# Scenario Reduction in Stochastic Programming

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## Introduction

Most approaches for solving stochastic programs of the form

$$\min \left\{ \int_\Xi f_0(x,\xi) P(d\xi) : x \in X \right\}$$

with a probability measure $P$ on $\Xi$ and a (normal) integrand $f_0$, require numerical integration techniques, i.e., replacing the integral by some quadrature formula

$$\int_\Xi f_0(x,\xi) P(d\xi) \approx \sum_{i=1}^{n} p_i f_0(x,\xi_i),$$

where $p_i = P(\{\xi_i\})$, $\sum_{i=1}^n p_i = 1$ and $\xi_i \in \Xi$, $i = 1, \ldots, n$.

Since $f_0$ is often expensive to compute, the number $n$ should be as small as possible. For the special case $p_i = \frac{1}{n}$, $i = 1, \ldots, n$, the best possible choice of elements $\xi_i \in \Xi$, $i = 1, \ldots, n$ (scenarios), for given $n$ is obtained by minimizing

$$\sup_{x \in X} \left| \int_\Xi f_0(x,\xi) P(d\xi) - \frac{1}{n} \sum_{i=1}^{n} f_0(x,\xi_i) \right|.$$

The latter optimization problem may be reformulated as a best approximation problem with respect to the (semi-) distance

$$d_{\mathcal{F}}(P,Q) := \sup_{f\in\mathcal{F}} \left| \int_{\Xi} f(\xi)(P-Q)(d\xi) \right|,$$

with $\mathcal{F} := \{f_0(x,\cdot) : x \in X\}$ and $Q$ varying in

$\mathcal{P}_n(\Xi) := \{Q : Q$ is a uniform probability measure, $|\mathrm{supp}(Q)| \leq n\}$.

Hence, it may also be reformulated as a semi-infinite program. It is also known as optimal quantization of $P$ with respect to the function class $\mathcal{F}$.

## Aim of the talk:

Solving the best approximation problem for function classes $\mathcal{F}$, which are relevant for mixed-integer two-stage stochastic programs.

**Additional motivation:** Scenario reduction methods are important for generating scenario trees.

# Linear two-stage stochastic programs

$$\min \left\{ \langle c, x \rangle + \int_{\Xi} \Phi(q(\xi), h(\xi) - T(\xi)x) P(d\xi) : x \in X \right\},$$

where $c \in \mathbb{R}^m$, $\Xi$ and $X$ are polyhedral subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, $P$ is a Borel probability measure on $\Xi$ and the $d \times m$-matrix $T(\cdot)$, vector $h(\cdot) \in \mathbb{R}^d$ are affine functions of $\xi$.

Furthermore, $\Phi$ and $D$ denote the infimum function of the linear second-stage program and its dual feasibility set, i.e.,

$$\begin{aligned} \Phi(u,t) &:= \inf\{\langle u, y \rangle : Wy = t, y \in Y\} \; ((u,t) \in \mathbb{R}^{\overline{m}} \times \mathbb{R}^d) \\ D &:= \{u \in \mathbb{R}^{\overline{m}} : \{z \in \mathbb{R}^d : W^\top z - u \in Y^*\} \neq \emptyset\}, \end{aligned}$$

where $q(\xi) \in \mathbb{R}^{\overline{m}}$ are the recourse costs, $W$ is the $d \times \overline{m}$ recourse matrix, $Y \subseteq \mathbb{R}^{\overline{m}}$ a polyhedral cone, $W^\top$ is the transposed of $W$ and $Y^*$ the polar cone of $Y$.

**Theorem:** (Walkup/Wets 69)
The function $\Phi(\cdot,\cdot)$ is finite and continuous on the polyhedral set $D \times W(Y)$. Furthermore, the function $\Phi(u,\cdot)$ is piecewise linear convex on the polyhedral set $W(Y)$ for fixed $u \in D$, and $\Phi(\cdot,t)$ is piecewise linear concave on $D$ for fixed $t \in W(Y)$.

**Assumptions:**

**(A1)** *relatively complete recourse*: for any $(\xi, x) \in \Xi \times X$, $h(\xi) - T(\xi)x \in W(Y)$;

**(A2)** *dual feasibility*: $q(\xi) \in D$ holds for all $\xi \in \Xi$.

**(A3)** *existence of second moments:* $\int_\Xi \|\xi\|^2 P(d\xi) < +\infty$.

Note that (A1) is satisfied if $W(Y) = \mathbb{R}^d$ (complete recourse). In general, (A1) and (A2) impose a condition on the support of $P$.

Recent extension to models with random recourse in Römisch-Wets 07.

## Scenario reduction

We consider discrete distributions $P$ with scenarios $\xi_i$ and probabilities $p_i$, $i = 1, \ldots, N$, and $Q$ being supported by a given subset of scenarios $\xi_j$, $j \notin J \subset \{1, \ldots, N\}$, of $P$.

Optimal reduction of a given scenario set $J$:
The best approximation of $P$ with respect to $\zeta_r$ by such a distribution $Q$ exists and is denoted by $Q^*$. It has the distance

$$\begin{aligned} D_J &:= \zeta_r(P, Q^*) = \min_Q \zeta_r(P, Q) = \sum_{i \in J} p_i \min_{j \notin J} \hat{c}_r(\xi_i, \xi_j) \\ &= \sum_{i \in J} p_i \min\{\sum_{k=1}^{n-1} c_r(\xi_{l_k}, \xi_{l_{k+1}}) : n \in \mathbb{N}, l_k \in \{1, \ldots, N\}, \\ &\qquad\qquad l_1 = i, l_n = j \notin J\} \end{aligned}$$

and the probabilities $q_j^* = p_j + \sum_{i \in J_j} p_i$, $\forall j \notin J$, where

$J_j := \{i \in J : j = j(i)\}$ and $j(i) \in \arg\min_{j \notin J} \hat{c}_r(\xi_i, \xi_j)$, $\forall i \in J$.

