Faltings height

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1 Notations

We use the following notations:

- K will denote a number field, R its ring of integers and $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group.
- for every group scheme G over a base S we denote by $s \colon S \longrightarrow G$ the identity sections. We use the same letter s for different group schemes, but this should not cause confusion.
- if A/K is an abelian variety, we denote by $\mathcal{A}$ its **Néron model** and by $\mathcal{A}^0$ the connected component of the identity. Recall from Eva's talk [Mar] that $\mathcal{A}$ is a smooth and separated group scheme over R such that
  (i) There is an identification $\mathcal{A}_K = A$.
  (ii) (**Néron mapping property**): for every smooth scheme S over R and the restriction to the generic fiber induces an isomorphism

$$\mathrm{Hom}_R(S, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_K(S_K, A)$$

- For every $t \in \mathrm{Spec}\ R$ we set $A_t \stackrel{\text{def}}{=} \mathcal{A}^0_t$. Recall from Eva's talk [Mar] that for every $t \in \mathrm{Spec}\ R$ we have the Chevalley decomposition

$$1 \longrightarrow A_t^{\mathrm{aff}} \longrightarrow A_t \longrightarrow A_t^{\mathrm{ab}} \longrightarrow 0$$

  that is functorial in A. We say that A is **semistable** or that it has **semistable reduction** if A_t^{aff} is a torus for every $t \in \mathrm{Spec}\ R$.

2 Motivation

Let A/K be an abelian variety. Then, for a prime ℓ we have defined the Tate module:

$$T_\ell A \stackrel{\text{def}}{=} \varprojlim_n A[\ell^n](\overline{K}) \qquad V_\ell A \stackrel{\text{def}}{=} T_\ell A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$$

During Niels' talk [Lin] we have conditionally proven these two theorems:

Theorem 2.1 (**Tate's conjecture**). *Let A and B be two abelian varieties over K and let ℓ be a prime. Then the natural map*

$$Hom(A, B) \otimes \mathbb{Z}_\ell \longrightarrow Hom_{\mathbb{Z}[\Gamma_K]}(T_\ell A, T_\ell B)$$

is an isomorphism.

Theorem 2.2 (**Semisimplicity Theorem**). *Let A be an abelian variety over K and let ℓ be a prime. Then the action of Γ_K on $V_\ell A$ is semisimple.*

What we needed for the proofs was this:

Lemma 2.1 (Key Lemma). *Let A be an abelian variety over K and ℓ a prime. Let $W \subseteq V_\ell(A)$ be a Γ_K-invariant subspace and for every $n > 0$ define the subgroups*

$$G_n = W \cap T_\ell(A)/\ell^n(W \cap T_\ell(A)) \hookrightarrow T_\ell(A)/\ell^n(T_\ell(A)) \cong A[\ell^n](\overline{K})$$

(these are actually finite subgroup schemes of A). Then the quotients A/G_n are only finitely many, up to isomorphism.

The strategy to prove this is to find an **height function** for abelian varieties, i.e. a function

$$h\colon \{ \text{ abelian varieties over } K \} \longrightarrow \mathbb{R}$$

such that

(1) the set $\{ A \mid \dim A = g,\ h(A) < N \}$ is finite up to isomorphism.

(2) h is bounded on A/G_n.

In this talk we are going to present a notion of height for abelian varieties due to Faltings, named (unsurprisingly) the **Faltings height**. But first we make a reduction:

2.1 Reduction to the semistable case

In Eva's talk [Mar], we have seen the following fundamental result about semistability:

Theorem 2.3. *Let A be an abelian variety over K.*

(i) there exists a finite field extension K'/K such that $A_{K'}$ has semistable reduction.

(ii) if A has semistable reduction, then for any finite field extension K'/K with ring of integers R', if we denote by $\mathcal{A}'$ the Neron model of $A_{K'}$, then $\mathcal{A}'^0 = \mathcal{A}^0_{R'}$.

