Stability of two- and multistage stochastic programs

W. Römisch

Humboldt-University Berlin
Institute of Mathematics
10099 Berlin, Germany

http://www.math.hu-berlin.de/~romisch

(H. Heitsch, C. Strugarek)

PARAOPT VIII, Cairo, Egypt, November 29, 2005

EDF Electricité de France

DFG research center Berlin
mathematics for key technologies

Multistage stochastic programs

Let $\xi = \{\xi_t\}_{t=1}^T$ be an $\mathbb{R}^d$-valued discrete-time stochastic data process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with ξ_1 deterministic. The stochastic decision x_t at period t is assumed to be measurable with respect to $\mathcal{F}_t := \sigma(\xi_1, \ldots, \xi_t)$ (nonanticipativity).

Multistage stochastic program:

$$\min \left\{ \mathbb{E}\Big[\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle\Big] \;\middle|\; \begin{array}{l} x_t \in X_t, \\ x_t \text{ is } \mathcal{F}_t - \text{measurable}, t = 1, \ldots, T, \\ A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} = h_t(\xi_t), t = 2, \ldots, T \end{array} \right\}$$

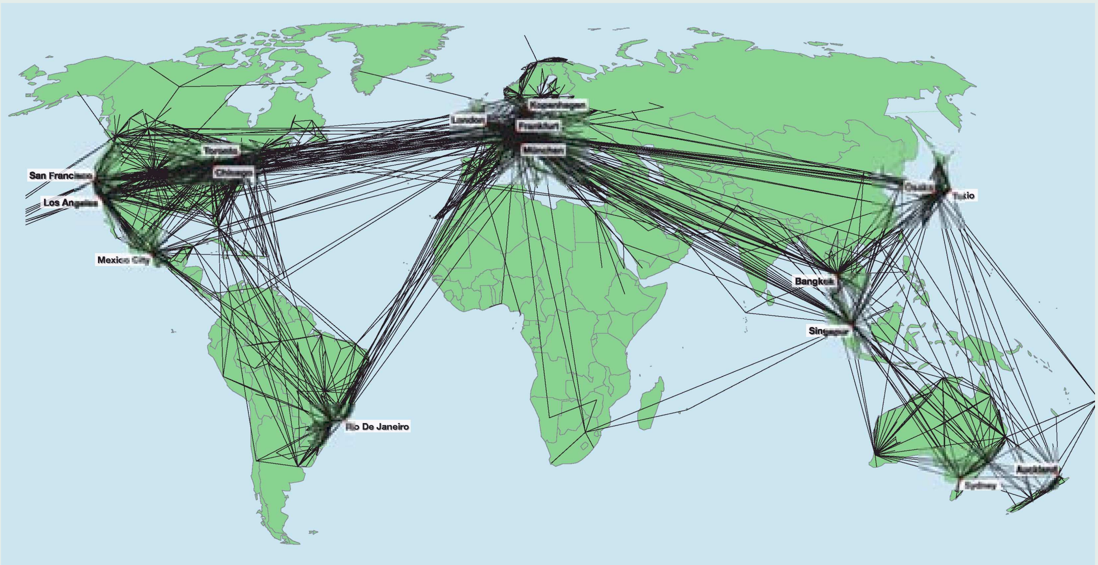
where X_t are nonempty and polyhedral set, $A_{t,0}$ are fixed matrices and $b_t(\cdot)$, $h_t(\cdot)$ and $A_{t,1}(\cdot)$ possibly depend affinely linearly on ξ_t, where ξ varies in a polyhedral set Ξ.

The model is (multiperiod) two-stage if $\mathcal{F}_t = \mathcal{F}$, $t = 2, \ldots, T$.

Stability of such models is not known so far (cf. the survey by Römisch 03).

Application: Airline Revenue Management

Origin&Destination (O&D) Revenue Management has become a standard instrument in airline industry. It considers the entire airline network and determines protection levels for all origin destination itineraries, fare classes, points of sale and data collection points (dcp's) of the booking horizon. Our model incorporates the stochastic nature of the passenger behaviour and represents a multi-stage stochastic program where its stages refer to the dcp's.



