Group schemes over fields and Abelian varieties

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We will work with schemes over a field k, of arbitrary characteristic and not necessarily algebraically closed. An algebraic variety will be a geometrically integral and separated scheme, of finite type over k.

1 Group schemes over Spec k

Definition 1.1 (Group scheme)**.** A group scheme over k is a scheme $G \longrightarrow \operatorname{Spec} k$ that is also group object in the category of schemes over k. This means that there are three morphisms

- the multiplication map $m\colon G \times_k G \longrightarrow G$.
- the identity element $e\colon \operatorname{Spec} k \longrightarrow G$.
- the inversion map $\iota\colon G \longrightarrow G$.

that satisfy the usual axioms of abstract groups.

Example 1.1 (**Additive group**). The additive group is defined to be $\mathbb{G}_{a,k} \overset{\text{def}}{=} \mathbb{A}^1_k = \operatorname{Spec} k[T]$,with the group law given by addition:

$$\begin{aligned} m&\colon \operatorname{Spec} k[T] \times_k \operatorname{Spec} k[T] \longrightarrow \operatorname{Spec} k[T] && T \mapsto T \otimes 1 + 1 \otimes T \\ e&\colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} k[T] && T \mapsto 0 \\ \iota&\colon \operatorname{Spec} k[T] \longrightarrow \operatorname{Spec} k[T] && T \mapsto -T \end{aligned}$$

Example 1.2 (**Multiplicative group**). The multiplicative group is defined to be $\mathbb{G}_{m,k} \overset{\text{def}}{=} \mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec} k[T, T^{-1}]$ with the group law given by multiplication:

$$\begin{aligned} m&\colon \operatorname{Spec} k[T,T^{-1}] \times_k \operatorname{Spec} k[T,T^{-1}] \longrightarrow \operatorname{Spec} k[T,T^{-1}] && T \mapsto T \otimes T \\ e&\colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} k[T,T^{-1}] && T \mapsto 1 \\ \iota&\colon \operatorname{Spec} k[T,T^{-1}] \longrightarrow \operatorname{Spec} k[T,T^{-1}] && T \mapsto T^{-1} \end{aligned}$$

Remark 1.1. If G is a group scheme, then the underlying topological space *is not* an abstract group in general (the usual translations are defined just for the rational points), however, for every scheme S, the set of morphisms $G(S) = \operatorname{Hom}(S, G)$ does in fact inherit the structure of an abstract group. For example, in the case of the additive group $\mathbb{G}_a$ we see that

$$\mathbb{G}_{a,k}(S) = \operatorname{Hom}_k(S, \operatorname{Spec} k[T]) = \mathcal{O}(S)$$

with the group structure induced by addition.

Conversely, using Yoneda's Lemma, one can prove that a group scheme is precisely a scheme G such that for every scheme S we have a group structure on $G(S)$:

$$m_S\colon G(S) \times G(S) \longrightarrow G(S) \qquad e_S \in G(S) \qquad \iota_S\colon G(S) \longrightarrow G(S)$$

that is functorial in S. We can rephrase this by saying that the functor of points $h_G\colon \mathbf{Sch}/k \longrightarrow \mathbf{Sets}$ factors through the forgetful functor $\mathbf{Groups} \longrightarrow \mathbf{Sets}$.

Definition 1.2. Let $G \longrightarrow \operatorname{Spec} k$ be a group scheme. Then G is said to be

1. *finite* if $G \longrightarrow \operatorname{Spec} k$ is finite.

2. *étale* if $G \longrightarrow \operatorname{Spec} k$ is étale.

3. *local* if it is finite and connected.

Definition 1.3 (Rank of a finite group scheme)**.** Let G be a finite group scheme over a field k. The rank of G is the integer $\dim_k \mathcal{O}(G)$.

Example 1.3 (**Constant group schemes**)*.* Let G be an abstract finite group. Then we can define a group scheme (G) in this way: as a scheme we set

$$(G) = \sqcup_{g \in G} \operatorname{Spec} k$$

We define the group structure via Yoneda: for every scheme S we see that $(G)(S)$ is the set of continuous functions $S \longrightarrow G$, where G has the discrete topology. It is clear that the group structure on G induce a group structure on $(G)(S)$ that gives in turn a group scheme structure.

1.1 Subgroup schemes, kernels

Definition 1.4 (Subgroup schemes)**.** Let G be a group scheme. A (normal) subgroup scheme of G is a closed subscheme $H \subseteq G$ such that $H(S)$ is a (normal) subgroup of $G(S)$ for every scheme S. Then H becomes a group scheme in its own right, with a group law inherited from that on G.

Remark 1.2*.* Equivalently, a subgroup scheme of G is a closed subscheme $j\colon H \hookrightarrow G$ such that

1. the restriction of the multiplication map $m\colon H \times_k H \longrightarrow G$ factors through H.

2. the restriction of the inversion map $\iota H \longrightarrow G$ factors through H.

3. the identity $e\colon \operatorname{Spec} k \longrightarrow G$ factors through H.

Example 1.4 (**Roots of unity**)*.* Let $n \in \mathbb{N}$ be a positive integer, then we can define $\mu_{n,k} \stackrel{\text{def}}{=} \operatorname{Spec} k[T]/(T^n - 1)$. For every scheme S we have $\mu_{n,k}(S) = \{ f \in \mathcal{O}_S(S) \mid f^n = 1 \}$ and this makes $\mu_{n,k}$ into a subgroup scheme of $\mathbb{G}_{m,k}$.

Moreover $\mu_{n,k}$ is a finite group scheme and it is always étale if k has characteristic zero. In characteristic $p > 0$ instead, $\mu_{n,k}$ is étale if and only if $p \nmid n$.

Example 1.5 (**Roots of zero**)*.* Suppose that k has characteristic $p > 0$. Then for every $m \in \mathbb{N}$ positive integer we define $\alpha_{p^m} \stackrel{\text{def}}{=} \operatorname{Spec} k[T]/(T^{p^m})$. For every scheme S we have $\alpha_{p^m}(S) = \{ f \in \mathcal{O}_S(S) \mid f^{p^m} = 0 \}$ and this makes α_{p^m} into a subgroup scheme of $\mathbb{G}_a$. It is clear that α_{p^m} is a finite local group scheme.

Example 1.6 (**Reduced group scheme**)*.* Suppose that k is a perfect field and let G be a group scheme of finite type over k. Then G_{red} is a subgroup scheme of G.