Let's study the behaviour of semistability under isogenies:

Proposition 2.1. *Let $f\colon A \longrightarrow B$ be an isogeny between two abelian varieties over K and let $F\colon \mathcal{A}^0 \longrightarrow \mathcal{B}^0$ be the corresponding map between the Néron models. Suppose that A has semistable reduction, then:*

(i) B has semistable reduction.

(ii) $F\colon \mathcal{A}^0 \longrightarrow \mathcal{B}^0$ is a flat and surjective morphism whose kernel $\mathcal{G}$ is a flat, quasi-finite and separated group scheme over R.

(iii) if $d = \deg f$, then $\mathcal{G}$ is killed by d.

Proof. (i) Let $f\colon A \longrightarrow B$ be an isogeny. Then we know that there is another isogeny $g\colon B \longrightarrow A$ such that $f \circ g = [d]_B$ and $g \circ f = [d]_A$ for $d = \deg f$. Then if we denote by $F\colon \mathcal{A}^0 \longrightarrow \mathcal{B}^0$ and $G\colon \mathcal{B}^0 \longrightarrow \mathcal{A}^0$ the extensions of f and g to the Néron models, we see that $F \circ G = [d]_{\mathcal{A}^0}$ and $G \circ F = [d]_{\mathcal{B}^0}$. Now consider for every $t \in T$ the induced maps on the Chevalley decompositions

$$\begin{tikzcd}
1 \arrow[r] & A_t^{\mathrm{aff}} \arrow[r] \arrow[d, "F_t^{\mathrm{aff}}"'] & A_t \arrow[r] \arrow[d, "F_t"'] & A_t^{\mathrm{ab}} \arrow[r] \arrow[d, "F_t^{\mathrm{ab}}"'] & 0 \\
1 \arrow[r] & B_t^{\mathrm{aff}} \arrow[r] \arrow[d, "G_t^{\mathrm{aff}}"'] & B_t \arrow[r] \arrow[d, "G_t"'] & B_t^{\mathrm{ab}} \arrow[r] \arrow[d, "G_t^{\mathrm{ab}}"'] & 0 \\
1 \arrow[r] & A_t^{\mathrm{aff}} \arrow[r] & A_t \arrow[r] & A_t^{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

On the abelian parts we see that

$$F_t^{ab} \circ G_t^{ab} = [d]_{B_t^{ab}} \qquad G_t^{ab} \circ F_t^{ab} = [d]_{A_t^{ab}}$$

and the first relation tells us that F_t^{ab} is surjective, since $[n]_{B_t^{ab}}$ is surjective on an abelian variety, whereas the second tells us that $\operatorname{Ker} F_t^{ab} \subseteq \operatorname{Ker} [n]_{A_t^{ab}}$, so that $\operatorname{Ker} F_t^{ab}$ is finite. This proves that F_t^{ab} is an isogeny and the same is true for G_t^{ab}. In particular, this tells us that $\dim A_t^{ab} = \dim B_t^{ab}$ and moreover we see that

$$\dim A_t = \dim A_K = \dim A = \dim B = \dim B_K = \dim B_t$$

by smoothness of the Néron models, so then see that $\dim A_t^{\text{aff}} = \dim B_t^{\text{aff}}$ as well. Now, suppose that B_t^{aff} has a nontrivial unipotent part U: then we can write $B_t^{\text{aff}} = U \times T$ (since it is a commutative affine group over a perfect field [Con, Proposition 2.16]) and since there are no nontrivial homomorphisms between a torus and an unipotent group [Con, Lemma 2.15], we see that $\operatorname{Im} F_t^{\text{aff}} \subseteq T$, that is of dimension strictly less than $\dim B_t^{\text{eff}} = \dim A_t^{\text{eff}}$. However this is absurd since

$$G_t^{\text{aff}} \circ F_t^{\text{aff}} = [d]_{A_t^{\text{aff}}}$$

and $[n]_{A_t^{\text{aff}}}$ is surjective, as A_t^{aff} is a torus.

