Rigid analytic spaces

Gregor Bruns

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Most of the material in this talk is stolen from [Bos14]. I claim no originality.

1 Overview

Throughout these notes let K be a complete non-Archimedean field. Here we will develop the notion of a classical rigid space X over K. It will consist of a set of points with values in an algebraic closure of K, together with a notion of analytic functions on subsets. These functions arise by piecing together elements of affinoid K-algebras.

The space X carries a canonical topology induced by the topology of K. As a consequence of Tate's Acyclicity Theorem 3.2 however, we cannot admit all open subsets of X to construct a structure sheaf of analytic functions. Instead we have to restrict ourselves to **admissible open subsets** and their **admissible open coverings**. Sheaves on X will be considered relative to this additional structure of a Grothendieck topology.

This entails, for instance, that there are non-zero abelian sheaves on a rigid space with all stalks being zero. Hence there seem to be not enough points in rigid analytic spaces. The idea of **Berkovich spaces** is to consider additional points with values in fields L (not necessarily finite over K) together with a choice of a non-Archimedean $\mathbb{R}$-valued absolute value extending the one on K. Although the construction in general is less natural than one would like, one obtains a Hausdorff topology and in general Berkovich spaces have many nice properties.

The notion of a classical rigid space can also be extended by considering rigid geometry in terms of formal schemes.

2 Affinoid spaces and affinoid subdomains

2.1 Affinoid spaces and the Zariski topology

Let A be an affinoid K-algebra. Every $f \in A$ can be viewed as a function on the set of maximal ideals of A in the following way. If x is a maximal ideal of A and A/x is the residue field then $f(x)$ is defined to be the residue class of f in A/x. If we embed K into an algebraic closure $\overline{K}$, then $f(x)$ is not defined uniquely (only up to conjugation over K), but the absolute value $|f(x)|$ is. We set $|f|_{\sup} = \sup_{x \in \operatorname{Max} A} |f(x)|$ where $\operatorname{Max} A$ is the set of maximal ideals of A.

Definition 2.1. Let A be an affinoid K-algebra. The **affinoid K-space** $\operatorname{Sp}(A)$ associated to A is the set of maximal ideals of A (aka the maximal spectrum of A) together with the K-algebra A of functions on it.

For now we have to restrict ourselves to maximal ideals since taking the whole spectrum of A leads to a lot of unwanted behaviour. For instance, there may be open subsets $U \subseteq X$ and closed subsets $Y \subseteq U$ such that after taking the closure $\overline{Y}$ in X we get $Y \subsetneq \overline{Y} \cap U$.

Definition 2.2. A **morphism** $\sigma\colon \operatorname{Sp}(A) \to \operatorname{Sp}(B)$ **of affinoid K-spaces** is induced by a morphism $\sigma^*\colon B \to A$ of affinoid K-algebras by setting

$$\sigma(\mathfrak{m}) = (\sigma^*)^{-1}(\mathfrak{m})$$

Note that $\sigma(\mathfrak{m})$ is maximal since we have $K \hookrightarrow B/\sigma^{*-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}$.

Now we can define the **Zariski topology** on $\operatorname{Sp}(A)$ as one would expect. The closed sets are given by

$$V(\mathfrak{a}) = \{x \in \operatorname{Sp}(A) \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\} = \{x \in \operatorname{Sp}(A) \mid \mathfrak{a} \subseteq x\}$$

where $\mathfrak{a} \subseteq A$ is any ideal. We then get the usual basis D_f of Zariski open subsets and we get an analogue of Hilbert's Nullstellensatz. We also can make sense of fiber products of affinoid K-spaces using completed tensor products.

2.2 The canonical topology and affinoid subdomains

Let $X = \operatorname{Sp}(A)$ be an affinoid K-space. Since the Zariski topology is very coarse, we are going to introduce another topology that is induced from the topology on K (which we already studied a bit). In particular it will share the strange feature of being totally disconnected. We consider sets of the form

$$X(f, \varepsilon) = \{x \in X : |f(x)| \leqslant \varepsilon\}$$

where $f \in A$ and $\varepsilon \in \mathbb{R}^+$. They will be our opens.