Indeed, since k is perfect, we see that G_{red} is geometrically reduced [TS15, Lemma 32.4.3] so that $G_{red} \times_k G_{red}$ is again geometrically reduced [TS15, Lemma 32.4.7]. Now consider the restriction of the multiplication map $m\colon G_{red} \times_k G_{red} \longrightarrow G$: since the domain is reduced, this must factor through G_{red}. It is clear that the inversion map and that the identity map factors through H, and then we are done.

As usual, many examples of subgroup schemes arise as kernel of group homomorphisms:

Definition 1.5 (**Homomorphism of group schemes**)**.** Let G, H be group schemes over k. A homomorphism of group schemes is a k-morphism of schemes $f\colon G \longrightarrow H$ that respects the multiplication in the usual way. Equivalently, the induced map $f_S\colon G(S) \longrightarrow H(S)$ is a homomorphism of abstract groups for every k-scheme S.

Definition 1.6 (Kernels). Let $f\colon G \longrightarrow H$ be an homomorphism of group schemes. Then we define $\operatorname{Ker} f$ as the scheme-theoretic fiber of f over $e_H\colon \operatorname{Spec} k \longrightarrow H$. For every scheme S one has

$$(\operatorname{Ker} f)(S) = \operatorname{Ker} (f_S\colon G(S) \longrightarrow H(S))$$

and then it is clear that $\operatorname{Ker} f$ is a subgroup scheme of G.

Example 1.7. Fix an $n \in \mathbb{N}$ positive integer and consider the morphism

$$m_n\colon \mathbb{G}_{m,k} \longrightarrow \mathbb{G}_{m,k} \qquad \text{induced by} \quad T \mapsto T^n$$

then it is clear that this is a homomorphism of group schemes and that $\mu_{n,k} = \operatorname{Ker} m_{n,k}$.

As another example, suppose that k has positive characteristic p, fix a positive integer m and consider the morphism

$$F_{m,k}\colon \mathbb{G}_{a,k} \longrightarrow \mathbb{G}_{a,k} \qquad \text{induced by} \quad T \mapsto T^{p^m}$$

then this is a homomorphism of group schemes and $\alpha_{m,k} = \operatorname{Ker} F_{m,k}$.

1.1.1 Frobenius

Speaking of homomorphisms, we want to introduce an useful tool and rich source of examples: the Frobenius morphism. In this section, we fix a field k of positive characteristic $p > 0$.

Definition 1.7 (Absolute Frobenius)**.** Let X be a scheme over k. Then we define a morphism

$$\operatorname{Frob}_X\colon X \longrightarrow X$$

that on every open affine subscheme $\operatorname{Spec} A \longrightarrow X$ acts as

$$A \longrightarrow A \qquad a \mapsto a^p$$

In general, the absolute Frobenius is not a morphism of k-schemes. To fix this problem one can introduce other notions of Frobenius:

Definition 1.8 (Arithmetic Frobenius)**.** Let X be a scheme over k. Then we denote by $X^{(}p)$ the fiber product

$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\operatorname{Frob}_k} & \operatorname{Spec} k \end{array}$$

The morphism $X^{(p)} \longrightarrow X$ is called the arithmetic Frobenius.

Example 1.8. If $X = \operatorname{Spec} k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$, then $X^{(p)} = \operatorname{Spec} k[X_1, \ldots, X_n]/(f_1^{(p)}, \ldots, f_n^{(p)})$ where $f_i^{(p)}$ is obtained from f_i by replacing the coefficients with their p-th power.

Moreover, the arithmetic Frobenius in this case is given by

$$k[X_1, \longrightarrow, X_n]/(f_1, \longrightarrow, f_n) \longrightarrow k[X_1, \ldots, X_n]/(f_1^{(p)}, \ldots, f_n^{(p)}) \qquad X_i \mapsto X_i \qquad a \mapsto a^p$$

In particular this is still not a morphism of k-schemes.

Definition 1.9 (Relative Frobenius)**.** Let X be a scheme over k. Then by the universal property of the fiber product, the absolute Frobenius Frob_X and the structure morphism $X \longrightarrow \operatorname{Spec} k$ give rise to a morphism of k-schemes

$$F^{(p)}\colon X \longrightarrow X^{(p)}$$

called the relative Frobenius of X.

Example 1.9. We continue with the previous example of $X = \text{Spec}\, k[X_1, \dots, X_n]/(f_1, \dots, f_n)$. In this case the relative Frobenius is induced by the map of k-algebras

$$k[X_1, \dots, X_n]/(f_1^{(p)}, \dots, f_n^{(p)}) \longrightarrow k[X_1, \dots, X_n]/(f_1, \dots, f_n) \qquad X_i \mapsto X_i^p$$

Lemma 1.1. *Let G be a group scheme over k. Then $G^(p)$ is a group scheme over k as well and the relative Frobenius $F^{(p)}\colon G \longrightarrow G^{(p)}$ is a group homomorphism.*

Proof. Basically one pulls back the operations on G to $G^{(p)}$ and checks that everything works. □

1.2 Quotients by finite group schemes

One of the first result that we learn in abstract algebra is the Isomorphism Theorem: namely that if we have a surjective homomorphism of abstract groups $f\colon G \longrightarrow H$ then we have an induced isomorphism $G/\text{Ker}\, f \xrightarrow{\sim} H$. Can we say something similar also for group schemes? Can we define the quotient for the action of a group in general?

Definition 1.10 (Actions of group schemes). An *action* of a k-group scheme G on a k-scheme X is a map

$$\mu\colon G \times_k X \longrightarrow X$$

that satisfies the usual compatibilities. This is the same as saying that for every k-scheme S we have an action

$$\mu_S\colon G(S) \times X(S) \longrightarrow X(S)$$

that is functorial in S. We say that the action is *free* if the map

$$(\mu, pr_2)\colon G \times_k X \longrightarrow X \times_k X$$

is a closed embedding.

Remark 1.3. Suppose that X is a k-group scheme and that $j\colon G \hookrightarrow X$ is a subgroup scheme: then G acts naturally on X via left-translations, meaning that the action is given as the composition

$$G \times X \xhookrightarrow{j \times \text{id}_X} X \times X \xrightarrow{m} X$$

In particular, we see that the action is free as the induced map $(\mu, pr_X)\colon G \times_k X \longrightarrow X \times_k X$ is obtained as the composition of

$$G \times X \xhookrightarrow{j \times \text{id}_X} X \times X \xrightarrow{(m, pr_2)} X \times X$$

and the first map is a closed embedding, whereas the second one is an isomorphism.