**Fortet-Mourier metrics:** $(r \geq 1)$

$$\zeta_r(P,Q) := \sup \left| \int_\Xi f(\xi)(P-Q)(d\xi) : f \in \mathcal{F}_r(\Xi) \right|,$$

where

$$\mathcal{F}_r(\Xi) := \{f : \Xi \mapsto \mathbb{R} : f(\xi) - f(\tilde{\xi}) \leq c_r(\xi,\tilde{\xi}),\ \forall \xi,\tilde{\xi} \in \Xi\},$$

$$c_r(\xi,\tilde{\xi}) := \max\{1, \|\xi\|^{r-1}, \|\tilde{\xi}\|^{r-1}\}\|\xi - \tilde{\xi}\| \quad (\xi,\tilde{\xi} \in \Xi).$$

**Proposition:** (Rachev/Rüschendorf 98)

$$\zeta_r(P,Q) = \inf\left\{ \int_{\Xi\times\Xi} \hat{c}_r(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) : \pi_1\eta = P, \pi_2\eta = Q \right\}$$

where $\hat{c}_r \leq c_r$ and $\hat{c}_r$ is the metric (reduced cost)

$$\hat{c}_r(\xi,\tilde{\xi}) := \inf\left\{ \sum_{i=1}^{n-1} c_r(\xi_{l_i},\xi_{l_{i+1}}) : n \in \mathbb{N}, \xi_{l_i} \in \Xi, \xi_{l_1} = \xi, \xi_{l_n} = \tilde{\xi} \right\}.$$

Determining the optimal scenario index set $J$ with prescribed cardinality $n$ is, however, a combinatorial optimization problem of set covering type:

$$\min \left\{ D_J = \sum_{i \in J} p_i \min_{j \notin J} \hat{c}_r(\xi_i, \xi_j) : J \subset \{1, \ldots, N\},\ |J| = N - n \right\}$$

Hence, the problem of finding the optimal set $J$ to delete is $\mathcal{NP}$-hard and polynomial time solution algorithms do not exist.

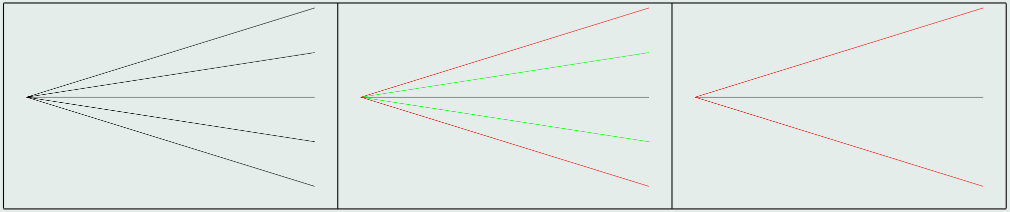
# Fast reduction heuristics

Starting point ($n = N-1$): $\min\limits_{l\in\{1,\dots,N\}} p_l \min\limits_{j\neq l} \hat{c}_r(\xi_l, \xi_j)$

**Algorithm 1:** (Backward reduction)

$$
\begin{aligned}
&\textbf{Step [0]:} && J^{[0]} := \emptyset\,.\\
&\textbf{Step [i]:} && l_i \in \arg\min_{l\notin J^{[i-1]}} \sum_{k\in J^{[i-1]}\cup\{l\}} p_k \min_{j\notin J^{[i-1]}\cup\{l\}} \hat{c}_r(\xi_k,\xi_j).\\
& && J^{[i]} := J^{[i-1]} \cup \{l_i\}\,.
\end{aligned}
$$

**Step [N-n+1]:** Optimal redistribution.



Starting point $(n=1)$: $\min\limits_{u \in \{1,\dots,N\}} \sum\limits_{k=1}^{N} p_k \hat{c}_r(\xi_k, \xi_u)$

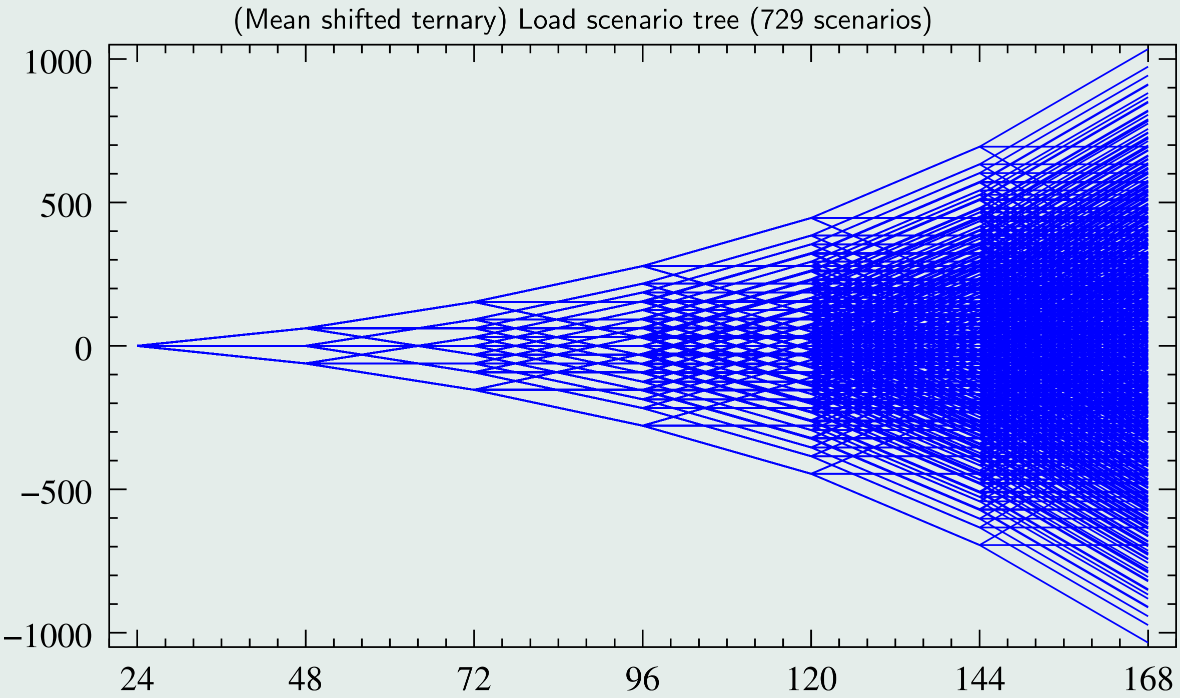
**Algorithm 2:** (Forward selection)