(ii) Observe that

$$F_t^{\text{aff}} \circ G_t^{\text{aff}} = [d]_{B_t^{\text{aff}}}$$

is surjective, since B_t^{aff} is a torus, and then F_t^{aff} is surjective as well. This proves that F_t is surjective, and we also know that it is flat (by generic flatness and translations) and finite, since $\dim A_t = \dim B_t$. Since this holds for every t, we see that F is clearly surjective and it is also flat by the Fiberwise Friterion for Flatness [TS15, Lemma 36.13.3].

Then we know immediately that $\operatorname{Ker} F$ is flat, and since for every $t \in \operatorname{Spec} R$ we have $(\operatorname{Ker} F)_t = \operatorname{Ker} F_t$, this shows also that $\operatorname{Ker} F$ is quasifinite (in general it is not finite: these fibers could be varying order in general, and $\operatorname{Ker} F$ is flat). Since $\mathcal{A}^0$ and $\mathcal{B}^0$ are separated over R, it also follows that $\operatorname{Ker} F$ is separated over R.

(iii) This follows immediately from the fact that $\mathcal{G} \subseteq \operatorname{Ker} [d]_{\mathcal{A}^0}$.

□

Now we make our reduction: first we need a lemma from Representation Theory:

Lemma 2.2. *Let G be an abstract group and let $H \triangleleft G$ be a normal subgroup of finite index. Then for a G-module V over a field k of characteristic zero, we have that V is semisimple as a G-module if and only if V is semisimple as an H-module.*

Proof. If V is semisimple as a G-module then it is also semisimple as an H module by Clifford's Theorem. The converse is an exercise in Representation Theory □

Lemma 2.3. *If Theorems 2.1 and 2.2 hold for a finite extension K'/K, then they hold for K as well.*

Proof. First observe that the Tate's Conjecture follows from the Semisimplicity Theorem (see for example [Lin]), so that it is enough to consider the latter. Let $K'' \supseteq K$ be a finite Galois extension containing K' and consider the absolute Galois groups $\Gamma_{K'}$ and $\Gamma_{K''}$: we know that $\Gamma_{K''}$ is a normal subgroup of finite index in both $\Gamma_{K'}$, so that $V_\ell(A)$ is a semisimple $\Gamma_{K''}$-module from the previous lemma, but again from the previous lemma it follows that $V_\ell(A)$ is a semisimple Γ_K-module. □

This, together with Theorem 2.3 (i) and Proposition 2.1 (i), shows that we we can look at an height function defined just over semistable abelian varieties

$$h\colon \{\, \text{semistable abelian varieties over } K \,\} \longrightarrow \mathbb{R}$$

so that in the following *all our abelian varieties will be semistable.*

3 Faltings height

The idea of height function is to give a measure of the arithmetic complexity of an object. Let's look at a stupid example:

Example 3.1. Write every rational number $r \in \mathbb{Q}$ as $r = \frac{a}{b}$, for a, b coprime integers. Then we define the height of r as

$$H(r) = \max\{\, |a|, |b| \,\}$$

In this way we have defined a function $H \colon \mathbb{Q} \longrightarrow \mathbb{R}_{>0}$ and moreover for every $N \in \mathbb{N}$ we have that

$$\{\, r \in \mathbb{Q} \mid H(r) < N \,\} \text{ is finite}$$

3.1 Height of a metrized line bundle

First recall that a **place** on R is an equivalence class of absolute values $v \colon K \longrightarrow \mathbb{R}$. Every place in R corresponds to one of the following normalized places:

