The Hitchin fibration in the Langlands program

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Notations

We consider:

- an algebraically closed field k,
- a positive integer r,
- a smooth, connected, projective curve X of genus g over k,
- a divisor $D = \sum_x d_x[x]$ on X that we assume effective ($d_x \geq 0, \ \forall x$) and of degree $d = \sum_x d_x \geq 2g - 2$,
- the support $|D| = \{x \in X \mid d_x \neq 0\}$ of D.

Hitchin only considers the case where $g > 0$ and D is a canonical divisor ($\mathcal{O}_X(D) = \Omega^1_X$).

The Hitchin algebraic stack

A Hitchin bundle is a pair $(\mathcal{E}, \theta)$ where:

- $\mathcal{E}$ is a rank r vector bundle on X,
- $\theta : \mathcal{E} \to \mathcal{E}(D) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is a twisted endomorphism (an endomorphism of the restriction of $\mathcal{E}$ to $X - |D|$ whose order of pole at any $x \in |D|$ is at most d_x).

The rank r vector bundles on X are parametrized by an algebraic stack (in the sense of Artin) $\mathcal{M}$.

Similarly, the Hitchin bundles of rank r over X are parametrized by an algebraic stack $\mathcal{N}$ and we have a morphism of algebraic stacks $\mathcal{N} \to \mathcal{M}$ which maps $(\mathcal{E}, \theta)$ to $\mathcal{E}$.

Basic properties of the Hitchin stack

The connected components of $\mathcal{M}$ and $\mathcal{N}$ are indexed by $\mathbb{Z}$: for each $e \in \mathbb{Z}$, the corresponding connected components $\mathcal{M}^e$ and $\mathcal{N}^e$ are the algebraic stacks of vector bundles $\mathcal{E}$ and Hitchin bundles $(\mathcal{E}, \theta)$ with $\deg(\mathcal{E}) = e$.

The algebraic stacks $\mathcal{M}^e$ and $\mathcal{N}^e$ are not of Deligne-Mumford type and are highly non separated: the automorphism groups are affine but not finite in general.

The algebraic stacks $\mathcal{M}^e$ and $\mathcal{N}^e$ are locally of finite type but are not quasi-compact.

By deformation theory it is easy to prove that the algebraic stacks $\mathcal{M}^e$ are smooth of dimension $r^2(g-1)$ over k.

Hitchin Fibration

Hitchin has introduced the affine scheme

$$\mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X(iD)).$$

Definition

The Hitchin fibration is the stack morphism

$$f : \mathcal{N} \to \mathcal{A}$$

which maps $(\mathcal{E}, \theta)$ to the characteristic polynomial

$$a = (-\mathrm{tr}(\theta), \ldots, (-1)^r \det(\theta))$$

of θ.

Smoothness of the generically regular semisimple locus

The stalk $\theta_\eta : \mathcal{E}_\eta \to \mathcal{E}_\eta$ of θ at the generic point η of the curve X is an endomorphism of a rank r vector space over $\kappa(\eta)$.

We may require that θ_η is regular semisimple, or equivalently that the discrimant of the characteristic polynomial of θ_η is non zero.

This last condition defines an open subset $\mathcal{A}^{\mathrm{reg}}$ of $\mathcal{A}$.

By deformation theory it is easy to prove:

Theorem

For any e the open substack

$$f^{-1}(\mathcal{A}^{\mathrm{reg}}) \cap \mathcal{N}^e \subset \mathcal{N}^e$$

is smooth of dimension $r^2 d$ over k.

Function fields

- Let us assume that k is the algebraic closure of a finite field $\mathbb{F}_q$ and that X is defined over $\mathbb{F}_q$.
- Let $F = \mathbb{F}_q(X)$ be the function field X. For every closed point x of X, let $\mathcal{O}_x$ be the completion of the local ring of X at x and $F_x = \mathrm{Frac}(\mathcal{O}_x)$.
- Weil has introduced the ring of adeles of F:

$$\mathbb{A} = \{a \in \prod_x F_x \mid a_x \in \mathcal{O}_x \text{ for all but finitely many } x\text{'s}\}.$$

- $\mathbb{A}$ contains $\mathcal{O} = \prod_x \mathcal{O}_x$ as a compact open maximal subring.
- As any rationnal function on X has only finitely many poles, F can be diagonaly embedded in $\mathbb{A}$ as a discrete subring.

Vector bundles on curves over finite fields

If X is defined over $\mathbb{F}_q$, the same is true for $\mathcal{M}$ and its connected components $\mathcal{M}^e$.