Definition 2.3. The **canonical topology** of X is the one generated by all $X(f, \varepsilon)$.

We write $X(f) = X(f, 1)$ and

$$X(f_1, \dots, f_r) = X(f_1) \cap \cdots X(f_r) = \{x \in X : |f_i(x)| \leqslant 1 \text{ for all } i = 1, \dots, r\}$$

for $f_1, \dots, f_r \in A$. These sets are called **Weierstraß domains** in X.

Lemma 2.4. The canonical topology on X is generated by the subsets $X(f)$ where $f \in A$. In particular, a subset $U \subseteq X$ is open if and only if it is a union of Weierstraß domains.

As an exercise in how to work with these things, here is a useful lemma that we can employ to prove the openness of various other sets.

Lemma 2.5. Let $f \in A$ and $x \in \mathrm{Sp}(A)$ such that $\varepsilon = |f(x)| > 0$. Then there is a $g \in A$ with $g(x) = 0$ such that $|f(y)| = \varepsilon$ for all $y \in X(g)$. In particular, $X(g)$ is an open neighborhood of x contained in $\{y \in X : |f(y)| = \varepsilon\}$.

Proof. Write $f(x)$ for the residue class of f in A/x and let $P(\zeta) \in K[\zeta]$ be the minimal polynomial of $f(x)$ over K. Factor $P(\zeta) = \prod_{i=1}^{n}(\zeta - \alpha_i)$ in $\overline{K}$ and choose an embedding $A/x \hookrightarrow \overline{K}$. Then we have

$$\varepsilon = |f(x)| = |\alpha_i|$$

for all i since all roots of the minimal polynomial are conjugate to each other, and the valuation is unique and does not depend on the embedding of A/x into $\overline{K}$.

Now consider $h = P(f) \in A$. Then $h(x) = P(f(x)) = 0$. Let $y \in X$ and assume $|f(y)| \neq \varepsilon$. Choose an embedding $A/y \hookrightarrow \overline{K}$. Then

$$|f(y) - \alpha_i| = \max(|f(y)|, |\alpha_i|) \geqslant |\alpha_i| = \varepsilon$$

for all i (we have equality in the non-Archimedean triangle inequality if both arguments do not have the same value), and therefore

$$|h(y)| = |P(f(y))| = \prod_{i=1}^{n} |f(y) - \alpha_i| \geqslant \varepsilon^n$$

Hence if $|h(y)| < \varepsilon^n$ then $|f(y)| = \varepsilon$. Taking any $c \in K^*$ with $|c| < \varepsilon^n$ we set $g = c^{-1}h$ and obtain that $|f(y)| = \varepsilon$ for any $y \in X(g)$. ∎

Corollary 2.6. For $f \in A$ and $\varepsilon \in \mathbb{R}^+$ the following sets are open in the canonical topology:

$$\{x \in \mathrm{Sp}(A) : f(x) \neq 0\}$$
$$\{x \in \mathrm{Sp}(A) : |f(x)| \leqslant \varepsilon\}$$
$$\{x \in \mathrm{Sp}(A) : |f(x)| = \varepsilon\}$$
$$\{x \in \mathrm{Sp}(A) : |f(x)| \geqslant \varepsilon\}$$

Proposition 2.7. *Let* $\varphi\colon \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ *be a morphism of affinoid* K*-spaces and let* $f_1, \ldots, f_r \in A$. *Then we have*

$$\varphi^{-1}(\mathrm{Sp}(A)(f_1, \ldots, f_r)) = \mathrm{Sp}(B)(\varphi^*(f_1), \ldots, \varphi^*(f_r))$$

In particular, φ *is continuous with respect to the canonical topology.*

Apart from Weierstraß domains there are some more general distinguished subsets of affinoid K-spaces. We call subsets of the form

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \left\{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\right\}$$

Laurent domains. Subsets of type

$$X\left(\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\right) = \left\{x \in X : |f_i(x)| \leqslant |f_0(x)|\right\}$$

for $f_0, \dots, f_r \in A$ without common zeros are called **rational domains**. The openness of these domains follows from our Lemma 2.5.