To have the multistage stochastic program well defined, we assume $x_t \in L_{r'}(\Omega, \mathcal{F}_t, I\!P; I\!R^{m_t})$ and $\xi_t \in L_r(\Omega, \mathcal{F}_t, I\!P; I\!R^d)$, where $r \geq 1$ and

$$r' := \begin{cases} \frac{r}{r-1} & , \text{ if only costs are random} \\ r & , \text{ if only right-hand sides are random} \\ r = 2 & , \text{ if only costs and right-hand sides are random} \\ \infty & , \text{ if all technology matrices are random and } r = T. \end{cases}$$

Then nonanticipativity may be expressed as

$$x \in \mathcal{N}_{na}$$

$$\mathcal{N}_{na} = \{x \in \times_{t=1}^{T} L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^{m_t}) : x_t = I\!E[x_t|\mathcal{F}_t]\,, \forall t\},$$

i.e., as a subspace constraint, by using the conditional expectations $I\!E[\cdot|\mathcal{F}_t]$ with respect to the σ-fields $\mathcal{F}_t$.

For $T = 2$ we have $\mathcal{N}_{na} = I\!R^{m_1} \times L_{r'}(\Omega, \mathcal{F}, P; I\!R^{m_2})$.

$\longrightarrow$ infinite-dimensional optimization problem

Dynamic programming

Theorem: (Evstigneev 76, Rockafellar/Wets 76)
Under weak assumptions the multistage stochastic program is equivalent to the (first-stage) convex minimization problem

$$\min \{ \int_{\Xi} f(x_1, \xi) P(d\xi) : x_1 \in X_1 \},$$

where f is an integrand on $\mathbb{R}^{m_1} \times \Xi$ given by

$$f(x_1, \xi) := \langle b_1(\xi_1), x_1 \rangle + \Phi_2(x_1, \xi^2),$$

$$\Phi_t(x_1, \ldots, x_{t-1}, \xi^t) := \inf \{ \langle b_t(\xi_t), x_t \rangle + \mathbb{E}\left[\Phi_{t+1}(x_1, \ldots, x_t, \xi^{t+1}) | \mathcal{F}_t\right] : x_t \in X_t, A_{t,0} x_t + A_{t,1} x_{t-1} = h_t(\xi_t) \}$$

for $t = 2, \ldots, T$, where $\Phi_{T+1}(x_1, \ldots, x_T, \xi^{T+1}) := 0$.

$\longrightarrow$The integrand f depends on the probability measure $\mathbb{P}$ in a nonlinear way !

Quantitative Stability

Let us introduce some notations. Let F denote the objective function defined on $L_r(\Omega, \mathcal{F}, I\!P; I\!R^s) \times L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^m) \to I\!R$ by $F(\xi, x) := I\!E[\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle]$, let

$$\mathcal{X}_t(x_{t-1}; \xi_t) := \{x_t \in X_t | A_{t,0} x_t + A_{t,1}(\xi_t) x_{t-1} = h_t(\xi_t)\}$$

denote the t-th feasibility set for every $t = 2, \ldots, T$ and

$$\mathcal{X}(\xi) := \{x \in \times_{t=1}^{T} L_{r'}(\Omega, \mathcal{F}_t, I\!P; I\!R^{m_t}) | x_1 \in X_1, x_t \in \mathcal{X}_t(x_{t-1}; \xi_t)\}$$

the set of feasible elements with input ξ.
Then the multistage stochastic program may be rewritten as

$$\min\{F(\xi, x) \, : \, x \in \mathcal{X}(\xi)\}.$$

Let $v(\xi)$ denote its optimal value and, for any $\alpha \geq 0$,

$$l_\alpha(F(\xi, \cdot)) := \{x \in \mathcal{X}(\xi) : F(\xi, x) \leq v(\xi) + \alpha\}$$

denote the α-level set of the stochastic program with input ξ.

The following conditions are imposed:

(A1) There exists a $\delta > 0$ such that for any $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, I\!P; I\!R^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$, any $t = 2, \dots, T$ and any $x_1 \in X_1$, $x_\tau \in \mathcal{X}_\tau(x_{\tau-1}; \tilde{\xi}_\tau)$, $\tau = 2, \dots, t-1$, the set $\mathcal{X}_t(x_{t-1}; \tilde{\xi}_t)$ is nonempty (relatively complete recourse locally around ξ).