Weil has given an adelic description of the category of $\mathbb{F}_q$-points of $\mathcal{M}^e$:

$$\mathcal{M}^e(\mathbb{F}_q) = [G(F)\backslash G(\mathbb{A})^e / G(\mathcal{O})].$$

Here $G = \mathrm{GL}(r)$ and $G(\mathbb{A})^e = \{g \in G(\mathbb{A}) \mid \mathrm{deg}(\mathrm{det}(g)) = -e\}$ and

$$\mathrm{deg} : \mathbb{A}^\times \to \mathbb{Z},\ a \mapsto \sum_x \mathrm{deg}(x) x(a_x)$$

is trivial on $\mathcal{O}^\times$ and on $F^\times$.

The Hitchin fibration over a finite field

Again, if X and D are defined over $\mathbb{F}_q$, the same is true for the Hitchin fibration $\mathcal{N} \xrightarrow{f} \mathcal{A}$ and we have an adelic description of the functor $f : \mathcal{N}(\mathbb{F}_q) \to \mathcal{A}(\mathbb{F}_q)$.

If $\mathfrak{g} = \mathrm{gl}(r)$ is the Lie algebra of G equipped with the adjoint action, and $\varpi^{-D} = (\varpi_x^{-d_x})_x \in \mathbb{A}^\times$ is any adele such that $x(\varpi^{-D}) = -d_x$ for every x, we have:

$$\mathcal{N}^e(\mathbb{F}_q) = [G(F)\backslash\{(g,\gamma) \in (G(\mathbb{A})^e/G(\mathcal{O})) \times \mathfrak{g}(F) \mid g^{-1}\gamma g \in \varpi^{-D}\mathfrak{g}(\mathcal{O})\}],$$

$$\mathcal{A}(\mathbb{F}_q) = \bigoplus_{i=1}^{r}(F \cap \varpi^{-iD}\mathcal{O})$$

and

$$f(g,\gamma) = \chi(\gamma) := (-\mathrm{tr}(\gamma), \ldots, (-1)^r \det(\gamma)).$$

Hitchin fibration and orbital integrals

For any $a \in \mathcal{A}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points of $f^{-1}(a)$ is (at least formally) a linear combination of global orbital integrals

$$\sum_{\substack{\gamma \\ \chi(\gamma)=a}} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})^e} \varphi(\mathrm{Ad}(g^{-1})(\gamma))\mathrm{d}g.$$

Here γ runs through a system of representatives of the adjoint orbits in $\mathfrak{g}(F)$ and φ is the characteristic function of $\varpi^{-D}\mathfrak{g}(\mathcal{O})$ in $\mathfrak{g}(\mathbb{A})$.

Any global orbital as above is essentially a product over the closed points x in X of the local orbital integrals

$$\int_{G_\gamma(F_x)\backslash G(F_x)} \varphi_x(\mathrm{Ad}(g_x^{-1})(\gamma))\mathrm{d}g_x$$

where φ_x is the characteristic function of $\varpi^{-d_x}\mathfrak{g}(\mathcal{O}_x)$ in $\mathfrak{g}(F_x)$

Other groups

By the Morita equivalence, a rank r vector bundle is nothing else than a G-torsor for $G = \mathrm{GL}(r)$, and all what we have said up to now make sense for an arbitrary connected reductive algebraic group G, for example $G = \mathrm{SL}(r)$, $\mathrm{Sp}(2r)$, $\mathrm{SO}(r)$, ..., E_8.

For simplicity, we restrict ourself here to split adjoint semisimple groups G over $\mathbb{F}_q$.

We fix a split maximal torus $T \subset G$. The Weyl group W_G of (G, T) acts on the Lie algebra $\mathfrak{t}$ of T. We may form the GIT quotient $\mathfrak{t}//W_G \cong \mathfrak{g}//G$.

For such G, we have the Hitchin fibration $f_G : \mathcal{N}_G \to \mathcal{A}_G$ with an adelic description as before. Here $\mathcal{A}_G$ may be viewed as the affine scheme of maps $a : X - |D| \to \mathfrak{t}//W_G$ whose order of pole at any $x \in X - |D|$ is at most d_x.

The elliptic part

For every proper standard Levi subgroup $T \subset M \subsetneq G$ we may consider the base $\mathcal{A}_M$ of the Hitchin fibration for M.

We have a morphism

$$i_M : \mathcal{A}_M \to \mathcal{A}_G$$

where $i_M(a_M) : X - |D| \to \mathfrak{t}//W_G$ is the composition of a_M and of the canonical projection $\mathfrak{t}//W_M \twoheadrightarrow \mathfrak{t}//W_G$.