Weierstraß, Laurent and rational domains are examples of affinoid subdomains of X. These form the class of subsets that will be ultimately relevant to us.

Definition 2.8. A subset $U \subseteq X$ is called an **affinoid subdomain** of X if there exists a morphism of affinoid K-spaces $\iota \colon X' \to X$ such that $\iota(X') \subseteq U$ and the following universal property holds: Any morphism of affinoid K-spaces $\varphi \colon Y \to X$ satisfying $\varphi(Y) \subseteq U$ admits a unique factorization through $\iota \colon X' \to X$ via a morphism of affinoid K-spaces $\varphi' \colon Y \to X'$.

This definition might seem strange at first since there is no mention of U being open. As a first indication that it is not so bad, we have:

Example 2.9. Points $\{x\} \subseteq X$ are not affinoid subdomains. The problem here is that the universal property does not hold. For simplicity assume $x \in X$ corresponds to the maximal ideal $\mathfrak{m}_x \subseteq A$ with residue field K and $\{x\}$ as an affinoid K-space comes from the algebra $A/\mathfrak{m}_x$. Then the inclusion $\{x\} \hookrightarrow X$ corresponds to the morphism $A \to A/\mathfrak{m}_x = K$. But we can also construct the map $\varphi \colon A \to A/\mathfrak{m}_x^2$. On the level of points this corresponds to a map with image $\{x\}$. But there is no way φ factors over $A \to K$.

If we want to be more rigorous, we have to exclude all possible affinoid K-space structures on $\{x\}$. We then need to show that the only ring which could have the universal property is the localization $A_{\mathfrak{m}_x}$, but this is not finitely generated and hence not K-affinoid.

By applying the universal property for the inclusions of each point $x \in U$ in U, we can actually show that ι is injective and $\iota(X') = U$. Hence we can identify the set U with the set of X' and get an induced structure of K-affinoid space on U. We will later see that any affinoid subdomain of X is indeed open with respect to the canonical topology.

Proposition 2.10. *Weierstraß, Laurent and rational domains in an affinoid K-space X are open affinoid subdomains.*

Proof. We only prove this for a Weierstraß domain $X(f) \subseteq X$, where f stands for a tuple $f_1, \dots, f_r \in A$. Consider the affinoid K-algebra $A\langle \zeta_1, \dots, \zeta_r \rangle$ of restricted power series and consider

$$A\langle f \rangle = A\langle f_1, \dots, f_r \rangle = A\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_i - f_i)$$

as an affinoid K-algebra. We have a canonical morphism $\iota^* \colon A \to A\langle f \rangle$ of affinoid K-algebras and an associated morphism $\iota \colon \mathrm{Sp}(A\langle f \rangle) \to X$ of affinoid K-spaces. We will show that $\mathrm{im}(\iota) \subseteq X(f)$ and ι satisfies the universal property.

Consider a morphism $\varphi \colon Y \to X$ of affinoid K-spaces with associated morphism $\varphi^* \colon A \to B$ of affinoid K-algebras. For any $y \in Y$ we get

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))|$$

since we have an inclusion $A/\varphi(y) \hookrightarrow B/y$ of finite extensions of K, induced by φ^*. Then $\varphi(Y) \subseteq X(f)$ if $|\varphi^*(f_i)|_{\sup} \leqslant 1$ for all $i = 1, \dots, r$. Now consider $\varphi = \iota$. Here $\iota^*(f_i)$ is the residue class of ζ_i in $A\langle f\rangle$, hence its Gauß norm is 1. Since the supremum norm of $\iota^*(f_i)$ is bounded by the residue norm, which in turn is bounded by the Gauß norm of $\iota^*(f_i)$, we have shown $\iota(X') \subseteq X(f)$.