(A2) The optimal value $v(\xi)$ is finite and the objective function F is level-bounded locally uniformly at ξ, i.e., for some $\alpha > 0$ there exists a $\delta > 0$ and a bounded subset B of $L_{r'}(\Omega, \mathcal{F}, I\!P; I\!R^m)$ such that $l_\alpha(F(\tilde{\xi}, \cdot))$ is nonempty and contained in B for all $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, I\!P; I\!R^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$.

(A3) $\xi \in L_r(\Omega, \mathcal{F}, I\!P; I\!R^s)$ for some $r \geq 1$.

Norm in L_r: $\|\xi\|_r := \left(\sum_{t=1}^{T} I\!E[\|\xi_t\|^r]\right)^{\frac{1}{r}}$

Theorem:

Let (A1), (A2) and (A3) be satisfied and X_1 be bounded.
Then there exist positive constants L, α and δ such that

$$|v(\xi) - v(\tilde{\xi})| \leq L(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}(\xi, \tilde{\xi}))$$

holds for all $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, I\!P; I\!R^s)$ with $\|\tilde{\xi} - \xi\|_r \leq \delta$.
Here, $D_{\mathrm{f}}(\xi, \tilde{\xi})$ denotes the filtration distance of ξ and $\tilde{\xi}$ defined by

$$D_{\mathrm{f}}(\xi, \tilde{\xi}) := \sup_{\varepsilon \in (0,\alpha]} \inf_{\substack{x \in l_\varepsilon(F(\xi,\cdot)) \\ \tilde{x} \in l_\varepsilon(F(\tilde{\xi},\cdot))}} \sum_{t=2}^{T-1} \max\{\|x_t - I\!E[x_t|\tilde{\mathcal{F}}_t]\|_{r'}, \|\tilde{x}_t - I\!E[\tilde{x}_t|\mathcal{F}_t]\|_{r'}\}$$

where $\mathcal{F}_t = \sigma(\xi_1, \ldots, \xi_t)$ and $\tilde{\mathcal{F}}_t = \sigma(\tilde{\xi}_1, \ldots, \tilde{\xi}_t)$, $t = 2, \ldots, T-1$.

Note that the filtration distance vanishes for multiperiod two-stage stochastic programs !

If solutions exist, the filtration distance is of the simplified form

$$D_{\rm f}(\xi,\tilde{\xi}) = \inf_{\substack{x\in l_0(F(\xi,\cdot))\\ \tilde{x}\in l_0(F(\tilde{\xi},\cdot))}} \sum_{t=2}^{T-1} \max\{\|x_t - \mathbb{E}[x_t|\tilde{\mathcal{F}}_t]\|_{r'}, \|\tilde{x}_t - \mathbb{E}[\tilde{x}_t|\mathcal{F}_t]\|_{r'}\}.$$

For example, solutions exist if Ω is finite or if $1 < r' < \infty$ implying that the spaces $L_{r'}$ are finite-dimensional or reflexive Banach spaces (hence, the level sets are compact or weakly sequentially compact).

Remark:

The continuity property of infima in the Theorem can be supplemented by a quantitative stability property of the set $S(\xi)$ of first stage solutions. Namely, there exists a constant $\hat{L} > 0$ such that

$$\sup_{x\in S(\tilde{\xi})} d(x, S(\xi)) \leq \Psi_{\xi}^{-1}(\hat{L}(\|\xi - \tilde{\xi}\|_r + D_{\rm f}(\xi,\tilde{\xi}))),$$