**Step [0]:** $J^{[0]} := \{1, \dots, N\}$.

**Step [i]:**
$$u_i \in \arg \min_{u \in J^{[i-1]}} \sum_{k \in J^{[i-1]} \setminus \{u\}} p_k \min_{j \notin J^{[i-1]} \setminus \{u\}} \hat{c}_r(\xi_k, \xi_j),$$
$$J^{[i]} := J^{[i-1]} \setminus \{u_i\}.$$

**Step [n+1]:** Optimal redistribution.

# **Example:** (Electrical load scenario tree)

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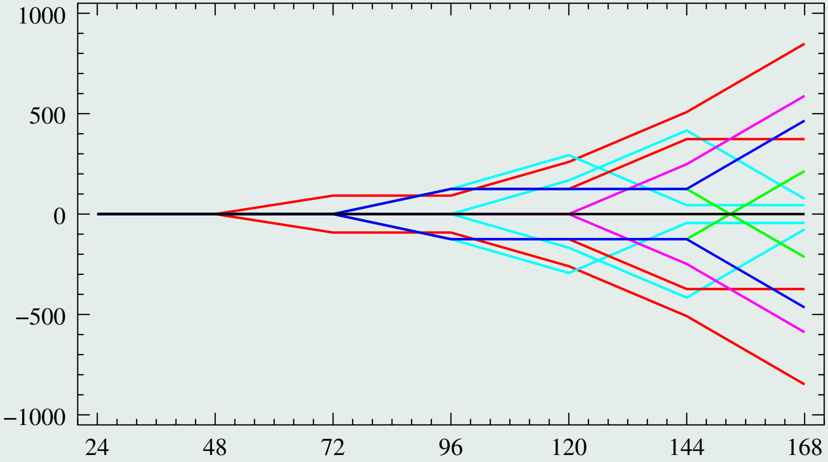
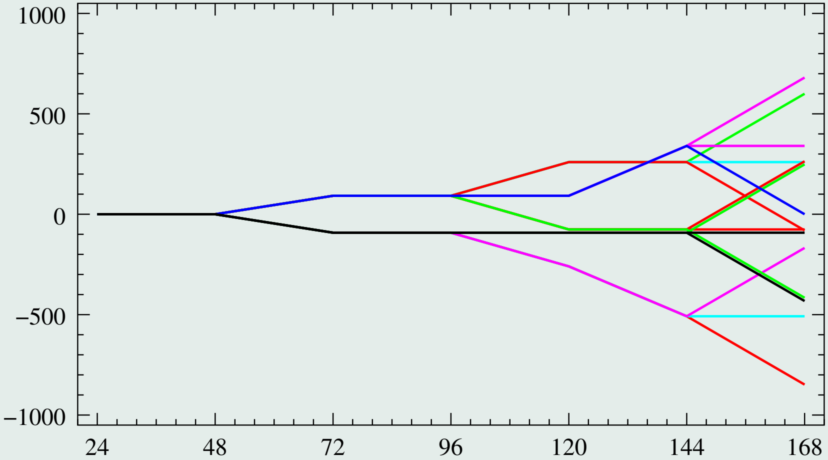
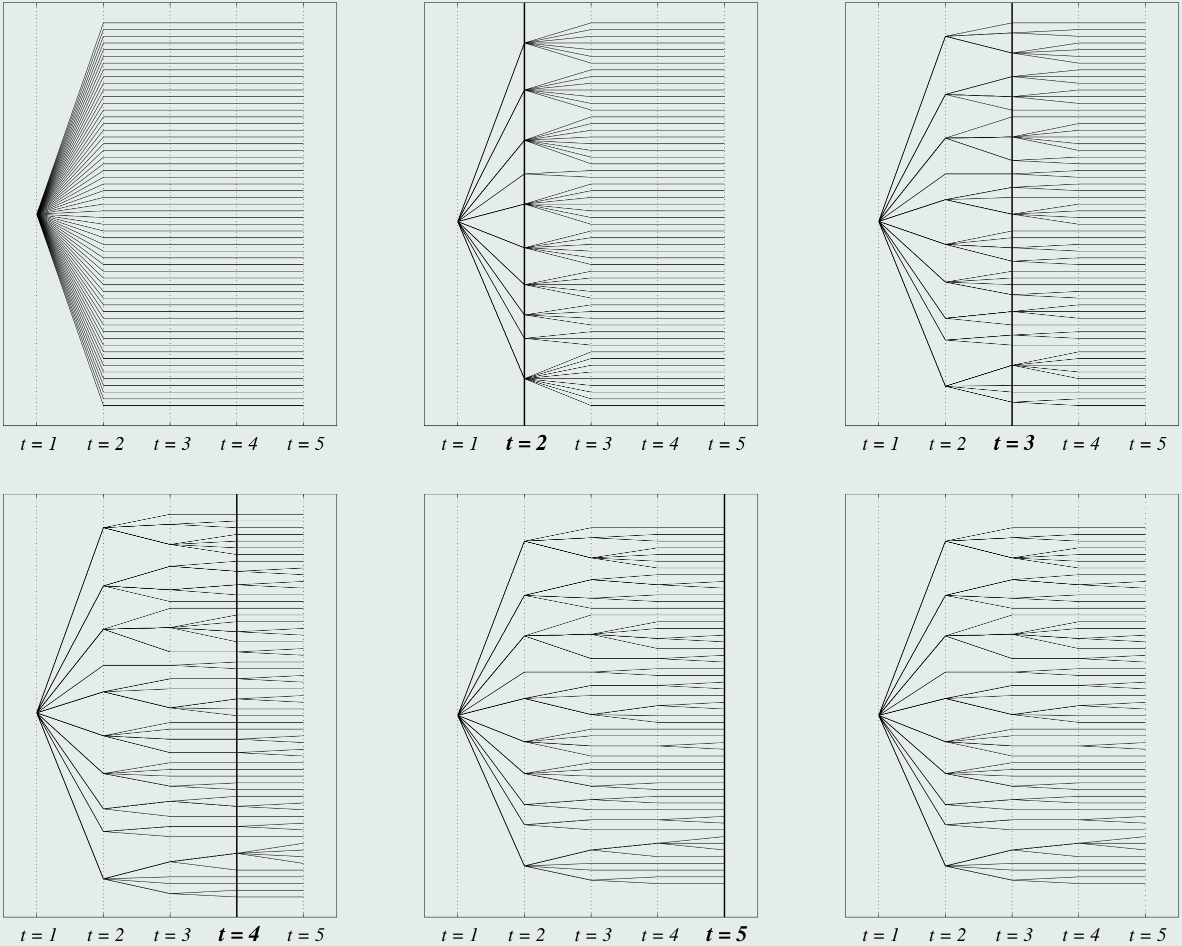
Reduced load scenario tree obtained by the forward selection method (15 scenarios)




