can replace both $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ by vector bundle resolutions, and for vector bundles the equalities is clear. □

3 Grothendieck-Verdier Duality

Now we will use the left and right derived functors we have constructed to formulate a theorem about Grothendieck-Verdier duality. If $\mathscr{F}$ is a (quasi)coherent sheaf and $n \in \mathbb{Z}$, we denote by $\mathscr{F}[n]$ the the complex that has $\mathscr{F}$ in degree n and all other terms are zero.

Theorem 3.1. *Let $f\colon X \to Y$ be a projective morphism of smooth projective varieties. Set $\dim f = \dim X - \dim Y$. Set $\omega_f = \omega_X \otimes f^*\omega_Y^*$. Let $\mathscr{F}^\bullet \in D^b(X)$ and $\mathscr{G}^\bullet \in D^b(Y)$. There exists a functorial isomorphism*

$$Rf_* R\mathscr{H}om(\mathscr{F}^\bullet, (Lf^*\mathscr{G}^\bullet) \otimes_L \omega_f[\dim f]) = R\mathscr{H}om(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet).$$

Note that we use the notation $\otimes_L$ for the (left) derived tensor product since we multiply objects in the derived category, but in fact, since ω_f is a line bundle and the right-hand side complex has only one nonzero term, we can compute this left derived tensor product by just tensoring each term on the left-hand side by ω_f and shifting the result.

Consider the following functor $f^!\colon D^b(Y) \to D^b(X)$: if $\mathscr{G}^\bullet \in D^b(Y)$, then $f^!\mathscr{G}^\bullet = (Lf^*\mathscr{G}^\bullet) \otimes_L \omega_f[\dim f]$. Then the isomorphism from Theorem 3.1 can be rewritten as follows: $Rf_*R\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}^\bullet) = R\mathscr{H}om(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet)$.

Corollary 3.2. *The functor $f^!$ is right adjoint to the functor Rf_*.*

Proof. Apply the functor $R\Gamma$ to both parts of Grothendieck-Verdier duality. It follows directly from the definition of f_* that $\Gamma(Y,-) \circ f_* = \Gamma(X,-)$. The functor f_* maps injective sheaves to flabby sheaves, and flabby sheaves form a positive adapted class for Γ. So, by Theorem 1.8, $R\Gamma(Y,-) \circ Rf_* = R\Gamma(X,-)$ (recall that flabby sheaves are adapted for Γ), and

$$R\Gamma(Y, R\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}^\bullet)) = R\Gamma(Y, R\mathscr{H}om(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet))$$

. By Proposition 2.17, we get

$$R\operatorname{Hom}(\mathscr{F}^\bullet, f^!\mathscr{G}^\bullet) = R\operatorname{Hom}(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet)$$

(note that we can apply Proposition 2.17, since $Rf_*\mathscr{F}^\bullet$ is (quasiisomorphic to) an object in $D^b(Y)$ for $\mathscr{F} \in D^b(X)$ and projective f despite we need to replace $F^\bullet$ by a (possibly unbounded and non-coherent) flabby resolution to compute $Rf_*\mathscr{F}^\bullet$).

Finally, it follows directly from the definition of $R\operatorname{Hom}$ that the zeroth cohomology in the $R\operatorname{Hom}$ complex is exactly $\operatorname{Hom}_{D^b(Y)}$. □

Let us get one more corollary of Grothendieck-Verdier duality. Call the following functor the *dualizing functor.* For $\mathscr{F}^\bullet \in D^b(X)$, set

$$\mathbb{D}_X(\mathscr{F}^\bullet) = R\mathscr{H}om(\mathscr{F}^\bullet, \omega_X[\dim X]) = \mathscr{F}^{\bullet L\vee} \otimes_L \omega_X[\dim X]$$

It has an inverse functor, which in fact coincides with $\mathbb{D}_X$:

$$\mathbb{D}_X(\mathbb{D}_X(\mathscr{F}^\bullet)) = (\mathscr{F}^{\bullet L\vee} \otimes_L \omega_X[\dim X])^{L\vee} \otimes_L \omega_X[\dim X] = \mathscr{F}^\bullet \otimes_L \omega_X^\vee[-\dim X] \otimes_L \omega_X[\dim X] = \mathscr{F}^\bullet.$$

Then the functor $f^!$ can be written as follows:

$$\mathbb{D}_X(Lf^*(\mathbb{D}_Y(\mathscr{G}^\bullet))) = (Lf^*(\mathscr{G}^{\bullet L\vee} \otimes_L \omega_Y[\dim Y]))^{L\vee} \otimes_L \omega_X[\dim X] =$$

$$Lf^*\mathscr{G}^\bullet \otimes_L \omega_Y^\vee[-\dim Y] \otimes_L \omega_X[\dim X] = Lf^*\mathscr{G}^\bullet \otimes_L \omega_f[\dim f] = f^!\mathscr{G}^\bullet.$$

If $\mathscr{F}^\bullet \in D^b(\operatorname{QCoh}(X))$, then we didn't prove Proposition 2.18 in this case, but we can set $\mathbb{D}_X(\mathscr{F}^\bullet) = R\mathscr{H}om(\mathscr{F}^\bullet, \omega_X[\dim X])$ without saying anything about tensor products.

Corollary 3.3. *(of Grothendieck-Verdier duality)*

$$Rf_* \circ \mathbb{D}_X = \mathbb{D}_Y \circ Rf_*.$$

Proof. Let $\mathscr{F}^\bullet \in D^b(X)$. Then

$$Rf_*(\mathbb{D}_X(\mathscr{F}^\bullet)) = Rf_*R\mathscr{H}om(\mathscr{F}^\bullet, \omega_X[\dim X]) = Rf_*R\mathscr{H}om(\mathscr{F}^\bullet, \omega_f[\dim f] \otimes_L Lf*(\omega_Y[\dim Y])).$$

On the other hand, $\mathbb{D}_Y(Rf_*\mathscr{F}^\bullet) = R\mathscr{H}om(Rf_*\mathscr{F}^\bullet, \omega_Y[\dim Y])$, and we see that the claim is exactly the Grothendieck-Verdier duality for $\mathscr{G}^\bullet = \omega_Y[\dim Y]$. □