Definition 1.11 (Orbit of an action). Let G be a k-group scheme acting on a k-scheme X via

$$\mu\colon G \times_k X \longrightarrow X$$

We say that two points $x, y \in X$ are equivalent under this action if there exist a point $z \in G \times_k X$ such that $pr_X(z) = x$ and $\mu(z) = y$. This is an equivalence relation and we call the equivalence class of a point $x \in X$ the *orbit* of x under the action of G.

Definition 1.12 (Sheaf of invariants). Let G be a k-group scheme acting over a k-scheme X via

$$\mu\colon G \times_k X \longrightarrow X$$

Denote by $\sim$ the equivalence relation that identifies the orbits of the action and let $\pi\colon X \longrightarrow X/\sim$ be the canonical projection of topological spaces. Then we can define a subsheaf $(\pi_* \mathcal{O}_X)^G$ of G-invariant functions this way: for every $U \subseteq X/\sim$ open subset, let $V = \pi^{-1}(U)$, then we say that

$$(\pi_* \mathcal{O}_X)^G(U) = \{\, f \in \mathcal{O}_X(V) \mid \mu^\#(f) = pr_X^\#(f) \,\}$$

Definition 1.13 (Invariant morphism). Let G be a k-group scheme acting over a k-scheme X. Then we say that a morphism of k-schemes, $f\colon X \longrightarrow Y$ is G-invariant if we have a commutative diagram

$$\begin{array}{ccc} G \times_k X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

This is equivalent to saying that for every k-scheme S the map $f_S\colon X(S) \longrightarrow Y(S)$ is constant on the orbits of $G(S)$.

We can build the quotient w.r.t. the action of a finite group scheme, under a certain additional hypothesis:

Theorem 1.1 (Quotient by finite group schemes)**.** *Let G be a finite k-group scheme acting on a scheme X, of finite type over k, such that the orbit of every closed point of X is contained in an affine open subset of X. Then there exists a morphism of schemes*

$$\pi\colon X \longrightarrow Y$$

with the following properties:

1. *as a topological space, (Y, π) is the quotient of X by the action of the underlying finite group.*
2. *the morphism $\pi\colon X \longrightarrow Y$ is G-invariant and the natural homomorphism $\mathcal{O}_Y \longrightarrow (\pi_*\mathcal{O}_X)^G$ is an isomorphism.*
3. *(Y, π) is determined up to isomorphism by the above properties. We will use the notation $Y = X/G$.*
4. *π is finite and surjective.*
5. *$(X/G, \pi)$ is a categorical quotient, meaning that for every G-invariant morphism $f\colon X \longrightarrow Z$ there exists an unique morphism $g\colon X/G \longrightarrow Z$ such that $f = g \circ \pi$.*
6. *suppose further that the action of G is free then:*
   - *π is a flat morphism of degree $rk(G)$.*
   - *the induced morphism*

     $$(\mu, pr_X)\colon G \times_k X \longrightarrow X \times_{X/G} X \subseteq X \times_k X$$

     is an isomorphism.

Proof. See [Mum08, Theorem 12.1] and [EMvdG, Theorem 4.16]. The basic idea is that we can reduce to the case in which $X = \operatorname{Spec} A$ is affine, and then we take as $X/G = \operatorname{Spec} A^G$, where $A^G = \{\, a \in A \mid \mu^\#(a) = 1 \otimes a \,\}$. □

Remark 1.4. In the previous College Seminar, Irfan stated this result when G is a finite group of automorphisms of an algebraic variety X: see [Kad, Theorem 9].

Remark 1.5. If X is a quasiprojective variety, then the condition of the theorem is satisfied. Indeed, any finite set of closed points in a projective space $\mathbb{P}^n_k$ is contained in an affine open subset, since we can always find a hypersurface (even a hyperplane if k is infinite) that does not pass through any of them.

Remark 1.6. The condition of $(\mu, pr_X)\colon G \times X \longrightarrow X \times_{X/G} X$ being an isomorphism means that for every scheme S, two S-valued points $x, y \in X(S)$ have the same image in $(X/G)(S)$ if and only if $y = \mu_S(g, x)$ for a certain $g \in G(S)$.

In particular, we can construct the quotient of a group by a finite group subscheme, recovering the First Isomorphism Theorem in this case:

Definition 1.14 (Epimorphisms)**.** A homomorphism of group schemes $f\colon X \longrightarrow Y$ is an *epimorphism* if it is surjective and the homomorphism $f^{\#}\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is surjective.

Corollary 1.1. [Mum08, Corollary 12.1] *In the hypotheses of Theorem 1.1 suppose moreover that X is a group scheme and G a finite normal subgroup acting on X by right-translations. Then X/G is again a group scheme, $\pi\colon X \longrightarrow X/G$ is a epimorphism and Ker $\pi = G$.*

Conversely, if $f\colon X \longrightarrow Y$ is a epimorphism of group schemes and $G =$ Ker f is finite, then $Y \cong X/G$.

Proof. We follow the proof of [Mum08]. For the first part, one checks that $\pi \times \pi\colon X \times_k X \longrightarrow X/G \times X/G$ is the quotient of $X \times_k X$ by $G \times G$ and that the composition

$$X \times_k X \xrightarrow{m} X \xrightarrow{\pi} X/G$$

is $G \times G$-invariant. Then we obtain a morphism

$$X/G \times_k X/G \longrightarrow X/G$$

and it can be verified that this makes X/G into a group scheme such that $\pi\colon X \longrightarrow X/G$ is a homomorphism. Then it is clear that this is an epimorphism since it is surjective and the homomorphism $\mathcal{O}_{X/G} \longrightarrow \pi^*\mathcal{O}_X$ factors as $\mathcal{O}_{X/G} \xrightarrow{\sim} (\pi_*\mathcal{O}_X)^G \hookrightarrow \pi_*\mathcal{O}_X$. To conclude, we need to show that Ker $\pi = G$: first observe that the action of G on X is free, and then we know from Remark 1.6 that for every scheme S a point $x \in X(S)$ belongs to (Ker $\pi)(S)$ if and only if there exists $g \in G(S)$ such that $\mu_S(g, e) = x$, this proves that $G(S) =$ (Ker $\pi)(S)$ for every S, and then $G =$ Ker π.