Reduced load scenario tree obtained by the backward reduction method (12 scenarios)



# Application: Scenario trees for multistage models




<Illustration> of the forward construction for T=5 time periods starting with 58 scenarios

# Mixed-integer two-stage stochastic programs

We consider

$$\min\left\{\langle c,x\rangle + \int_\Xi \Phi(q(\xi),h(\xi)-T(\xi)x)P(d\xi) : x\in X\right\},$$

where $\Phi$ is given by

$$\Phi(u,t) := \inf\left\{\langle u_1,y_1\rangle + \langle u_2,y_2\rangle \,\middle|\, \begin{array}{l} W_1y_1+W_2y_2=t \\ y_1\in\mathbb{R}_+^{m_1},\, y_2\in\mathbb{Z}_+^{m_2} \end{array}\right\}$$

for all pairs $(u,t)\in\mathbb{R}^{m_1+m_2}\times\mathbb{R}^r$, and $c\in\mathbb{R}^m$, $X$ is a closed subset of $\mathbb{R}^m$, $\Xi$ a polyhedron in $\mathbb{R}^s$, $T\in\mathbb{R}^{r\times m}$, $W_1\in\mathbb{Q}^{r\times m_1}$, $W_2\in\mathbb{Q}^{r\times m_2}$, and $q(\xi)\in\mathbb{R}^{m_1+m_2}$ and $h(\xi)\in\mathbb{R}^r$ are affine functions of $\xi$, and $P$ is a Borel probability measure such that

$$\int_\Xi \|\xi\|^2 P(d\xi) < +\infty.$$

In addition, we assume relatively complete recourse and dual feasibility.

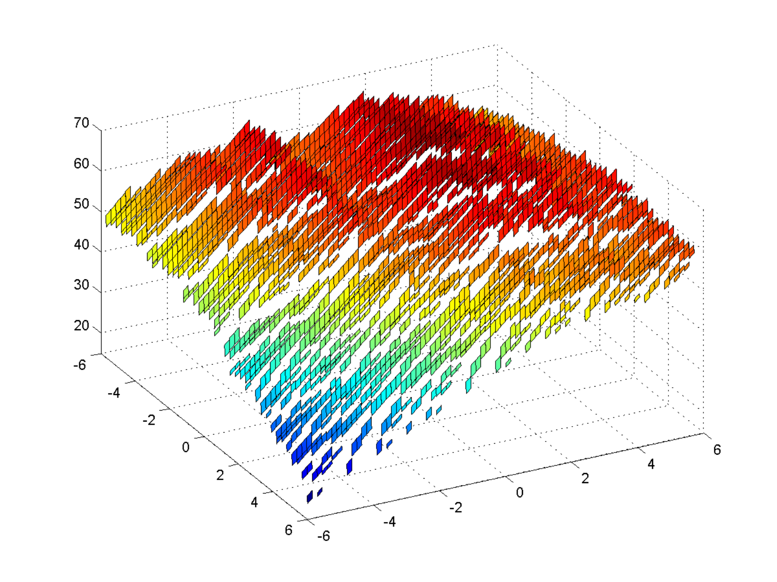
**Example 1:** (Schultz-Stougie-van der Vlerk 98)

## Stochastic multi-knapsack problem:

$\min = \max$, $m = 2$, $m_1 = 0$, $m_2 = 4$, $c = (1.5, 4)$, $X = [-5, 5]^2$,
$h(\xi) = \xi$, $q(\xi) \equiv q = (16, 19, 23, 28)$, $y_i \in \{0, 1\}$, $i = 1, 2, 3, 4$,
$P \sim \mathcal{U}(5, 5.5, \ldots, 14.5, 15\}$ (discrete)

Second stage problem: MILP with 1764 Boolean variables and 882 constraints.