- **finite** places: for every maximal ideal $\mathfrak{p} \in \operatorname{Spec} R$ we define
$$|\cdot|_{\mathfrak{p}} \colon K \longrightarrow \mathbb{R} \qquad |a|_{\mathfrak{p}} = [R : \mathfrak{p}]^{-\operatorname{ord}_{\mathfrak{p}}(a)}$$
- **infinite real** places: for every real embedding $j \colon K \hookrightarrow \mathbb{R}$ we define
$$|\cdot|_j \colon K \longrightarrow \mathbb{R} \qquad |a|_j = |j(a)|$$
where the absolute value on the right is the usual absolute value on $\mathbb{R}$.
- **infinite complex** places: for every couple of conjugate non-real embeddings $j, \bar{j} \colon K \hookrightarrow \mathbb{C}$ we define
$$|\cdot|_{j,\bar{j}} \colon K \longrightarrow \mathbb{R} \qquad |a|_{j,\bar{j}} = |j(a)|^2$$

If v is a place, then we write $v \mid \infty$ to say that v is an infinite place and $v \nmid \infty$ to say that v is a finite place. Moreover, if $d \in \mathbb{Z}$, we write that $v \mid d$ to say that v is finite and that the corresponding maximal ideal $\mathfrak{p}$ divides d in R. If $v \mid \infty$, then we denote by K_v the completion of K w.r.t. v: we have that $K_v \cong \mathbb{R}, \mathbb{C}$ depending on v being real or complex.

We have the following result:

Proposition 3.1 (Product formula). *For every $a \in K$ it holds that*

$$\prod_{v \text{ place of } R} |a|_v = 1$$

Now we can define the notion of metrized line bundle on $\operatorname{Spec} R$: recall that there is a correspondence between fractional ideals of R in K, projective modules of rank 1 over R and line bundles on $\operatorname{Spec} R$, and in the following we will sometimes identify there objects.

Let M be such a line bundle and suppose that for every infinite place v of R we have a norm $\|\cdot\|_v$ on $M \otimes_R K_v$ (observe that this is a vector space, real or complex according to v).

Moreover, for every finite valuation v, let R_v be the localization of R at the prime corresponding to v. Then is a DVR and $M_v = M \otimes_R R_v$ is a projective module of rank 1 over R_v, hence it is free: write $M_v = R_v \cdot m_v$ for a certain $m_v \in M_v$. Then for every $m \in M$, $m \neq 0$ we denote by $\frac{m}{m_v}$ the unique element of K such that $m = \frac{m}{m_v} \cdot m_v$ in $M_v \otimes_{R_v} K$.

Definition 3.1 (Height of a metrized line bundle)**.** With notations as before, we define the number

$$H(M) \overset{\text{def}}{=} \frac{1}{\prod_{v \nmid \infty} \left| \frac{m}{m_v} \right|_v \prod_{v \mid \infty} \|m\|_v^{\varepsilon_v}} \qquad \varepsilon_v = \begin{cases} 1, & \text{if } v \text{ is real} \\ 2, & \text{if } v \text{ is complex} \end{cases}$$

where we can take any $m \in M, m \neq 0$. Then we define the **height** of M as

$$h(M) \overset{\text{def}}{=} \frac{1}{[K : \mathbb{Q}]} \log H(M)$$

Remark 3.1. The above is well-defined, i.e. it is independent of m and of the m_v.

Indeed, it is clear that the height is independent of the m_v, because choosing another generator m'_v changes $\frac{m}{m'_v}$ by multiplication with an invertible element. To see that it is independent of m, consider M as a fractional ideal of R: then every other element of M is of the form am for a certain $a \in K$. Now we see that

$$\prod_{v\nmid\infty}\left|\frac{am}{m_v}\right|_v \prod_{v\mid\infty}\|am\|_v = \prod_{v\nmid\infty}|a_v|\left|\frac{a}{m}\right|_v \prod_{v\mid\infty}|a|_v\|m\|_v \overset{\text{Prop 3.1}}{=} \prod_{v\nmid\infty}\left|\frac{m}{m_v}\right|_v \prod_{v\mid\infty}\|m\|_v$$

Remark 3.2. The above proof shows that to compute $h(M)$ we can choose any element of $K \otimes_R M$.