For the second part, we first prove that the morphism $f\colon X \longrightarrow Y$ is finite: thanks to [Vis08, Proposition 1.15], we can suppose $k = \bar{k}$. Then we see that the morphism is quasifinite, since every fiber is a translate of G, and then it is generically finite by [TS15, Lemma 36.31.7], using homogeneity, it follows that f is finite. In particular, pulling back affine open subset on Y, we see that the orbit of every closed point of X is contained in an affine open, so that we can form the quotient $\pi\colon X \longrightarrow X/G$. One shows that f is G-invariant so that it factors through a morphism $g\colon X/G \longrightarrow Y$. Now, we know from the first part that X/G is a group scheme and then it is easy to see that $g\colon X/G \longrightarrow Y$ is an epimorphism with trivial kernel. So we are reduced to proving that an epimorphism $f\colon X \longrightarrow Y$ with trivial kernel is an isomorphism: we know already that it is finite, and that $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is injective, so that it is enough to prove surjectivity. We can again assume that $k = \bar{k}$, and now Nakayama shows that surjectivity is equivalent to the fact that the fiber over closed points are closed points with reduced structure, but this is true since these fibers are all translated of Ker $f =$ Spec k. □

Remark 1.7. In the following, we will say that

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

is a short exact sequence of group schemes if $G \longrightarrow G''$ is an epimorphism and $G' \longrightarrow G$ is its kernel.

Lemma 1.2. *Let*

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

be a short exact sequence of finite group schemes over a field k. Then

$$rk(G) = rk(G')rk(G'')$$

Proof. Follows immediately from the fact that $G \longrightarrow G''$ is a flat morphism of degree rk(G') (1.1). □

2 Finite group schemes

Now we want to focus on finite group schemes since they will be very important for abelian varieties.

2.1 The connected component of the identity

Let G be a finite group scheme over k. Then the connected component G^0 containing the identity is both open and closed, so that it has a natural scheme structure.

Proposition 2.1. *Let G be a finite group scheme over k. Then:*

1. *G^0 is a closed subgroup scheme of G.*
2. *for every field extension K/k, we have $(G_K)^0 = (G^0)_K$ so that G^0 is geometrically irreducible.*

Proof.
1. First we observe that G^0 is geometrically connected, since it is connected and contains a rational point [TS15, Lemma 32.5.14]. Hence $G^0 \times_k G^0$ is again connected [TS15, Lemma 32.5.4], so that its image under the multiplication map is connected and contains the identity, meaning that we have an induced map
$$m\colon G^0 \times_k G^0 \longrightarrow G^0$$
and this implies that G^0 is a subgroup scheme of G.
2. This follows from the fact that G^0 is a geometrically connected point.

Remark 2.1. It is not true that every connected component of G is geometrically connected: indeed, consider the finite group scheme $G = \mu_{3,\mathbb{Q}} = \text{Spec } \mathbb{Q}[X]/(X^3-1)$, then this scheme has two points, but one splits into two different points when passing to the algebraic closure.

□

2.2 Etale finite group schemes

We begin with some remarks about finite étale schemes over a field in general:

Lemma 2.1. *Let X be a scheme over k. Then X is finite and étale if and only if it is of the form*
$$X = \bigsqcup_{i=1}^{n} \mathit{Spec}\ k_i$$
where k_i/k is a finite separable extension for every $i = 1, \dots, n$.

Proof. It is enough to prove the lemma when X is connected. Suppose first that $X = \text{Spec } k'$, with k'/k a finite and separable extension. Then, thanks to the Primitive Element Theorem, we have that $k' = k(\alpha) = k[X]/(f(X))$, where $f(X)$ is a separable irreducible polynomial: then it is easy to see that $\Omega^1_{k'/k} = 0/$ Indeed we know that
$$\Omega^1_{k'/k} = k' \cdot dX/f'(\alpha) \cdot dX \qquad \text{and } f'(\alpha) \neq 0$$
and this proves that $\text{Spec } k' \longrightarrow \text{Spec } k$ is étale.

Conversely suppose that we have an affine scheme Spec A that is connected and finite over Spec k: since A has dimension 0, every prime ideal is maximal so that every point in Spec A is closed. Then since there are finitely many of these points, this means that the complement of each point is again closed. Since Spec A is connected, this means that it consists of an unique point. Now suppose moreover that Spec A is étale over k: then it is reduced, and since A has an unique prime ideal, this means that $A = k'$ is a finite field extension of k. If k'/k is not separable, then there exists an element $\alpha \in k'$ whose minimal polynomial $f(X)$ over k is such that $f(\alpha) = f'(\alpha) = 0$, i.e. $f(X)$ factorizes in $k'[X]$ as $f(X) = (X-\alpha)^2 \times g(X)$. However, since being étale is stable under base change, we know that $\text{Spec } k' \otimes k[X]/(f(X)) \longrightarrow \text{Spec } k[X]/(f(X))$ must be étale, and since $k[X]/(f(X))$ is a field, this implies that $k' \otimes k[X]/(f(X))$ is reduced. Now we come to the absurd, since $k' \otimes_k k[X]/(f(X)) \cong k'[X]/(f(X))$. □

Now, if we denote by $\Gamma_k = \mathrm{Gal}(k^{sep}/k)$ the **absolute Galois group** of k, we have a natural action of Γ_k as an abstract group on Spec k^{sep}. Moreover, suppose that X is an étale finite scheme over k, then the action of Γ_k extends to an action on $\mathrm{Hom}_k(\mathrm{Spec}\ k^{sep}, X)$.

Remark 2.2. We can consider the group Γ_k as a topological group via the Krull topology, i.e., the subgroup of the form $\mathrm{Gal}(k^{sep}/L)$, with L/k a finite Galois extension form a fundamental system of neighborhoods of the identiny. Equivalently , this topology is obtained by considering the Γ_k as a profinite group

$$\Gamma_k = \varprojlim \mathrm{Gal}(L/k)$$

where the limit is over all the finite Galois extension.

Then let X be an étale finite scheme over k, and consider the set $\mathrm{Hom}_k(\mathrm{Spec}\ k^{sep}, X)$ with the discrete topology. Then it is easy to see that the natural action

$$\mu \colon \Gamma_k \times \mathrm{Hom}_k(\mathrm{Spec}\ k^{sep}, X) \longrightarrow \mathrm{Hom}_k(\mathrm{Spec}\ k^{sep}, X)$$

is continuous.