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad W = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 1 & 3 & 2 \end{pmatrix}$$



The function $\Phi$ is well understood (Blair-Jeroslow 77, Bank-Mandel 88) and according to (Schultz 96, Römisch-Vigerske 07) the function class $\mathcal{F}$ is contained in

$$\mathcal{F}_{2,\mathcal{B}}(\Xi) := \{f\mathbb{1}_B : f \in \mathcal{F}_2(\Xi), B \in \mathcal{B}\},$$

where $\mathcal{B}$ is a class of (convex) polyhedra in $\Xi$ with a uniformly bounded number of faces containing all sets of the form

$$\{\xi \in \Xi : h(\xi) \in Tx + B\},$$

where $x \in X$ and $B$ is a polyhedron in $\mathbb{R}^r$ each of whose facets, i.e., $(r-1)$-dimensional faces, is parallel to a facet of the cone $\operatorname{pos} W_1 = \{W_1 y_1 : y_1 \in \mathbb{R}_+^{m_1}\}$ or of the unit cube $[0,1]^r$.
Here, $\mathbb{1}_B$ denotes the characteristic function of the set $B$ and the class $\mathcal{F}_2(\Xi)$ consists of all continuous functions $f : \Xi \to \mathbb{R}$ such that the estimates

$$|f(\xi)| \le \max\{1, \|\xi\|^2\} \text{ and } f(\xi) - f(\tilde{\xi}) \le \max\{1, \|\xi\|, \|\tilde{\xi}\|\}\|\xi - \tilde{\xi}\|$$

hold true for all $\xi, \tilde{\xi} \in \Xi$.

**Proposition:**
In case $\mathcal{F} = \mathcal{F}_{2,\mathcal{B}}(\Xi)$, convergence with respect to the metric $d_{\mathcal{F}}$ is equivalent to convergence with respect to $\zeta_2$ (Fortet-Mourier metric of order $2$) and $\alpha_{\mathcal{B}}$ ($\mathcal{B}$-discrepancy), where

$$\alpha_{\mathcal{B}}(P,Q) := \sup_{B\in\mathcal{B}} |P(B) - Q(B)|$$

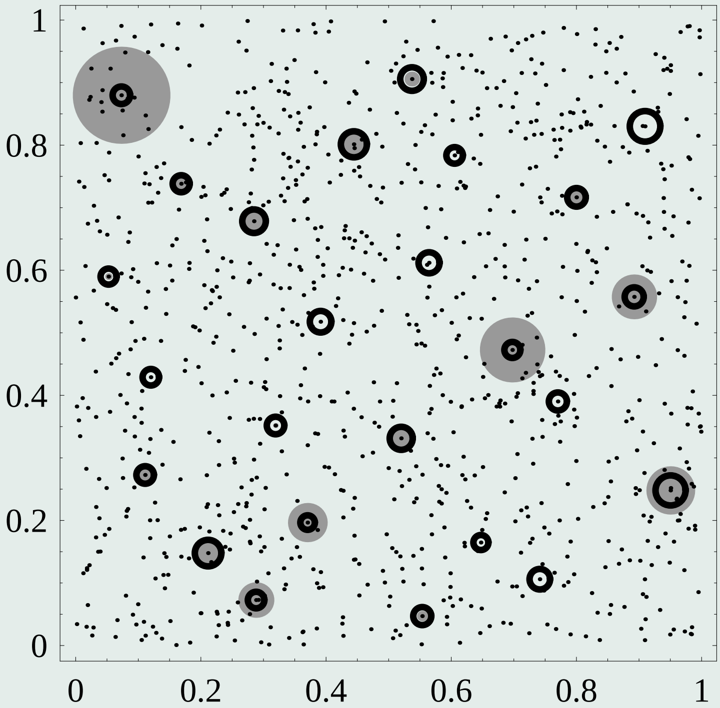
If the set $\Xi$ is bounded, it even holds

$$\alpha_{\mathcal{B}}(P,Q) \leq d_{\mathcal{F}}(P,Q) \leq C\alpha_{\mathcal{B}}(P,Q)^{\frac{1}{s+1}}$$

with some constant $C$ depending on $\Xi$.

In the following, we consider the situation $r = s$ and $h(\xi) = \xi$, and denote the class $\mathcal{B}$ by $\mathcal{B}_{\text{poly(W)}}$. Special cases are $\mathcal{B}_{\text{rect}}$ (rectangular discrepancy) for the pure integer situation and $\mathcal{B}_{\text{cell}}$ (cell discrepancy). Cells are sets of the form $(-\infty, \xi]$ in $\mathbb{R}^s$.

# Influence of different metrics: $\alpha_{\mathcal{B}_{\text{rect}}}$ versus $\zeta_2$



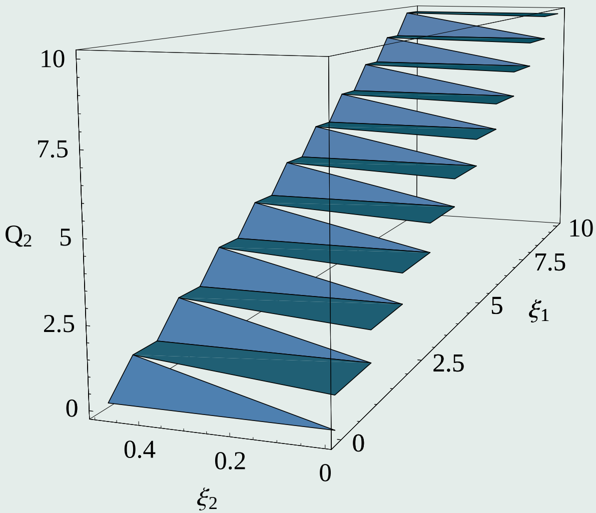

25 scenarios chosen by Quasi Monte Carlo out of 1000 samples from the uniform distribution on $[0,1]^2$ and optimal probabilities adjusted w.r.t. $\lambda\alpha_{\mathcal{B}_{\text{rect}}} + (1-\lambda)\zeta_2$ for $\lambda = 1$ (gray balls) and $\lambda = 0.9$ (black circles)