Remark 3.3. Suppose that M is actually free over R and that $M = R \cdot m$. Then fr every $v \nmid \infty$ as a generator of M_v we can take the element $m_v = \frac{m}{1}$ itself and we see that

$$H(M) = \frac{1}{\prod_{v\mid\infty}\|m\|_v}$$

so that

$$h(M) = -\frac{1}{[K:\mathbb{Q}]}\sum_{v\mid\infty}\varepsilon_v \log\|m\|_v$$

Lemma 3.1. *Let K'/K be a finite field extension with ring of integers R'. Then*

$$h(M \otimes_R R') = h(M)$$

with the natural structure of metrized line bundle on $M \otimes_R R'$.

Proof. Exercise in Algebraic Number Theory. □

Lemma 3.2. *Let M and N be two metrized line bundles. Then $h(M \otimes_R N) = h(M) + h(N)$ (with the natrural structure of metrized line bundle on $M \otimes_R N$).*

Proof. Easy check. □

3.2 Faltings height of an abelian variety

Now, for any abelian variety A over K of dimension g we have the Néron model $\mathcal{A}$ and the connected part $\mathcal{A}^0$. These are group schemes over R, with identity section

$$s\colon \operatorname{Spec} R \longrightarrow \mathcal{A}^0 \subseteq \mathcal{A}$$

Then, consider the relative canonical sheaf $\omega_{\mathcal{A}/R} = \Omega^g_{\mathcal{A}/R}$: this is a line bundle on $\mathcal{A}$ that pulls back to a line bundle $s^*\omega_{\mathcal{A}/R}$ on Spec R.

We want to make this into a metrized line bundle: observe that we have

$$s^*\omega_{\mathcal{A}/R} \otimes_R K = s^*\omega_{A/K} = H^0(A, \omega_{A/K})$$

so that, if we consider an infinite place v we see that

$$s^*\omega_{\mathcal{A}/R} \otimes_R K_v = H^0(A_{K_v}, \omega_{A_{K_v}/K_v})$$

and here we can put the norm

$$\|\cdot\|_v\colon H^0(A_{K_v}, \omega_{A_{K_v}/K_v}) \longrightarrow \mathbb{R} \qquad \|\omega\|_v \overset{\text{def}}{=} \left|\int_{A(\overline{K_v})} \omega \wedge \overline{\omega}\right|^{\frac{1}{2}}$$

Remark 3.4. Observe that $\overline{K_v} = \mathbb{C}$, and indeed fixing an embedding $K \hookrightarrow \mathbb{C}$ makes A into an abelian variety over $\mathbb{C}$. Under this point of view, the norm corresponds more ore less to the volume of A. For example, when $A = \mathbb{C}/\Gamma$ is an elliptic curve, taking the holomorphic differential dz we see that

$$\left|\int_{\mathbb{C}/\Gamma} dz \wedge d\bar{z}\right| = \left|\int_{\mathbb{C}/\Gamma} (dx + idy) \wedge (dx - idy)\right| = \left|\int_{\mathbb{C}/\Gamma} -2idx \wedge dy\right| = 2\left|\int_{\mathbb{C}/\Gamma} dx \wedge dy\right| = 2 \cdot \mathrm{Vol}(\mathbb{C}/\Gamma)$$

Definition 3.2 (Faltings height)**.** We define the **Faltings height** of the abelian variety A/K as

$$h(A) = h(s^*\omega_{\mathcal{A}/R}) = \frac{1}{[K:\mathbb{Q}]} \log H(s^*\omega_{\mathcal{A}/R})$$

Lemma 3.3. *The Faltings height is unchanged under finite field extensions.*

Proof. Let A be a semistable abelian variety defined over a number field K and let K'/K be a finite field extension with ring of integers R'. If $\mathcal{A}'$ is the Néron model of $\mathcal{A}_{K'}$, then we know from Theorem 2.3 that $\mathcal{A}'^0 = \mathcal{A}^0_{R'}$. This implies that $s^*\omega_{\mathcal{A}'/R'} = s^*\omega_{\mathcal{A}/R} \otimes_R R'$ so that we can conclude by Lemma 3.1. □