We still have to show the universal property for ι^*, i.e. each morphism $\varphi^*\colon A \to B$ of affinoid K-algebras satisfying $|\varphi^*(f_i)|_{\sup} \leqslant 1$ for all $i = 1, \dots, r$ admits a unique factorization through ι^*. We can extend φ^* to a morphism $A\langle\zeta\rangle \to B$ by mapping $\zeta_i \mapsto \varphi^*(f_i)$. Then all $\zeta_i - f_i$ belong to the kernel and we get an induced morphism $A\langle f\rangle \to B$ which is seen to be the required factorization. Uniqueness follows from the fact that the image of A is dense in $A\langle f\rangle$. ∎

We call these open affinoid subdomains **special**. Here are some results about properties of (special) affinoid subdomains that one would expect or hope to have.

Proposition 2.11. *Let $\varphi\colon Y \to X$ be a morphism of affinoid K-spaces and let $X' \hookrightarrow X$ be an affinoid subdomain. Then $Y' = \varphi^{-1}(X')$ is an affinoid subdomain of Y and we have $Y' = Y \times_X X'$.*

If X' is a Weierstraß, Laurent or rational subdomain of X then the corresponding statement is true for Y' as an affinoid subdomain of Y. The defining functions of X' pull back to the defining functions of Y'.

Idea of proof. The first assertion follows from the universal property of affinoid subdomains, together with the fact that fibered products exist for affinoid K-spaces. The second part is mainly an application of Proposition 2.7. ∎

Proposition 2.12. *Let $U, V \subseteq X$ be affinoid subdomains of X. Then $U \cap V$ is an affinoid subdomain of X as well. Furthermore, if U and V are Weierstraß (resp. Laurent, resp. rational) subdomains then the same is true for $U \cap V$.*

Proposition 2.13. *Let $U \to X$ be a morphism of affinoid K-spaces defining U as an affinoid subdomain of X. Then U is open in X and the canonical topology of X restricts to the one of U.*

Theorem 2.14 (Gerritzen–Grauert)**.** *Let X be an affinoid K-space and $U \subseteq X$ an affinoid subdomain. Then U is a finite union of rational subdomains of X.*

3 Tate's Acyclity Theorem

We let X be an affinoid K-space and $\mathfrak{T}$ be the category of affinoid subdomains in X, where the morphisms are the inclusions. Then it makes sense to consider (pre)sheaves on $\mathfrak{T}$ and we can in particular ask whether the presheaf $\mathcal{O}_X$ of affinoid functions on X is a sheaf.

The first sheaf condition, that locally zero functions are globally zero, is satisfied for $\mathcal{O}_X$. On the other hand, because the canonical topology on X is totally disconnected, we usually cannot glue local sections together. However we will see that the gluing condition is satisfied for *finite* coverings.

Let $\mathcal{F}$ be a presheaf on X and $\mathfrak{U} = (U_i)_{i \in I}$ be a covering of X by affinoid subdomains U_i. We say that $\mathcal{F}$ is a **$\mathfrak{U}$-sheaf** if for all affinoid subdomains $U \subseteq X$ the sequence

$$\mathcal{O}_X(U) \to \prod_{i \in I} \mathcal{O}_X(U_i \cap U) \rightrightarrows \prod_{i,j \in I} \mathcal{O}_X(U_i \cap U_j \cap U)$$

is exact.

Theorem 3.1 (Tate)**.** *Let X be an affinoid K-space. The presheaf $\mathcal{O}_X$ of affinoid functions is a $\mathfrak{U}$-sheaf on X for all finite coverings $\mathfrak{U} = (U_i)_{i \in I}$ of X by affinoid subdomains $U_i \subseteq X$.*

Idea of proof. One first reduces to Laurent domains, then to Laurent domains generated by one function, and proves it for these domains by direct computation. ■

We say that a covering $\mathfrak{U}$ is **$\mathcal{F}$-acyclic** if $\mathcal{F}$ satisfies the sheaf properties for $\mathfrak{U}$ and the Čech cohomology groups $H^q(\mathfrak{U}, \mathcal{F})$ vanish for $q > 0$.