We can define a Γ_k**-set** as a set with a continuous action of Γ_k (where we consider the set with the discrete topology). Then we have seen that we have a map

$$\{ \text{ finite étale } k-\text{schemes } \} \longrightarrow \{ \text{ finite } \Gamma_k - \text{sets } \} \qquad X \mapsto \mathrm{Hom}_k(\mathrm{Spec}\ k^{sep}, X)$$

Actually it is easy to see that the above is actually a functor, when in the second category we take the morphism to be the Γ_k-invariant maps. The important point is that this functor gives an equivalence of categories.

Theorem 2.1. *The above functor is an equivalence of categories.*

Idea of proof. The idea is that we can describe a quasi-inverse as follows: let S be a finite set together with a continuous action of Γ_k. Then S is the disjoint union of its orbits $S = \sqcup_{j=1}^n S_i$: fix an element $s_i \in S_i$ for every i. Then it is easy to see that the stabilizer G_i of s_i in Γ_k is an open subgroup and we can take $k_i = \mathrm{Fix}(G_i) \subseteq k^{sep}$. This is a finite separable extension of k: indeed, since G_i is open, it contains a subgroup of the form $\mathrm{Gal}(k^{sep}/L_i)$ for a finite Galois extension L_i/k, so that $k_i \subseteq \mathrm{Fix}(\mathrm{Gal}(k^{sep}/L_i)) = L_i$. Then we can define the quasi-inverse as

$$S \mapsto \bigsqcup_{i=1}^n \mathrm{Spec}\ k_i$$

□

Now, define a Γ_k**-group** as an abstract group with the discrete topology on which Γ_k acts continuously by group automorphisms. It is clear that the group objects in the category of Γ_k-sets are precisely the Γ_k groups, so that we get:

Corollary 2.1. *The functor*

$$\{ \textit{ finite étale } k-\textit{ group schemes } \} \longrightarrow \{ \textit{ finite } \Gamma_k - \textit{groups } \} \qquad X \mapsto \mathit{Hom}_k(\mathit{Spec}\ k^{sep}, X)$$

is an equivalence of categories

Proof. The above equivalence of categories induces an equivalence between group objects. □

Corollary 2.2. *Let G be a finite étale group scheme. Then $G_{k^{sep}}$ is isomorphic to a k^{sep}-constant group scheme.*

Proof. Follows from the fact that the absolute Galois group of k^{sep} is the trivial group. □

Now we use this equivalence to associate to any (finite) group scheme an étale group scheme: more precisely, let G be a finite group scheme over k and consider the set

$$\pi_0(G) = \{ \text{ connected components of } G_{k^{sep}} \} = \{ \text{ connected components of } G_{\overline{k}} \}$$

where the last equality should be interpreted as saying that the map $X_{\overline{k}} \longrightarrow X_{k^{sep}}$ induces a bijection between the connected components: this follows from the fact that any connected over a separably closed field is also geometrically connected (cfr. [TS15, Lemma 32.5.14]). Then we can give $\pi_0(X)$ the structure of an étale group scheme over k.

Proposition 2.2. *Let G be a finite group scheme over k. Then*

1. *there is a natural structure of an étale group scheme on $\pi_0(G)$. We denote this group scheme by $G_{ét}$.*

2. *there is a canonical short exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G_{ét} \longrightarrow 1$$

3. *if k is perfect, then the map $G \longrightarrow G_{ét}$ induces an isomorphism $G_{red} \cong G_{ét}$, so that the above sequence splits. In particular, if G is commutative, then $G \cong G^0 \times G_{ét}$.*

Proof. 1. Consider first $\pi_0(G)$ as the set of connected components of $G_{k^{sep}}$. Then there is an obvious action of Γ_k on it, and it is easy to see that this action is continuous, so that we can regard $\pi_0(X)$ as an étale scheme over k. Moreover, looking at $\pi_0(X)$ as the set of connected components of $G_{\overline{k}}$, this has a natural structure of abstract group, given by the multiplication of $\overline{k}$-points over each connected component. One then proves that the action of Γ_k is given by group automorphisms, so that $\pi_0(G)$ becomes an étale group scheme.

2. By construction, $(G_{ét})_{k^{sep}}$ is the constant group scheme modeled over $\pi_0(G)$: then we have the natural morphism

$$X_{k^{sep}} \longrightarrow (G_{ét})_{k^{sep}}$$

that sends each connected component of $X_{k^{sep}}$ to the point it represents in $(G_{ét})_{k^{sep}}$. It is easy to see that this is an epimorphism of group schemes and that the kernel is precisely $(G_{k^{sep}})^0$, so that we have an exact sequence

$$1 \longrightarrow (G_{k^{sep}})^0 \longrightarrow G \longrightarrow (G_{ét})_{k^{sep}} \longrightarrow 1$$

Now, recalling that $(G_{k^{sep}})^0 = (G^0)_{k^{sep}}$, and using Galois descent [EMvdG, Exercise 3.9], we get the desired exact sequence.

3. To see that the induced map $G_{red} \longrightarrow G_{ét}$ is an isomorphism, we can suppose that $k = \overline{k}$. Then we see that G_{red} is étale, and in particular it must be isomorphic to a constant group scheme: now it is easy to convince oneself that the conclusion is true.

□

Corollary 2.3. *Let G be a finite group over a field k. Then G is étale if and only if G^0 is geometrically reduced.*

Proof. It is clear that if G is smooth then G^0 is geometrically reduced, since it is an open subscheme. For the converse, we can assume that $k = \overline{k}$, but then G^0 is the trivial subscheme and the short exact sequence of Proposition 2.2 tells us that $G \cong G_{ét}$. □

We conclude this section by showing that every finite group scheme in characteristic zero is étale. In general, it is a theorem of Cartier that every group scheme locally of finite type over a field of characteristic zero is smooth (see [EMvdG, Theorem 3.20]).

Proposition 2.3. *Let G be a finite group scheme over a field k of characteristic 0. Then G is étale.*

Proof. The proof follows [EMvdG, Theorem 3.20]. Thanks to Corollary

2.3 Local group schemes

Now we want to say something about the counterpart of étale group schemes: local group schemes. We know from Proposition 2.3 that every local group scheme over a field of characteristic zero is just the trivial group, so that in the following we will fix a field k of characteristic $p > 0$.