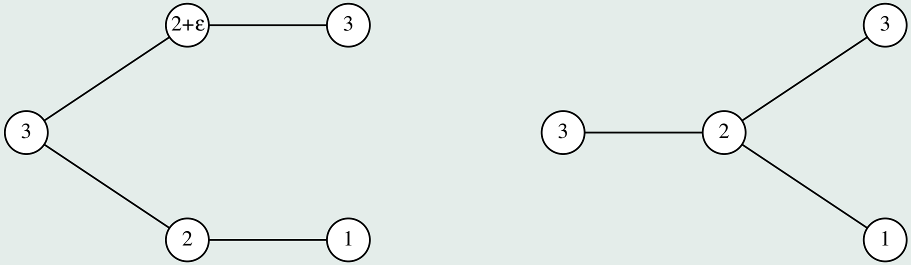
where $\Psi_\xi(\tau) := \inf\{\mathbb{E}[f(x_1,\xi)] - v(\xi) : d(x_1, S(\xi)) \geq \tau, x_1 \in X_1\}$ with $\Psi_{\xi}^{-1}(\alpha) := \sup\{\tau \in \mathbb{R}_+ : \Psi_\xi(\tau) \leq \alpha\}$ $(\alpha \in \mathbb{R}_+)$ is the growth function of the original problem near its solution set $S(\xi)$.

The following example shows that the filtration distance D_{f} is indispensable for the stability result to hold.

Example: (Optimal purchase under uncertainty)
The decisions x_t correspond to the amounts to be purchased at each time period with uncertain prices are ξ_t, $t = 1, \ldots, T$, and such that a prescribed amount a is achieved at the end of a given time horizon. The problem is of the form

$$\min \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \xi_t x_t\right] \,\middle|\, \begin{array}{l} (x_t, s_t) \in X_t = \mathbb{R}_+^2, \\ (x_t, s_t) \text{ is } (\xi_1, \ldots, \xi_t)\text{-measurable}, \\ s_t - s_{t-1} = x_t,\ t = 2, \ldots, T, \\ s_1 = 0,\ s_T = a. \end{array} \right\},$$

where the state variable s_t corresponds to the amount at time t.
Let $T := 3$ and ξ_ε denote the stochastic price process having the two scenarios $\xi_\varepsilon^1 = (3, 2+\varepsilon, 3)$ $(\varepsilon \in (0,1))$ and $\xi_\varepsilon^2 = (3, 2, 1)$ each endowed with probability $\frac{1}{2}$. Let $\tilde{\xi}$ denote the approximation of ξ_ε given by the two scenarios $\tilde{\xi}^1 = (3, 2, 3)$ and $\tilde{\xi}^2 = (3, 2, 1)$ with the same probabilities $\frac{1}{2}$.



Scenario trees for ξ_ε (left) and $\tilde{\xi}$

We obtain

$$\begin{aligned} v(\xi_\varepsilon) &= \frac{1}{2}((2+\varepsilon)a + a) = \frac{3+\varepsilon}{2}a \\ v(\tilde{\xi}) &= 2a\,, \qquad \text{but} \\ \|\xi_\varepsilon - \tilde{\xi}\|_1 &\leq \frac{1}{2}(0+\varepsilon+0) + \frac{1}{2}(0+0+0) = \frac{\varepsilon}{2}. \end{aligned}$$

Hence, the multistage stochastic purchasing model is not stable with respect to $\|\cdot\|_1$.

However, the estimate for $|v(\xi) - v(\tilde{\xi})|$ in the stability theorem is valid with $L = 1$ since $D_{\mathrm{f}}(\xi, \tilde{\xi}) = \frac{a}{2}$.

Conclusions

The stability result has important consequences for the construction of scenario trees ξ_{tr} as approximations of the original process ξ. The tree ξ_{tr} should be selected such that

$$\|\xi - \xi_{\mathrm{tr}}\|_r \qquad \text{and} \qquad D_{\mathrm{f}}(\xi, \xi_{\mathrm{tr}})$$

are smaller than some tolerance. This problem may be solved for ξ having scenarios ξ^i and probabilities p_i, $i = 1, \ldots, N$.

Application: Airline revenue management (continued)

Let ξ^i be passenger demand scenarios for a single flight (LH400, A340-300) with $d = 14$ fare classes and the booking time horizon with $T = 18$ obtained by (re)sampling from historical data (N=300). An implementation of a (forward) tree construction leads to the following scenario tree with 150 scenarios, 1190 nodes and branching at all $t = 1, \ldots, 18$.