## Example 2:

We consider the following mixed-integer two-stage stochastic program: Let $m = 1$, $s = 2$, $c = 0$, $X := \{0\}$, $\Xi = [0, 10] \times [0, 0.5]$, the probability measure $P$ consists of $N = 1000$ uniformly weighted points, sampled from the uniform distribution on $\Xi$, and

$$\begin{aligned}\Phi(t) &= \inf\{2y_1 + y_2 : y_1 + y_2 \geq t_1, y_1 \leq t_2, y_1 \in \mathbb{R}_+, y_2 \in \mathbb{Z}_+\} \\ &= \begin{cases} \lfloor t_1 \rfloor + 1 & , \text{ if } t_1 - \lfloor t_1 \rfloor > t_2, \\ \lfloor t_1 \rfloor + 2(t_1 - \lfloor t_1 \rfloor) & , \text{ otherwise.} \end{cases}\end{aligned}$$

The function $\xi \mapsto \Phi(\xi)$ from $\Xi$ to $\mathbb{R}$ is shown in



While $\Phi$ is discontinuous, introducing slack variables and writing the linear program in standard form entails that the continuous variable $y_1 \in \mathbb{R}^3_+$ is assigned to the recourse matrix

$$W_1 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Hence, the closures of the regions of continuity of $\Phi$ are indeed contained in the family $\mathcal{B}_{\mathsf{poly(W)}}$, i.e., they are polyhedra each of whose facets parallels a facet of pos $W_1$ or of the unit cube.

## Scenario reduction

We consider a probability measure $P$ with finite support $\{\xi^1, \ldots, \xi^N\}$ and set $p_i := P(\{\xi^i\}) > 0$ for $i = 1, \ldots, N$. Denoting by $\delta_\xi$ the Dirac measure placing mass one at the point $\xi$, the measure $P$ has the form

$$P = \sum_{i=1}^{N} p_i \delta_{\xi^i}.$$

The problem of optimal scenario reduction consists in determining a discrete probability measure $Q$ on $\mathbb{R}^s$ supported by a subset of $\{\xi^1, \ldots, \xi^N\}$ and deviating from $P$ as little as possible with respect to $\alpha_{\mathcal{B}}$. It can be written as

$$\min \left\{ \alpha_{\mathcal{B}} \left( \sum_{i=1}^{N} p_i \delta_{\xi^i}, \sum_{j=1}^{n} q_j \delta_{\eta^j} \right) \middle| \begin{array}{l} \{\eta^1, \ldots, \eta^n\} \subset \{\xi^1, \ldots, \xi^N\} \\ q_j \geq 0 \; j = 1, \ldots, n, \sum_{j=1}^{n} q_j = 1 \end{array} \right\}.$$

This optimization problem may be decomposed into an outer problem for determining $\operatorname{supp}(Q) = \eta$ and an inner problem for choosing the probabilities $q_j$, $j = 1, \ldots, n$.

To this end, we denote

$$\alpha_{\mathcal{B}}(P,(\eta,q)) := \alpha_{\mathcal{B}}\left(\sum_{i=1}^{N} p_i\delta_{\xi^i}, \sum_{j=1}^{n} q_j\delta_{\eta^j}\right)$$

$$S_n := \{q \in \mathbb{R}^n : q_j \geq 0, j = 1, \ldots, n, \sum\nolimits_{j=1}^{n} q_j = 1\}.$$

Then the scenario reduction problem may be rewritten as

$$\min_{\eta}\{\min_{q \in S_n} \alpha_{\mathcal{B}}(P,(\eta,q)) : \eta \subset \{\xi^1, \ldots, \xi^N\}, |\eta| = n\}$$

with the inner problem (optimal redistribution)

$$\min\{\alpha_{\mathcal{B}}(P,(\eta,q)) : q \in S_n\}$$

for the fixed support $\eta$. The outer problem is a combinatorial optimization problem (NP hard) while the inner problem may be reformulated as a linear program.

We assume for the sake of notational simplicity, that $\eta = \{\xi^1, \ldots, \xi^n\}$. Then the inner problem is of the form:

$$\min\{\alpha_{\mathcal{B}}(P, (\{\xi^1, \ldots, \xi^n\}, q)) : q \in S_n\}$$

The finiteness of the support of $P$ allows to define for $B \in \mathcal{B}$ the critical index set $I(B)$ by

$$I(B) := \{i \in \{1, \ldots, N\} : \ \xi^i \in B\}$$

and to write

$$|P(B) - Q(B)| = \left| \sum_{i \in I(B)} p_i - \sum_{j \in I(B) \cap \{1,\ldots,n\}} q_j \right| .$$

Furthermore, we define the system of critical index sets of $\mathcal{B}$ as

$$\mathcal{I}_{\mathcal{B}} := \{I(B) : B \in \mathcal{B}\}.$$

Thus, the $\mathcal{B}$-discrepancy between $P$ and $Q$ may be reformulated as follows:

$$\alpha_{\mathcal{B}}(P, Q) = \max_{I \in \mathcal{I}_{\mathcal{B}}} \left| \sum_{i \in I} p_i - \sum_{j \in I \cap \{1,\ldots,n\}} q_j \right| .$$

This allows to solve the inner problem by means of the following linear program:

$$\min \left\{ t \;\middle|\; \begin{array}{l} q \in S_n, I \in \mathcal{I}_\mathcal{B} \\ -\sum_{j \in I \cap \{1,\ldots,n\}} q_j \leq t - \sum_{i \in I} p_i \\ \sum_{j \in I \cap \{1,\ldots,n\}} q_j \leq t + \sum_{i \in I} p_i \end{array} \right\}$$

Since $|\mathcal{I}_\mathcal{B}| \leq 2^N$, the number of inequalities is too large to solve this LP numerically.
However, whenever two critical index sets share the same intersection with the set $\{1, \ldots, n\}$, only the right-hand sides of the related inequalities differ. Thus, it is possible to pass to the minimum of all right-hand sides corresponding to the same left-hand side.