By definition, the elliptic locus $\mathcal{A}_G^{\mathrm{ell}} \subset \mathcal{A}_G^{\mathrm{reg}}$ is the complementary open subset of the union over all the proper standard Levi subgroups M, of $i_M(\mathcal{A}_M) \cap \mathcal{A}_G^{\mathrm{reg}}$.

Theorem (Faltings)

The open substack $f_G^{-1}(\mathcal{A}_G^{\mathrm{ell}}) \subset \mathcal{N}_G$ is of Deligne-Mumford type and smooth over $\mathbb{F}_q$.

The morphism $f_G : f_G^{-1}(\mathcal{A}_G^{\mathrm{ell}}) \to \mathcal{A}_G^{\mathrm{ell}}$ is proper.

Langlands duality

Let $X^*(T)$ be the character group of our split maximal torus T in the split adjoint semisimple group G over $\mathbb{F}_q$.

The Langlands dual of (G, T) is the simply connected complex semisimple group $\hat{G}$, equipped with a maximal torus $\hat{T}$, which is defined in the following way:

- $\hat{T} = \mathbb{C}^\times \otimes X^*(T)$,
- the roots of $\hat{T}$ in $\hat{G}$ are the coroots of T in G and vice versa.

For example:

G	$\hat{G}$
$\mathrm{PSL}(r)$	$\mathrm{SL}(r)$
$\mathrm{PSp}(2r)$	$\mathrm{Spin}(2r+1)$
$\mathrm{SO}(2r+1)$	$\mathrm{Sp}(2r)$
$\mathrm{PO}(2r)$	$\mathrm{Spin}(2r)$

Endoscopic groups

For any $s \in \hat{T}$ we may consider the centralizer $Z_s(\hat{G})$ of s in $\hat{G}$:

- $Z_s(\hat{G})$ is a connected reductive group since $\hat{G}$ is simply connected,
- $Z_s(\hat{G})$ also admits $\hat{T}$ as a maximal torus.

The centralizer $Z_s(\hat{G})$ is no more semisimple in general, but Langlands duality makes sense for reductive groups.

Definition (Langlands)

For each $s \in \hat{T}$, the endoscopic group H_s of G is "the" reductive group over $\mathbb{F}_q$ admitting T as a maximal torus, whose dual is $Z_s(\hat{G})$.

The endoscopic group H_s is said elliptic if $Z_s(\hat{G})$ is not contained in any proper Levi subgroup of $\hat{G}$.

Endoscopic strata

If $H = H_s$ is an endoscopic group, the Weyl group W_H of (H, T) is naturally embedded in W_G since it is equal to the Weyl group of $(Z_s(\hat{G}), \hat{T}) \subset (\hat{G}, \hat{T})$.

Therefore, as for a Levi subgroup, we have a morphism

$$i_H : \mathcal{A}_H \to \mathcal{A}_G$$

where $i_H(a_H) : X - |D| \to \mathfrak{t}/\!/W_G$ is the composition of a_H and of the canonical projection $\mathfrak{t}/\!/W_H \twoheadrightarrow \mathfrak{t}/\!/W_G$.

The endoscopic group H is elliptic if and only if

$$i_H^{-1}(\mathcal{A}_G^{\mathrm{ell}}) \neq \emptyset.$$

Relative cohomology of the Hitchin fibration

Let us fix a prime number ℓ invertible in $\mathbb{F}_q$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

Over $\mathcal{A}_G^{\mathrm{ell}}$, we may consider the complex of ℓ-adic sheaves

$$Rf_{G,*}\overline{\mathbb{Q}}_\ell$$

and the direct sum of its perverse cohomology sheaves

$${}^{\mathrm{p}}\mathcal{H}^\bullet(f_G) := \bigoplus_i {}^{\mathrm{p}}\mathcal{H}^i(Rf_{G,*}\overline{\mathbb{Q}}_\ell)$$

As $f_G^{-1}(\mathcal{A}_G^{\mathrm{ell}})$ is smooth over $\mathbb{F}_q$ and $f_G : f_G^{-1}(\mathcal{A}_G^{\mathrm{ell}}) \to \mathcal{A}_G^{\mathrm{ell}}$ is proper, by Deligne's theorem we know that these perverse cohomology sheaves are all pure and geometrically semisimple.