Theorem 3.2 (Tate's Acyclity Theorem)**.** *Let X be an affinoid K-space and $\mathfrak{U}$ be a finite covering of X by affinoid subdomains. Then $\mathfrak{U}$ is acyclic with respect to the presheaf $\mathcal{O}_X$.*

4 Grothendieck topologies on affinoid spaces

We have seen that $\mathcal{O}_X$ on an affinoid K-space carrying the canonical topology will usually not be a sheaf. Instead of further improving our spaces, we will rethink our definition of sheaf, or more precisely our notion of open cover. As a rough idea, we are going to adapt the notion of cover so as to only include finite unions of affinoid subdomains. We have already seen in the last section that $\mathcal{O}_X$ is a sheaf with respect to these sort of covers. In this way we arrive at a more restrictive notion of analytic function than our naive idea in the beginning: they can only be pieced together from finitely many locally analytic functions on affinoid subdomains.

To formalize our ideas, we make a series of definitions.

Definition 4.1 (Grothendieck topology)**.** A **Grothendieck topology** $\mathfrak{T}$ is a category $\operatorname{Cat}\mathfrak{T}$ together with a set $\operatorname{Cov}\mathfrak{T}$ of families $(U_i \to U)_{i \in I}$ of morphisms in $\operatorname{Cat}\mathfrak{T}$, which we call the **coverings**, such that:

1. If $\phi\colon U \to V$ is an isomorphism in $\operatorname{Cat}\mathfrak{T}$ then $(\phi) \in \operatorname{Cov}\mathfrak{T}$.
2. If $(U_i \to U)_{i \in I}$ and $(V_{ij} \to U_i)_{j \in J_i}$ are coverings, then $(V_{ij} \to U_i \to U)_{i \in I, j \in J_i}$ is a covering.
3. If $(U_i \to U)_{i \in I}$ is a covering and $V \to U$ a morphism in $\operatorname{Cat}\mathfrak{T}$ then the fiber products $U_i \times_U V$ exist in $\operatorname{Cat}\mathfrak{T}$ and $(U_i \times_U V \to V)_{i \in I}$ is a covering.

Grothendieck topologies are not topologies in the usual sense. Instead of capturing the properties of open sets, they only know about what it means to be an (open) cover.

Example 4.2. Let X be a topological space. Define a Grothendieck topology $\mathfrak{T}$ by letting $\operatorname{Cat}\mathfrak{T}$ be the usual category $\operatorname{Top}(X)$, i.e. the objects are the open sets and all the morphisms are given by inclusions of open sets. Furthermore, let $\operatorname{Cov}\mathfrak{T}$ be given by the usual open covers, i.e. a family $(U_i)_{i\in I}$ of open subsets of U is an open cover of U if the map $\bigcup_{i\in I} U_i \to U$ induced by the inclusions is surjective.

Observe that for $U_1, U_2 \subseteq U$ open sets we have $U_1 \times_U U_2 = U_1 \cap U_2$. Then the second condition in the definition of Grothendieck topology means that covers can be refined and the third condition means that given an inclusion of open sets $V \hookrightarrow U$ and $(U_i)_{i\in I}$ a covering of U, the family $(U_i \cap V)_{i\in I}$ is a covering of V.

It turns out that Grothendieck topologies provide exactly the framework we need to define (pre)sheaves.

Definition 4.3. Let $\mathfrak{T}$ be a Grothendieck topology and $\mathfrak{C}$ a category where products exist. A **presheaf** on $\mathfrak{T}$ with values in $\mathfrak{C}$ is a contravariant functor $\mathcal{F}\colon \operatorname{Cat}\mathfrak{T} \to \mathfrak{C}$. We say that $\mathcal{F}$ is a **sheaf** if

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for all coverings $(U_i \to U)_{i\in I}$ in $\operatorname{Cov}\mathfrak{T}$.

Remark 4.4. To every presheaf we can associate an associated sheaf satisfying a universal property. This process is called **sheafification**. Unlike for topological spaces, we cannot define the sheafification by considering functions on the disjoint union $\coprod_{x\in X} \mathcal{F}_x$ of stalks, since there might be sheaves w.r.t the Grothendieck topology that are nonzero yet have zero stalks at all points of X. Instead, the construction relies on iterating the construction of a sheaf version of a Čech complex.