Remark 3.5. The definition of Faltings height makes sense for any abelian variety over K, not necessarily semistable. however, it is not true that it is invariant under finite field extensions, since the Néron model could behave badly. That's why for any abelian variety A/K one defines the **stable Faltings height** of A as

$$h_F(A) = h(A_{K'})$$

where K'/K is any finite extension such that $A_{K'}$ has semistable reduction. However since all our abelian varieties are semistable, we will not consider this notion.

3.3 Reduction to the principally polarized case

Recall that we want to prove the following two facts:

(1′) up to isomorphism, there are finitely many semistable abelian varieties A/K of dimension g such that $h(A) < N$.

(2′) if A/K is a semistable abelian variety then $\{ h(A/G_n) \mid n > 0 \}$ is bounded.

Now we want to show how to reduce to the case of principally polarized abelian varieties:

Proposition 3.2. *Let A, B be two abelian varieties over K. Then*

$$h(A \times B) = h(A) + h(B)$$

Proof. This follows from the fact that the Néron model of $A \times B$ is the product of the Néron models $\mathcal{A} \times \mathcal{B}$. So that

$$s^*\omega_{\mathcal{A}\times\mathcal{B}/R} = s^*(\omega_{\mathcal{A}/R} \boxtimes \omega_{\mathcal{B}/R}) = s^*\omega_{\mathcal{A}/R} \otimes s^*\omega_{\mathcal{B}/R}$$

and this is an isomorphism of metrized line bundles, so that we conclude by Lemma 3.2. □

Proposition 3.3. *Let A/K be an abelian variety. Then*

$$h(A) = h(A^\vee)$$

Proof. See [FWG+92, Proposition III.3.7]. □

Corollary 3.1. *Suppose that the following holds:*

(1 ″) up to isomorphism of polarized abelian varieties, there are finitely many semistable principally polarized abelian varieties A/K of dimension g such that $h(A) < N$.

Then (1′) above holds as well.

Proof. Recall from Gregor's talk [Bru, Theorem 6.2] that if A/K is an abelian variety that admits a principal polarization, then up to isomorphism (of polarized varieties) there are only finitely many principally polarized abelian varieties (A, L). So that it is enough to prove that the statement

(1 ″′) up to isomorphism, there are finitely many semistable principally polarizable abelian varieties A/K of dimension g such that $h(A) < N$.

implies our statement (1′). Recall again from Gregor's talk [Bru, Theorem 4.1], that if A is any abelian variety, then $(A \times A^\vee)^4$ is principally polarized. Moreover we see from Proposition 3.3 that $h((A \times A^\vee)^4) = 8 \cdot (A)$ and $\dim(A \times A^\vee)^4 = 8 \cdot \dim A$. Then to conclude it is enough to show that up to isomorphism there are finitely many abelian variety A/K such that $(A \times A^\vee)^4 \cong Z$ for a fixed Z. To prove this, it is enough to prove that $(A \times A^\vee)^4$ has finitely many factors, up to isomorphism, but this is true by [Bru, Theorem 6.4]. □

4 The isogeny formula

Now we want to give a formula for how the Faltings height changes under isogeny: let $f\colon A \longrightarrow B$ be an isogeny between abelian variety over K and let $F\colon \mathcal{A}^0 \longrightarrow \mathcal{B}^0$ be the induced map on the Néron models.