Ngô's support theorem

Any irreducible perverse sheaf K on $\mathcal{A}_G^{\mathrm{ell}}$ has a support

$$\mathrm{Supp}(K) \subset \mathcal{A}_G^{\mathrm{ell}}$$

which is an irreducible closed subset. Moreover K is the intersection complex of $\mathrm{Supp}(K)$ with value in an irreducible ℓ-adic local system on a dense open subset of $\mathrm{Supp}(K)$.

Theorem (Ngô Bao Châu)

The support of any irreducible constituent of ${}^{\mathrm{p}}\mathcal{H}^\bullet(f_G)$ *is equal to* $\mathrm{A}_G^{\mathrm{ell}} \cap i_H(\mathcal{A}_H)$ *for some elliptic endoscopic group* H.

Moreover, the constituents with support $\mathrm{A}_G^{\mathrm{ell}} \cap i_H(\mathcal{A}_H)$ *can be computed explicitly in term of* ${}^{\mathrm{p}}\mathcal{H}^\bullet(f_H)$.

Fundamental Lemma

We may now consider the function "trace of Frobenius" of $Rf_{G,*}\overline{\mathbb{Q}}_\ell$.

Then, using Ngô's support theorem we immediately get a global version of the Langlands-Shelstad Fundamental Lemma for Lie algebras over functions fields.

A standard global-to-local argument gives the Langlands-Shelstad Fundamental Lemma for Lie algebras in the equal characteristic case.

Finally, thanks to the work of Waldspurger we get the Langlands-Shelstad Fundamental Lemma for groups in unequal characteristics.

Chaudouard-L.'s extension of Ngô's work

Outside the elliptic locus, the Hitchin fibration is neither of finite type nor separated. One first needs to truncate it. We do that by using Mumford stability.

Our truncated Hitchin fibration is again proper with a smooth total space, so that we can again apply Deligne's purity theorem.

Our key result is that Ngô's support theorem extends to the whole $\mathcal{A}_G^{\mathrm{reg}}$: *the support of any irreducible constituent of the relative cohomology of the truncated Hitchin fibration meets* $\mathcal{A}_G^{\mathrm{ell}}$.

As a consequence we get the weighted Fundamental Lemma which has been conjectured by Arthur.

The Langlands Correspondence

If F is a global field (i.e. a number field or a function field), the automorphic representations of a split reductive group G over F should be parametrized by homomorphisms $L_F \to \hat{G}$ where L_F is some suitable "extension" of $\mathrm{Gal}(\overline{F}/F)$.

For $G = \mathrm{GL}(1)$, the Langlands correspondence is nothing else than the abelian class field theory.

If F is a function field the correspondence has been established by Drinfeld for $G = \mathrm{GL}(2)$ and by Lafforgue for $G = \mathrm{GL}(r)$ with $r > 2$.

The geometric Langlands program

As before, let X be defined over a finite field $\mathbb{F}_q$ and F be its function field.

The automorphic forms are functions on $G(F)\backslash G(\mathbb{A})$. One way to produce such a function is to take the trace of Frobenius of a complex of ℓ-adic sheaves on the algebraic stack $\mathcal{M}_G$.

Similarly, one way to obtain an homomorphism $L_F \to \hat{G}$ is to take the generic fiber of an ℓ-adic $\hat{G}$-local system on X.

The work of Drinfeld and Gaitsgory

If X is now a complex algebraic curve, the Langlands correspondence suggests that there should be an equivalence between a certain derived category of $\mathcal{D}$-modules on $\mathcal{M}_G$ and a certain derived category of quasi-coherent sheaves on the moduli stack $\mathcal{L}_{\hat{G}}$ of $\hat{G}$-local systems on X.

In this geometric program the Hitchin fibration and especially its fiber at 0 (the global nilpotent cone) play a major role.

The Hausel and Rodriguez-Villegas conjecture

Finally, let me mention some joint work in progress with Pierre-Henri Chaudouard, the goal of which is a conjecture by Hausel and Rodriguez-Villegas.

Here we restrict ourself to $G = \mathrm{GL}(r)$.

The total stack of the Hitchin fibration might be truncated using Mumford stability, or equivalently in the same way than Arthur is truncating the trace formula.

Then it makes sense to count the number of points of the truncated stack over a finite field.

This counting can be reduced to the counting of points of the truncated global nilpotent cone.

Truncated nilpotent orbital integrals

For each Hichin bundle $(\mathcal{E}, \theta)$ with θ nilpotent, we may consider the Jordan type of θ_η.

The truncated global nilpotent cone can be stratified by this Jordan type.

The counting of points of the different strata amounts to compute the global truncated nilpotent orbital integrals which enter in Arthur's trace formula.

Up to now we have results for the regular nilpotent orbits and other particular cases.