To this end, we introduce the following reduced system of critical index sets

$$\mathcal{I}_\mathcal{B}^* := \{I(B) \cap \{1, \ldots, n\} : B \in \mathcal{B}\}.$$

Thereby, every member $J \in \mathcal{I}_{\mathcal{B}}^*$ of the reduced system is associated with a family $\varphi(J) \subset \mathcal{I}_{\mathcal{B}}$ of critical index sets, all of which share the same intersection with $\{1, \dots, n\}$:

$$\varphi(J) := \{I \in \mathcal{I}_{\mathcal{B}} : J = I \cap \{1, \dots, n\}\} \quad (J \in \mathcal{I}_{\mathcal{B}}^*).$$

Finally, we consider the quantities

$$\gamma^J := \max_{I \in \varphi(J)} \sum_{i \in I} p_i \quad \text{and} \quad \gamma_J := \min_{I \in \varphi(J)} \sum_{i \in I} p_i \quad (J \in \mathcal{I}_{\mathcal{B}}^*),$$

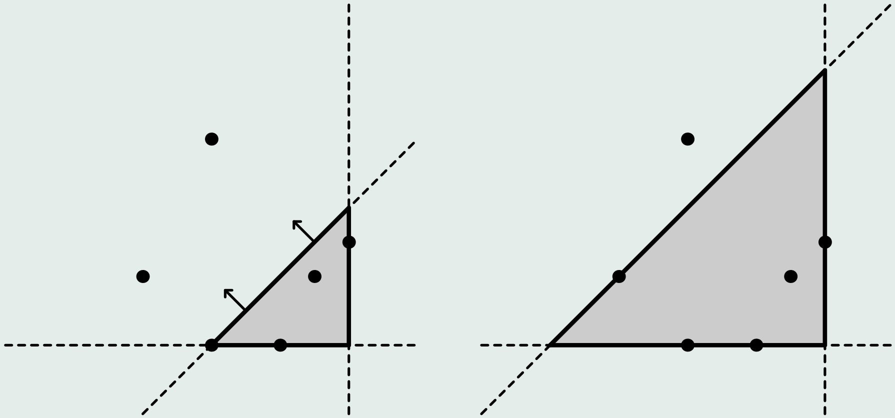
to write the linear program as

$$\min \left\{ t \;\middle|\; \begin{array}{l} q \in S_n, J \in \mathcal{I}_{\mathcal{B}}^* \\ -\sum_{j \in J} q_j \le t - \gamma^J \\ \sum_{j \in J} q_j \le t + \gamma_J \end{array} \right\}$$

Now we have $|\mathcal{I}_{\mathcal{B}}^*| \le 2^n$ and, hence, drastically reduced the maximum number of inequalities. This can make the LP solvable at least for moderate values of $n$.

## How to determine $\mathcal{I}^*_{\mathcal{B}}$, $\gamma_J$ and $\gamma^J$?

**Observation:**
$\mathcal{I}^*_{\mathcal{B}}$, $\gamma_J$ and $\gamma^J$ are determined by those polyhedra (belonging to $\mathcal{P}$), each of whose facets contains an element of $\{\xi^1, \ldots, \xi^n\}$, such that it can not be enlarged without changing its interior's intersection with $\{\xi^1, \ldots, \xi^n\}$. The polyhedra in $\mathcal{P}$ are called supporting.



Non supporting polyhedron (left) and supporting polyhedron (right). The dots represent the remaining scenarios $\xi^1, \ldots, \xi^n$

## Proposition:

$$
\begin{aligned}
\mathcal{I}^*_{\mathcal{B}} &= \{J \subseteq \{1,\dots,n\} : \exists B \in \mathcal{P}, \cup_{j\in J}\{\xi^j\} = \{\xi^1,\dots,\xi^n\} \cap \operatorname{int} B\} \\
\gamma^J &= \max\{P(\operatorname{int} B) : B \in \mathcal{P}, \cup_{j\in J}\{\xi^j\} = \{\xi^1,\dots,\xi^n\} \cap \operatorname{int} B\} \\
\gamma_J &= \sum_{i\in I} p_i \quad \text{with} \quad I \subseteq \{1,\dots,N\} \quad \text{defined by} \\
I &:= \left\{ i : \min_{j\in J} \langle m^l, \xi^j\rangle \le \langle m^l, \xi^i\rangle \le \max_{j\in J} \langle m^l, \xi^j\rangle, l = 1,\dots,k \right\},
\end{aligned}
$$

where $m^j$, $j = 1,\dots,k$, are the rows of a matrix $M \in \mathbb{R}^{k\times s}$ having the property that every polyhedron $B \in \mathcal{B}_{\text{poly(W)}}$ can be written as

$$B = \{\xi \in \mathbb{R}^s : \underline{a}^B \le M\xi \le \bar{a}^B\}$$

for some $\underline{a}^B$ and $\bar{a}^B$ in $\overline{\mathbb{R}}^k$.