If X is a set and $\mathfrak{T}$ a Grothendieck topology on it, then we call X a **G-topological space**. The case that is of interest to us is of course when X is an affinoid K-space and $\mathfrak{T}$ is some Grothendieck topology such that the open covers are defined using affinoid subdomains and $\mathcal{O}_X$ is a sheaf with respect to $\mathfrak{T}$. There are two main variants of this:

Definition 4.5 (Weak Grothendieck topology)**.** Let X be an affinoid K-space. Let $\operatorname{Cat}\mathfrak{T}$ be the category of affinoid subdomains of X with the inclusions as morphisms. The set $\operatorname{Cov}\mathfrak{T}$ consists of all finite families $(U_i \to U)_{i\in I}$ of inclusions of affinoid subdomains of X such that $U = \bigcup_{i\in I} U_i$. We call $\mathfrak{T}$ the **weak Grothendieck topology on** X.

Tate's Acyclicity Theorem 3.2 states that $\mathcal{O}_X$ is a sheaf for the weak Grothendieck topology. To capture even more structure of X, we can admit some more open sets and coverings:

Definition 4.6 (Strong Grothendieck topology)**.** Let X be an affinoid K-space. The **strong Grothendieck topology on** X is given by the following data.

1. A subset $U \subseteq X$ is called admissible open if there is a covering $U = \bigcup_{i\in I} U_i$ of U by affinoid subdomains $U_i \subseteq X$ such that for all morphisms of affinoid K-spaces $\varphi\colon Z \to X$ satisfying $\varphi(Z) \subseteq U$ the covering $(\varphi^{-1}(U_i))_{i\in I}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

2. A covering $V = \bigcup_{j \in J} V_j$ of some admissible open subset $V \subseteq X$ by means of admissible open sets V_j is called admissible if for each morphism of affinoid K-spaces $\varphi\colon Z \to X$ satisfying $\varphi(Z) \subseteq V$ the covering $(\varphi^{-1}(V_j))_{j \in J}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

It is necessary to show that the strong Grothendieck topology deserves its name. Fortunately, this is not so hard.

Proposition 4.7. *Let X be an affinoid K-space. The strong Grothendieck topology is indeed a Grothendieck topology on X, satisfying the following conditions:*

(G_0) *$\emptyset$ and X are admissible open.*

(G_1) *Let $(U_i)_{i \in I}$ be an admissible covering of an admissible open subset $U \subseteq X$. Furthermore, let $V \subseteq U$ be a subset such that $V \cap U_i$ is admissible open for all $i \in I$. Then V is admissible open in X.*

(G_2) *Let $(U_i)_{i \in I}$ be a covering of an admissible open set $U \subseteq X$ by admissible open subsets $U_i \subseteq X$ such that $(U_i)_{i \in I}$ admits an admissible covering of U as refinement. Then $(U_i)_{i \in I}$ itself is admissible.*

We say that a morphism $\varphi\colon Z \to X$ of affinoid K-spaces equipped with Grothendieck topologies is **continuous** if the inverse image $\varphi^{-1}(U)$ of any admissible open subset $U \subseteq X$ is admissible open in Z, and the inverse image of any admissible covering in X is an admissible covering in Z. Now Proposition 2.11 shows that all morphisms between affinoid K-spaces are continuous with respect to the weak Grothendieck topologies. As one would hope, this extends to the strong topologies as well.

Proposition 4.8. *Let $Y \to X$ be a morphism of affinoid K-spaces. Then φ is continuous with respect to the strong Grothendieck topologies on X and Y.*

In order for our definitions to be reasonable, the strong Grothendieck topology should also at least improve on the Zariski topology. This is indeed the case:

Lemma 4.9. Let X be an affinoid K-space. Then the strong Grothendieck topology on X is finer than the Zariski topology, i.e. every Zariski open subset $U \subseteq X$ is admissible open and every Zariski covering is admissible.