Proposition 4.1 (Isogeny formula). *With the above notations, we have that*

$$h(B) - h(A) = \frac{1}{2}\log(\deg f) - \frac{1}{[K:\mathbb{Q}]}\log(|s^*\Omega^1_{\mathcal{G}/R}|)$$

Proof. We split the proof in two parts:

(a) First one checks that both sides of the formula are invariant under finite field extensions K'/K. Then, extending to the Hilbert class field of K, we can assume that $s^*\omega_{\mathcal{A}/R}$ and $s^*\omega_{\mathcal{B}/R}$ are both *free* over R: say that $s^*\omega_{\mathcal{A}/R} = R \cdot \omega_A$ and $s^*\omega_{\mathcal{B}/R} = R \cdot \omega_B$ for certain differentials ω_A and ω_B on A and B respectively. In particular, we know from Remark 3.3 that

$$h(A) = -\frac{1}{[K:\mathbb{Q}]}\sum_{v|\infty}\varepsilon_v \log\left|\int_{A(\overline{K_v})}\omega_A \wedge \overline{\omega_A}\right|^{\frac{1}{2}} = -\frac{1}{2[K:\mathbb{Q}]}\sum_{v|\infty}\varepsilon_v \log\left|\int_{A(\overline{K_v})}\omega_A \wedge \overline{\omega_A}\right|$$

and the same for $h(B)$.

Now fix an infinite place $v \mid \infty$ and a corresponding embedding $j_v\colon K \hookrightarrow \mathbb{C}$: if we look at $A(\overline{K_v})$ and $B(\overline{K_v})$ as complex tori, we see that

$$\int_{B(\overline{K_v})}\omega_B \wedge \overline{\omega_B} = \int_{f(A(\overline{K_v}))}\omega_B \wedge \overline{\omega_B} = \frac{1}{(\deg f)}\int_{A(\overline{K_v})} f^*\omega_B \wedge \overline{f^*\omega_B}$$

Since $f^*\omega_B \in R \cdot \omega_A$, we can write $f^*\omega_B = a \cdot \omega_A$ for a certain $a \in R$. Then we see that

$$\frac{1}{(\deg f)}\int_{A(\overline{K_v})} f^*\omega_B \wedge \overline{f^*\omega_B} = \frac{j_v(a)\cdot\overline{j_v(a)}}{(\deg f)}\int_{A(\overline{K_v})}\omega_A \wedge \overline{\omega_A} = \frac{|j_v(a)|^2}{(\deg f)}\int_{A(\overline{K_v})}\omega_A \wedge \overline{\omega_A}$$

Now we can compute

$$\begin{aligned}
2[K:\mathbb{Q}](h(B) - h(A)) &= \sum_{v|\infty}\varepsilon_v\left[-\log\left|\int_{B(\overline{K_v})}\omega_B \wedge \overline{\omega_B}\right| + \log\left|\int_{A(\overline{K_v})}\omega_A \wedge \overline{\omega_A}\right|\right] \\
&= \sum_{v|\infty}\varepsilon_v\left[-2\log|j_v(a)| + \log(\deg f)\right] \\
&= [K:\mathbb{Q}]\log(\deg f) - 2\log\prod_{v|\infty}|j_v(a)|^{\varepsilon_v} \\
&= [K:\mathbb{Q}]\log(\deg f) - 2\log|\mathrm{Norm}_{K/\mathbb{Q}}(a)|
\end{aligned}$$

(b) To conclude, we just need to show that $|\mathrm{Norm}_{K/\mathbb{Q}}(a)| = |s^*\Omega^1_{\mathcal{G}/R}|$. Consider the canonical exact sequence of sheaves on $\mathcal{A}^0$

$$F^*\Omega^1_{\mathcal{B}^0/R} \longrightarrow \Omega^1_{\mathcal{A}^0/R} \longrightarrow \Omega^1_{\mathcal{A}^0/\mathcal{B}^0} \longrightarrow 0$$