Note that $|\mathcal{P}| \le \binom{n+2}{2}^k$!
For $n = 5$, $k = 3$ and $n = 20$, $k = 12$, the latter is equal to $3375$ and $7.36 \cdot 10^{27}$, respectively.

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## Numerical results

Optimal redistribution w.r.t. the polyhedral discrepancy $\alpha_{\mathcal{B}_{poly(W)}}$:

| | k | n=5 | n=10 | n=15 | n=20 |
|---|---|---|---|---|---|
| | cell | 0.01 | 0.01 | 0.01 | 0.05 |
| $\mathbb{R}^3$ | 3 | 0.01 | 0.04 | 0.56 | 6.02 |
| N=100 | 6 | 0.03 | 1.03 | 14.18 | 157.51 |
| | 9 | 0.15 | 7.36 | 94.49 | 948.17 |
| | cell | 0.01 | 0.01 | 0.05 | 0.30 |
| $\mathbb{R}^4$ | 4 | 0.01 | 0.19 | 1.83 | 17.22 |
| N=100 | 8 | 0.11 | 5.66 | 59.28 | 521.31 |
| | 12 | 0.67 | 39.86 | 374.15 | 3509.34 |
| | cell | 0.01 | 0.01 | 0.01 | 0.07 |
| $\mathbb{R}^3$ | 3 | 0.01 | 0.05 | 0.53 | 4.28 |
| N=200 | 6 | 0.03 | 0.76 | 11.80 | 132.21 |
| | 9 | 0.12 | 4.22 | 78.49 | 815.79 |
| | cell | 0.01 | 0.01 | 0.06 | 0.29 |
| $\mathbb{R}^4$ | 4 | 0.01 | 0.20 | 2.56 | 41.73 |
| N=200 | 8 | 0.11 | 4.44 | 73.70 | 1042.78 |
| | 12 | 0.74 | 28.29 | 473.72 | 6337.68 |

Running times [sec] of the optimal redistribution algorithm

## Example 2: (continued)

The distribution $P$ is approximated by different methods:

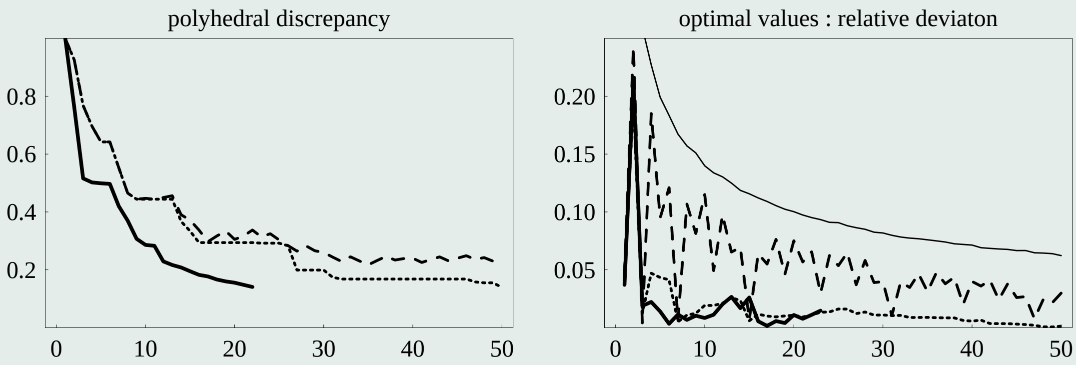
- **random sampling**: $10,000$ random samples of size $n$ from $P$, i.e., every sample consists of $n$ equally weighted points. The approximate problem was solved for each sample and the average relative deviation of the optimal value to the optimal value of the initial problem has been computed.
- **Quasi Monte Carlo (QMC)**: The first $n$ numbers of the Halton sequences with bases $2$ and $3$ provide $n$ equally weighted points in $\mathbb{R}^2$. The resulting discrepancy to the initial measure has been computed for *fixed* probability weights. The approximate problem has been solved.
- **adjusted Quasi Monte Carlo**: The probabilities of the Halton points have been adjusted by the optimal redistribution algorithm to obtain a minimal polyhedral discrepancy to $P$. The approximate problem has been solved.
- **Forward selection**:

**Step [0]:** $J^{[0]} := \varnothing$.

**Step [i]:** $l_i \in \mathrm{argmin}_{l \notin J^{[i-1]}} \inf_{q \in S_i} \alpha_{\mathcal{B}}(P, (\{\xi^{l_1}, \ldots, \xi^{l_{i-1}}, \xi^{l}\}, q)),\ J^{[i]} := J^{[i-1]} \cup \{l_i\}$.

**Step [n+1]:** Minimize $\alpha_{\mathcal{B}}(\{P, (\xi^{l_1}, \ldots, \xi^{l_n}\}, q))$ s.t. $q \in S_n$.

**Conclusion:** Random sampling performs badly, (next neighbor) QMC is somewhat better, (next neighbor) QMC and readjusting the probabilities to the correct discrepancy decreases significantly the approximation error. Forward selection provides good results, but is very slow due to the optimal redistribution after each step.




Relative error of the optimal value $\frac{|v-\tilde{v}|}{|v|}$, depending on $n$ for forward selection (bold), sampling (thin), Quasi-Monte Carlo (dashed) and *readjusted* Quasi-Monte Carlo (dotted).

## Conclusions and outlook

- There exist reasonable fast heuristics for linear two-stage stochastic programs,
- The heuristics apply to generate scenario trees for multistage stochastic programs,
- For mixed-integer two-stage stochastic programs similar heuristics exist, but they are more expensive and restricted to moderate dimensions,
- Development of mixed heuristics based on the (rectangular) discrepancy and Fortet-Mourier metrics,
- Hence, there is hope for generating scenario trees for mixed-integer multistage models.

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