Let $\mathfrak{T}$ and $\mathfrak{T}'$ be two Grothendieck topologies on X such that

- $\mathfrak{T}'$ is a refinement of $\mathfrak{T}$,
- each $\mathfrak{T}'$-open $U \subseteq X$ admits a $\mathfrak{T}'$-covering $(U_i)_{i \in I}$ where all U_i are $\mathfrak{T}$-open in X,
- each $\mathfrak{T}'$-covering of a $\mathfrak{T}$-open subset $U \subseteq X$ admits a $\mathfrak{T}$-covering as a refinement

Then we may uniquely (up to isomorphism) extend a $\mathfrak{T}$-sheaf $\mathcal{F}$ on X to a $\mathfrak{T}'$-sheaf $\mathcal{F}'$. In fact $\mathcal{F}'$ is given by

$$U \mapsto \varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, \mathcal{F})$$

where the colimit is taken over all $\mathfrak{T}'$-coverings of U consisting of $\mathfrak{T}$-open sets. Then one may check that $\mathcal{F}'$ extends $\mathcal{F}$ and is a sheaf as well.

Proposition 4.10. *Let X be an affinoid K-space. Then any sheaf $\mathcal{F}$ on X with respect to the weak Grothendieck topology admits a unique extension with respect to the strong Grothendieck topology. In particular this is true for the presheaf $\mathcal{O}_X$ of affinoid functions.*

This shows that there is a unique way to extend the sheaf $\mathcal{O}_X$ in the weak Grothendieck topology to the strong Grothendieck topology. The resulting sheaf is called **the sheaf of rigid analytic functions on** X and denoted $\mathcal{O}_X$ as well. From now on, given an affinoid K-space X, we will use the strong Grothendieck topology and the sheaf of rigid analytic functions on X.

5 Rigid analytic spaces

Recall that a **ringed** K**-space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on X.

Definition 5.1. A **G-ringed** K**-space** is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on X. If in addition all stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local rings then $(X, \mathcal{O}_X)$ is called a **locally G-ringed** K**-space**.

Morphisms of G-ringed K-spaces are defined as one would expect. One then constructs a functor from the category of affinoid K-spaces into locally G-ringed K-spaces and shows that it is fully faithful.

Definition 5.2. A **rigid analytic** K**-space** is a locally G-ringed K-space $(X, \mathcal{O}_X)$ such that

1. the G-topology of X satisfies conditions (G_0), (G_1) and (G_2) of Proposition 4.7,
2. X admits an admissible covering $(X_i)_{i \in I}$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K-space for all $i \in I$.

A **morphism of rigid analytic** K**-spaces** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism in the sense of locally G-ringed K-spaces.

As done for schemes, one can show that rigid analytic K-spaces can be glued together from local pieces and transition morphisms. The same is true for morphisms between rigid analytic K-spaces. We also have:

Lemma 5.3. Fibered products $X \times_Z Y$ exist in the category of rigid analytic K-spaces.

Since the usual definition of connectedness does not make sense for rigid analytic K-spaces, one supplants it by the following.

Definition 5.4. A rigid analytic K-space X is called **connected** if there does not exist an admissible covering (X_1, X_2) of X where $X_1, X_2 \subseteq X$ are non-empty admissible open subspaces with $X_1 \cap X_2 = \emptyset$.

From Tate's Acyclicity Theorem 3.2 it follows that an affinoid K-space $\mathrm{Sp}(A)$ is connected if and only if A cannot be written as a non-trivial cartesian product of two K-algebras.

6 GAGA

Similarly to the analytification functor over $\mathbb{C}$ there is a "rigid analytification" functor over complete non-Archimedean fields.

Definition 6.1. Let $(Z, \mathcal{O}_Z)$ be a K-scheme of locally finite type. A **rigid analytification** of $(Z, \mathcal{O}_Z)$ is a rigid K-space $(Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}})$ together with a morphism of locally G-ringed K-spaces $\iota\colon (Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}}) \to (Z, \mathcal{O}_Z)$ satisfying the following universal property: Given a rigid K-space $(Y, \mathcal{O}_Y)$ and a morphism of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ the latter factors through ι via a unique morphism of rigid K-spaces $(Y, \mathcal{O}_Y) \to (Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}})$.

Proposition 6.2. *Every K-scheme Z locally of finite type admits an analytification $Z^{\mathrm{rig}} \to Z$. Furthermore, the underlying map of sets identifies the points of Z^{rig} with the closed points of Z.*

The universal property of rigid analytifications implies that we can also analytify morphisms between K-schemes.