Then pulling back to Spec R we get (notice that $F \circ s = s$):

$$s^*\Omega^1_{\mathcal{B}^0/R} \longrightarrow s^*\Omega^1_{\mathcal{A}^0/R} \longrightarrow s^*\Omega^1_{\mathcal{A}^0/\mathcal{B}^0} \longrightarrow 0$$

now, if we consider the generic point Spec K we see that $s^*\Omega^1_{\mathcal{A}^0/\mathcal{B}^0} \otimes_R K = \Omega^1_{A/B} = 0$ (indeed, the isogeny $f\colon A \longrightarrow B$ is etale since K has characteristic zero). This shows that $s^*\Omega^1_{\mathcal{A}^0/\mathcal{B}^0}$ has finite support, and then we can consider it as a finite abelian group: looking at $s^*\Omega^1_{\mathcal{B}^0/R}$ and $s^*\Omega^1_{\mathcal{A}^0/R}$ as free $\mathbb{Z}$-modules , we see that

$$\begin{aligned} |s^*\Omega_{\mathcal{A}^0/\mathcal{B}^0}| &= |\det_{\mathbb{Z}}(s^*\Omega^1_{\mathcal{B}^0/R} \longrightarrow s^*\Omega^1_{\mathcal{A}^0/R})| \\ &= |\det_{\mathbb{Z}}(s^*\omega_{\mathcal{B}^0/R} \longrightarrow s^*\omega_{\mathcal{A}^0/R})| \\ &= |\det_{\mathbb{Z}}(R \xrightarrow{\cdot a} R)| = |\mathrm{Norm}_{K/\mathbb{Q}}(a)| \end{aligned}$$

Now, to conclude it's enough to show that $s^*\Omega^1_{\mathcal{A}^0/\mathcal{B}^0} \cong s^*\Omega^1_{\mathcal{G}/R}$ but this follows immediately by base change.

□

Now we want to study better the term $|s^*\Omega^1_{\mathcal{G}/R}|$: observe that it has finite support on Spec R, by the above proof, and we would like to identify this support.

Lemma 4.1. *Let k be a field of characteristic $p > 0$ and let G be a finite group scheme over k, killed by d. Suppose that $p \nmid d$, then G is étale.*

Proof. Recall [Ago, Proposition 2.2] that we have an exact sequence of group schemes

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 1$$

so that it is enough to prove that $G^0 = 0$. We see that G^0 is again killed by d and moreover we know [Ago, Proposition 2.4] that its rank is of the form p^r for a certain $r \geq 0$. In particular p^r and d are coprime, so that we can find $a, b \in \mathbb{Z}$ such that $ad + bp^r = 1$: however, we also know that G^0 is killed by p^r [Ago, Corollary 2.4], so that it must be $G^0 = 1$. □

Proposition 4.2. *With notations as in the previous section, let $d = \deg f$, and for every place v above d let R_v be the localization of R at v, $\widehat{R_v}$ its completion and $\widehat{\mathcal{G}_v} = \mathcal{G} \otimes_R \widehat{R_v}$. Then*

(i) the support of $s^*\Omega^1_{\mathcal{G}/R}$ lies over the complement of Spec R_d.

(ii) we have that

$$|s^*\Omega^1_{\mathcal{G}/R}| = \prod_{v|d} |s^*\Omega^1_{\widehat{\mathcal{G}_v}/\widehat{R_v}}|$$

Proof. 1. This follows immediately from the previous Lemma 4.1, once we recall that $\mathcal{G}$ is killed by d (Proposition 2.1).

2. From the previous point it follows that $s^*\Omega^1_{\mathcal{G}/R} = \bigoplus_{v|d} s^*\Omega^1_{\mathcal{G}/R} \otimes_R R_v$. However, since $s^*\Omega^1_{\mathcal{G}/R} \otimes_R R_v$ has finite length we see that $s^*\Omega^1_{\mathcal{G}/R} \otimes R_v = (s^*\Omega^1_{\mathcal{G}/R} \otimes R_v) \otimes_{R_v} \widehat{R_v} = s^*\Omega^1_{\widehat{\mathcal{G}_v}/\widehat{R_v}}$ □

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