Proposition 2.4. *Let G be a local group scheme over k. Then $rk(G) = p^m$ for a certain m.*

Idea of proof. We proceed by induction on the rank of G. If the rank is 1, we are happy. If the rank is bigger, consider the relative Frobenius $F\colon G \longrightarrow G^{(p)}$ and set $G[F] = \mathrm{Ker}\, F$. Then we have the exact sequence

$$1 \longrightarrow G[F] \longrightarrow G \longrightarrow G/G[F] \longrightarrow 1$$

and both $G[F]$ and $G/G[F]$ are local group schemes, so if both $G[F]$ and $G/G[F]$ are nontrivial, we can use inductive hypothesis and conclude. Then, juggling aound with differentials, one can actually prove that $\mathrm{rk}(G[F]) = p^d$, where $d = \dim T_{G,e}$ and since G has rank bigger than 1 we must have $d > 0$ and we are done. □

We can use this result to reprove Lagrange theorem for group schemes:

Corollary 2.4 (Lagrange Theorem for finite group schemes)**.** *Let k be ay field and let G be a finite group scheme over k, of rank r. Then, the multiplication-by-r morphism $[r]\colon G \longrightarrow G$ is constant, and equal to $[0]$.*

Idea of proof. Thanks to the exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G_{ét} \longrightarrow 1$$

we are reduced to the case of G étale or G local. If G is étale, we can base change to k^{sep}, so that we get a constant group scheme, and the our result follows from Lagrange's theorem for abstract finite groups. Suppose then that k has characteristic $p > 0$ and that G is a local group: set $G = \mathrm{Spec}\, A$ and let $\mathfrak{m} \subsetneq A$ be the maximal ideal corresponding to the rational point. Then considering the chain of descending subspaces $A \supsetneq \mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \dots$ we see that $\mathfrak{m}^r = 0$. The previous Proposition tells us that $r = p^n$ for a certain n: working a bit with m^* as in the proof of Proposition 2.3, one can prove that $[p^n]^*(\mathfrak{m}) \subseteq \mathfrak{m}^{p^n} = 0$. Then this means that $[p^n]$ factors through the identity, which proves our claim. □

3 Recap on abelian varieties

Definition 3.1 (Abelian variety)**.** An abelian variety is a proper group variety (over k).

Example 3.1. Some standard examples of abelian varieties are:

- every elliptic curve E/k is an abelian variety.
- if $\Lambda \subseteq \mathbb{C}^g$ is a lattice, then the quotient $X = \mathbb{C}^g/\Lambda$ is a complex manifold called a complex torus. Every complex torus that is also projective is also an abelian variety over $\mathbb{C}$. In particular, every complex torus of dimension 1 is an abelian variety, as it is an elliptic curve.
- if C/k is a smooth curve, then $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ is an abelian variety over k.

Remark 3.1. As suggested by the previous examples, every abelian variety X is smooth: to prove this, we can assume thanks to [Vis08, Proposition 1.15] that $k = \bar{k}$, and then, since X is integral we know that there is an open subset of smooth points. Moving around this open subset via translations, we see that every point is smooth.

The properness hypothesis is crucial, as it gives very strong restrictions on the structure of X:

Example 3.2. One can prove that every abelian variety over $\mathbb{C}$ is actually complex torus: that is, a complex manifold of the form $\mathbb{C}^g/\Lambda$, where $\Lambda \subseteq \mathbb{C}^g$ is a lattice. This was proven by Emre in the previous college seminar: see [Ser].

Proposition 3.1. *If $f\colon X \longrightarrow Y$ is a morphism of abelian variety, then there exists an unique homomorphism of group schemes $g\colon X \longrightarrow Y$ such that $f = t_{f(e)} \circ g$.*

Proof. Proven by Angela in the last College Seminar: see [Ort, Corollary 5]. The crucial tool is the Rigidity Lemma [Ort, Lemma 4]. □

Proposition 3.2. *Every abelian variety is a commutative group scheme.*

Proof. Proven by Angela in the last College Seminar: see [Ort, Corollary 6]. The crucial tool is the Rigidity Lemma [Ort, Lemma 4]. □

Definition 3.2 (Multiplication-by-n)**.** In particular, we see that the multiplication by n map on an abelian variety X is an homomorphism: we denote it by $[n]_X\colon X \longrightarrow X$ and we define $X[n] \overset{\text{def}}{=} \text{Ker } [n]_X$.

Two particularly important results are

Proposition 3.3. *Let L be any line bundle on an abelian variety X. Then for every $n \in \mathbb{Z}$ one has*

$$[n]_X^* L \cong L^{\frac{n^2+n}{2}} \otimes [-1]_X^* L^{\frac{n^2-n}{2}}$$

Proof. Proven by Angela in the last College Seminar: see [Ort, Theorem 17]. The crucial tool is the Theorem of the Cube [Ort, Theorem 12]. □

Proposition 3.4 (Theorem of the Square)**.** *Let L be any line bundle on an abelian variety X and let $x, y \in X(k)$ be two rational points. Then*

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L$$

Proof. Proven by Angela in the last College Seminar: see [Ort, Theorem 18]. The crucial tool is the Theorem of the Cube [Ort, Theorem 12]. □

Proposition 3.5. *Every abelian variety is projective.*

Proof. We have seen this in Irfan's talk (see [Kad, Corollary 4]) in the case of $k = \bar{k}$. Then the result follows from any k thanks to [Mil, Proposition 6.6(c)]. □

3.1 Isogenies

Now we define a very important class of morphisms between abelian varieties:

Proposition 3.6. [EMvdG, Proposition 5.2] *Let $f\colon X \longrightarrow Y$ be an homomorphism of abelian varieties, then TFAE:*

1. $\dim X = \dim Y$ *and f is surjective.*
2. $\dim X = \dim Y$ *and Ker f is a finite subgroup scheme.*
3. *f is a finite, flat and surjective morphism.*

Proof. Thanks to [Vis08, Proposition 1.15], we can reduce to the case $k = \bar{k}$ and then this result was proven by Irfan in [Kad, Proposition 6]. The main idea is that f is generically flat, as Y is reduced, and then one can use the group structure to show that f is actually flat everywhere and then use the theorem on the dimension of fibers. □

Definition 3.3 (Isogeny)**.** An homomorphism of abelian varieties $f\colon X \longrightarrow Y$ satisfying one of the previous equivalent conditions is called an isogeny.