Corollary 6.3. Rigid analytification defines a functor from the category of K-schemes of locally finite type to the category of rigid K-spaces. It is called the **GAGA functor**.

It is worth noting that, just like in the complex case, even more is true. If X is a proper K-scheme then the GAGA functor induces an equivalence between coherent $\mathcal{O}_X$-modules and coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-modules (which we haven't defined) and the cohomology groups agree. Furthermore, closed rigid subvarieties of X^{rig} arise as rigid analytifications of subvarieties of X and all morphisms between rigidly analytified K-schemes are algebraic.

As an example, we will construct rigid affine n-space $\mathbb{A}_K^{n,\mathrm{rig}}$. The idea is to take the infinite union of balls of increasing radius and glue them together by the canonical maps. Let $r > 0$ and $T_n(r)$ be the K-algebra of all power series $\sum_\nu a_\nu \zeta^\nu$ where $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\lim_\nu a_\nu r^{|\nu|} = 0$. This is the ring of all power series converging on the ball of radius r. Choose any $c \in K$ with $|c| > 1$ and set $T_n^{(i)} = T_n(|c|^i)$. This algebra may be identified with the Tate algebra $K\langle c^{-i}\zeta\rangle$. We have inclusions

$$K[\zeta] \hookrightarrow \cdots \hookrightarrow T_n^{(2)} \hookrightarrow T_n^{(1)} \hookrightarrow T_n^{(0)} = T_n \tag{6.1}$$

and this corresponds to inclusions of affinoid subdomains

$$\mathbb{B}^n = \operatorname{Sp} T_n^{(0)} \hookrightarrow \operatorname{Sp} T_n^{(1)} \hookrightarrow \operatorname{Sp} T_n^{(2)} \hookrightarrow \cdots$$

We can interpret $\operatorname{Sp} T_n^{(i)}$ as the n-dimensional ball of radius $|c^i|$. Taking the union of all $\operatorname{Sp} T_n^{(i)}$ and gluing them along the inclusions yields the rigid K-space $\mathbb{A}_K^{n,\mathrm{rig}}$. It comes equipped with an admissible covering $\mathbb{A}_K^{n,\mathrm{rig}} = \bigcup_{i=0}^{\infty} \operatorname{Sp} T_n^{(i)}$.

Since intuitively the power series with $\lim_\nu a_\nu \infty^\nu = 0$ are just the polynomials (i.e. $a_\nu = 0$ for $|\nu| \gg 0$) the following result is not surprising:

Lemma 6.4. The inclusions in (6.1) induce inclusions of maximal spectra

$$\operatorname{Max} T_n^{(0)} \subseteq \operatorname{Max} T_n^{(1)} \subseteq \cdots \subseteq \operatorname{Max} K[\zeta]$$

and $\operatorname{Max} K[\zeta] = \bigcup_{i=0}^{\infty} \operatorname{Max} T_n^{(i)}$. This means that on the level of sets, the points of $\mathbb{A}_K^{n,\mathrm{rig}}$ correspond to the closed points of $\mathbb{A}_K^n$.

Similarly, we can construct the rigid analytification of any affine K-scheme of finite type. Let $X = \operatorname{Spec} K[\zeta]/\mathfrak{a}$ for an ideal $\mathfrak{a} \subseteq K[\zeta]$. Then we consider the maps

$$K[\zeta]/\mathfrak{a} \to \cdots \to T_n^{(2)}/(\mathfrak{a}) \to T_n^{(1)}/(\mathfrak{a}) \to T_n^{(0)}/(\mathfrak{a})$$

and this corresponds to inclusions of affinoid subdomains

$$\operatorname{Sp} T_n^{(0)}/(\mathfrak{a}) \hookrightarrow \operatorname{Sp} T_n^{(1)}/(\mathfrak{a}) \hookrightarrow \operatorname{Sp} T_n^{(2)}/(\mathfrak{a}) \hookrightarrow \cdots$$

and again we construct the rigid analytification of X by gluing together all these pieces.

References

[Bos14] S. Bosch, *Lectures on formal and rigid geometry*. Springer, 2014 (cit. on p. 1).