Remark 3.2. The degree of an isogeny $f\colon X \longrightarrow Y$ is its degree as a finite map, i.e. $\deg f = [K(X) : K(Y)]$. Since f is flat, this coincides with the degree over each point, and in particular $\deg f = \text{rk}(\text{Ker } f)$.

Remark 3.3. Every isogeny is an epimorphism, since it is finite and surjective between integral schemes. Hence, Corollary 1.1, we see that every isogeny $f\colon X \longrightarrow Y$ induces an isomorphism $X/G \cong Y$, so that there is a correspondence between isogenies $X \longrightarrow Y$ and finite subgroup schemes of X.

In this correspondence, we can also work out the isogenies corresponding to étale and local subgroups:

Definition 3.4 (Separable morphisms)**.** A dominant morphism of integral schemes $f\colon X \longrightarrow Y$ is called separable if it induces a separable extension between the fields of rational functions.

Proposition 3.7 (Separable isogenies)**.** [EMvdG, Proposition 5.6] *Let $f\colon X \longrightarrow Y$ be an isogeny of abelian varieties. Then TFAE:*

1. *f is étale.*

2. *Ker f is an étale subgroup scheme.*

3. *f is separable.*

An isogeny satisfying one of these conditions is called separable.

Proof. (1) $\implies$ (2) + (3) This follows since being étale is preserved under base change.

(2), (3) $\implies$ (1) Consider on X the set $U = \{\, x \in X \mid f \text{ is étale at } x \,\}$. This corresponds to the set $\{\, x \in X \mid \Omega^1_{X/Y,x} = 0 \,\}$ and since $\Omega^1_{X/Y}$ is coherent, we see that U is open. Since π is finite, this means that the set $V = Y \setminus \pi(X \setminus U)$ is open on Y, and moreover $f^{-1}(V) \longrightarrow V$ is étale. Now, in both cases U is nonempty, so that the generic point of Y belongs to V, and then we have proved that f is generically étale. By homogeneity, it follows that f is étale everywhere. □

Definition 3.5 (Purely inseparable morphism)**.** A morphism of schemes $f\colon X \longrightarrow Y$ is called purely inseparable if it satisfies one of the following equivalent conditions:

1. f is universally injective.

2. f is injective and for every point $x \in X$, the induced field extension $k(x)/k(f(x))$ is purely separable.

3. for every field k'/k, the induced map $X(k') \longrightarrow Y(k')$ is injective

Proposition 3.8 (Purely inseparable isogenies)**.** *Let $f\colon X \longrightarrow Y$ be an isogeny of abelian varieties. Then TFAE:*

1. *f is purely inseparable.*

2. *Ker f is a local subgroup scheme.*

3. *f the induced field extension between rational functions is purely inseparable.*

Proof. (1) $\implies$ (2) + (3) is clear.

(3) $\implies$ (2) we can factor the isogeny f as the composition $X \longrightarrow X/(\operatorname{Ker} f)^0 \longrightarrow Y$. Then we know that the kernel of the second isogeny is $\operatorname{Ker} f/(\operatorname{Ker} f)^0$, that is étale, so that $X/(\operatorname{Ker} f)^0 \longrightarrow Y$ is a separable isogeny. But then it induces a field extension that is both separable and purely inseparable, so that it must be of degree 1, hence an isomorphism.

(2) $\implies$ (1) Since $(\operatorname{Ker} f)^0_{k'} = ((\operatorname{Ker} f)_k)^0$, we can assume that $k' = k$. Then it is easy. □

Remark 3.4. In the previous College Seminar, when $k = \overline{k}$, Irfan proved that there is a correspondence between finite constant subgroups K of an abelian variety X and separable isogenies $f\colon X \longrightarrow Y$ (a separable isogeny is an isogeny that induces a separable extension between the fields of rational functions). This result generalizes that one in two ways:

1. If k is not algebraically closed, we could have separable isogenies whose kernel is not given by a constant subgroups. However, we note that if G is a finite étale group scheme, then $G_{\overline{k}}$ becomes a constant group scheme.

2. If k is algebraically closed but has characteristic $p > 0$ we could get isogenies that are not separable. For example the relative Frobenius is a purely inseparable isogeny of degree p^g, where g is the dimension of the abelian variety (see [EMvdG, Proposition 5.15]).

3.1.1 The isogenies $[n]_X$

A very important example of an isogeny is given by the multiplication by n map.

Proposition 3.9. *Let X be an abelian variety of dimension g. Then for every integer $n \in \mathbb{Z}, n \neq 0$ the map*

$$[n]_X \colon X \longrightarrow X$$

is an isogeny of degree n^{2g}. Moreover

1. *if $\operatorname{char} k = 0$, then $[n]_X$ is always separable.*

2. *if $\operatorname{char} k = p > 0$ then $[n]_X$ is separable if and only if $p \nmid n$.*

Proof. The first part of this result was proven by Irfan in [Kad, Proposition 7]: the main tool is that if L is a symmetric line bundle on X (meaning that $[-1]_X^* L \cong L$), then $[n]_X^* L \cong L^{n^2}$.

The second part is obvious when $\operatorname{char} k = 0$, as every field extension is separable in this case. In general, one can prove that the differential on the tangent spaces at e is given precisely by multiplication by n:

$$d[n]_{X,e} \colon T_e X \longrightarrow T_e X \qquad v \mapsto n \cdot v$$

so that this is an isomorphism exactly when $p \nmid n$. This proves that if $p \mid n$, then $[n]_X$ is not separable, and using homogeneity, this also proves that it is separable when $p \nmid n$. □

Corollary 3.1. *Let X be an abelian variety of dimension g and let n be a nonzero integer. If $\operatorname{char} k = 0$ or if $\operatorname{char} k = p > 0$ with $p \nmid n$, then $X[n](\overline{k}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.*

Proof. We know that $X[n](\overline{k})$ is a finite abelian group of cardinality n^{2g} and moreover, for every divisor m of n, the set of elements of $X[n](\overline{k})$ whose order divided m is precisely $X[m](\overline{k})$, so that it has cardinality m^{2g}. Using the structure theorem for finite abelian groups we see that it must be $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

□

Instead, the p-torsion part can behave strangely in characteristic p. We have this result:

Proposition 3.10. *Let k a field of positive characteristic $p > 0$ and let X be an abelian variety of dimension g. Then there exist an integer $0 \leq r \leq g$, such that*

$$X[p^m](k^{sep}) \cong (\mathbb{Z}/p^m\mathbb{Z})^r \qquad \text{for every } m \geq 0$$

Moreover, as group shemes we have

$$X[p^m] \cong (\mathbb{Z}/p^r\mathbb{Z})^r \times (\mu_{p^n})^r \times G_n^0$$

for a certain local group scheme G_n^0.

Proof. See [Mum08, pag. 137]. □

Definition 3.6 (p-rank of an abelian variety)**.** The integer r defined before is called the p-rank of an abelian variety. In the case of elliptic curves, if $r = 1$ the curve is called ordinary, whereas if $r = 0$ the curve is called supersingular.

Example 3.3. Suppose that char $k = 2$. And consider the elliptic curve

$$E_1 = \{ x_1^2 x_2 + x_0 x_1 x_2 + x_0^3 + x_2^3 = 0 \}$$

with origin $O = [0, 1, 0]$. Then E is ordinary (2-torsion points corresponds to tangents passing through O).

Example 3.4. Suppose that char $k = 2$ and consider the elliptic curve

$$E_0 = \{ x_1^2 x_2 + x_1 x_2^2 + x_0^3 = 0 \}$$

with origin $O = [0, 1, 0]$. Then E is supersingular (again, look at the tangents).

Proposition 3.11. *Let $f \colon X \longrightarrow Y$ be an isogeny between abelian varieties. Then there exists another isogeny $g \colon Y \longrightarrow X$ such that*

$$f \circ g = [n]_Y \qquad g \circ f = [n]_X$$

Proof. We know that Ker f is a finite group subscheme of X, and then it can be shown that Ker f is annihilated by its rank, in particular, there is an $m \in \mathbb{N}$ such that Ker $f \subseteq X[m]$. Then this means that the morphism $[m]_X \colon X \longrightarrow X$ can be factorized as $X \xrightarrow{f} Y \xrightarrow{g} X$ for a certain homomorphism g. If we prove that $f \circ g = [n]_Y$ as well, we are done. Now, let $y \in Y(\bar{k})$ be a geometric point: then we see that $y = f(x)$ for a geometric point $x \in X(\bar{k})$ and we are done. □

Corollary 3.2. *The relation of being isogenous is an equivalence relation.*

3.2 The dual abelian variety

A fundamental tool in the theory of smooth curves is given by the Jacobian: this is the set of line bundles of degree zero, and it has a natural structure of an abelian variety. Moreover, for an elliptic curve, there is an isomorphism between the curve and its Jacobian. Can we do something similar also for abelian varieties?

Let X be an abelian variety and let L be a line bundle on X. We can define the map

$$\phi_L(k) \colon X(k) \longrightarrow \mathrm{Pic}(X) \qquad x \mapsto t_x^* L \otimes L^{-1}$$

and the Theorem of the Square tells us precisely that this is an homomorphism of abstract groups. This map behaves well w.r.t. extension of the base field, so that for every field extension k'/k we can define a subgroup

$$K(L)(k') = \mathrm{Ker}\ \phi_L(k') = \{ x \in X(k') \mid t_x^* L \cong L \text{ on } X_{k'} \}$$

and a class of line bundles

Definition 3.7. We define

$$\mathrm{Pic}\ ^0(X) \stackrel{\text{def}}{=} \{ L \in \mathrm{Pic}\ (X) \mid \phi_L(\bar{k}) = 0 \} = \{ L \in \mathrm{Pic}\ (X) \mid K(L)(\bar{k}) = 0 \}$$

These line bundles are parametrized naturally by another abelian variety, called the dual variety to X:

Theorem 3.1 (Dual abelian variety)**.** *Let X be an abelian variety over k. There exist another abelian variety $X^\vee$ defined over k and a line bundle $\mathcal{P}$ on $X \times X^\vee$ that satisfy the following universal property: for every k-scheme T and for every line bundle L on $X \times T$ such that*

1. *$L_{|X \times \{ t \}} \in Pic^0(A_t)$ for every $t \in T$ closed point.*
2. *$L_{\{ e \} \times T} \cong \mathcal{O}_T$.*

there exists an unique homomorphism $f \colon T \longrightarrow X^\vee$ such that $L = (id_X \times f)^* \mathcal{P}$.

Definition 3.8 (Dual abelian variety, Poincare' bundle)**.** The abelian variety $X^\vee$ is called the dual abelian variety of X and the line bundle $\mathcal{P}$ is called the Poincare' bundle.

Construction of the dual abelian variety. Over an algebraically closed field, Niels constructed the dual abelian variety in the last College Seminar: see [Lin]. We can follow the same procedure for any field: the idea is to take an ample line bundle L and check that there exist a finite subgroup scheme $K(L) \subseteq X$ whose closed points correspond to the abstract subgroups defined before. Then one proves that $X^\vee \stackrel{\text{def}}{=} X/K(L)$ is the dual abelian variety.

Another, less hands-on construction uses the Picard functor of Grothendieck, see [EMvdG]. □

Remark 3.5. Now fix a line bundle L on X and consider the line bundle $\Lambda(L) \cong m^*L \otimes pr_1^*L \otimes pr_2^*L$ on $X \times X$. Then one checks that $\Lambda(L)_{|X\times\{x\}} \cong t_x^*L \otimes L^{-1}$ for every closed point $x \in X$, and moreover $\Lambda(L)_{|\{e\}\times X} \cong \mathcal{O}_X$. Then by the universal property of the dual abelian variety, this corresponds to a morphism

$$\phi_L \colon X \longrightarrow X^\vee$$

Proposition 3.12. *The morphism ϕ_L is an homomorphism of abelian varieties. Moreover, if L is an ample bundle, it is an isogeny.*

Theorem 3.2 (Properties of the dual)**.**

1. *The Poincarè bundle is symmetric, meaning that $(X, \mathcal{P})$ is canonically the dual abelian variety of $X^\vee$.*
2. *If $f \colon X \longrightarrow Y$ is a homomorphism of abelian varieties, then it induces a homomorphism of abelian varieties $f^\vee \colon Y^\vee \longrightarrow X^\vee$ that on geomtric points corresponds to the pullback.*
3. *If $f \colon X \longrightarrow Y$ is an isogeny, then $f^\vee$ is an isogeny as well.*
4. *Let $f \colon X \longrightarrow Y$ be a morphism and let L be a line bundle on Y. Then $\phi_{f^*L} = f^\vee \circ \phi_L \circ f$.*

Proof. See [Mum08],[EMvdG] or [Lin]. □

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