Stability, sensitivity and limit theorems of stochastic dominance constrained optimization models

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Introduction and contents

The use of stochastic orderings as a modeling tool has become standard in theory and applications of stochastic optimization. Much of the theory is developed and many successful applications are known.

Research topics:

- Multivariate concepts and analysis,
- scenario generation and approximation schemes,
- analysis of (Quasi-) Monte Carlo approximations,
- numerical methods and decomposition schemes.

Contents of the talk:

(1) Introduction, stochastic dominance

(2) Quantitative stability results

(3) Sensitivity of optimal values

(4) Limit theorems for empirical approximations

Optimization models with stochastic dominance constraints

We consider the optimization model

$$\min \{f(x) : x \in D,\, G(x,\xi) \succeq_{(k)} Y\},$$

where $k \in \mathbb{N}$, D is a nonempty convex closed subset of $\mathbb{R}^m$, Ξ a convex closed subset of $\mathbb{R}^s$, $f : \mathbb{R}^m \to \mathbb{R}$ is convex, ξ is a random vector with support Ξ and Y a real random variable on some probability space both having finite moments of order $k-1$, and $G : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}$ is continuous, concave with respect to the first argument and satisfies the linear growth condition

$$|G(x,\xi)| \leq C(B) \max\{1, \|\xi\|\} \quad (x \in B, \xi \in \Xi)$$

for every bounded subset $B \subset \mathbb{R}^m$ and some constant $C(B)$ (depending on B). The random variable Y plays the role of a benchmark outcome.

D. Dentcheva, A. Ruszczyński: Optimization with stochastic dominance constraints, *SIAM J. Optim.* 14 (2003), 548–566.

Stochastic dominance relation $\succeq_{(k)}$

$$X \succeq_{(k)} Y \quad \Leftrightarrow \quad F_X^{(k)}(\eta) \leq F_Y^{(k)}(\eta) \quad (\forall \eta \in \mathbb{R})$$

where X and Y are real random variables belonging to $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, \mathbb{P})$ with norm $\|\cdot\|_{k-1}$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By $\mathcal{L}_0$ we denote consistently the space of all scalar random variables.

Let P_X denote the probability distribution of X and $F_X^{(1)} = F_X$ its distribution function, i.e.,

$$F_X^{(1)}(\eta) = \mathbb{P}(\{X \leq \eta\}) = \int_{-\infty}^{\eta} P_X(d\xi) \quad (\forall \eta \in \mathbb{R})$$

and

$$F_X^{(k+1)}(\eta) = \int_{-\infty}^{\eta} F_X^{(k)}(\xi)d(\xi) = \int_{-\infty}^{\eta} \frac{(\eta-\xi)^k}{k!} P_X(d\xi)$$
$$= \frac{1}{k!}\|\max\{0, \eta - X\}\|_k^k \quad (\forall \eta \in \mathbb{R}),$$

where

$$\|X\|_k = \left(\mathbb{E}(|X|^k)\right)^{\frac{1}{k}} \quad (\forall k \geq 1).$$

A. Müller and D. Stoyan: *Comparison Methods for Stochastic Models and Risks*, Wiley, Chichester, 2002.

The original problem is equivalent to its split variable formulation

$$\min\left\{f(x) : x \in D,\, G(x,\xi) \geq X,\, F_X^{(k)}(\eta) \leq F_Y^{(k)}(\eta), \forall \eta \in I\right\}$$

by introducing a new real random variable X and the constraint

$$G(x,\xi) \geq X \quad \mathbb{P}\text{–almost surely}.$$

This formulation motivates the need of two different metrics for handling the two constraints of different nature:

The almost sure constraint $G(x,\xi) \geq X$ ($\mathbb{P}$-a.s.) and the functional constraint $F_X^{(k)}(\cdot) \leq F_Y^{(k)}(\cdot)$, respectively.

D. Dentcheva, A. Ruszczyński: Optimality and duality theory for stochastic optimization problems with nonlinear dominance constraints, *Math. Progr.* 99 (2004), 329–350.

Properties:

(i) Equivalent characterization of $\succeq_{(2)}$:

$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

for each nondecreasing concave utility $u : \mathbb{R} \to \mathbb{R}$ such that the expectations are finite.

(ii) The function $F_X^{(k)} : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $k \geq 1$ and convex for $k \geq 2$.

(iii) For every $k \in \mathbb{N}$ the SD relation $\succeq_{(k)}$ introduces a partial ordering in $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, \mathbb{P})$ which is not generated by a convex cone if Y is not deterministic.

Extensions: By imposing appropriate assumptions all results remain valid for the following two extended situations:

(a) finite number of kth order stochastic dominance constraints,

(b) the objective f is replaced by an expectation function of the form $\mathbb{E}[g(\cdot\,, \xi)]$ where g is a real-valued function defined on $\mathbb{R}^m \times \mathbb{R}^s$.

The case of discrete distributions:

Let ξ_j, X_j and Y_j the scenarios of ξ, X and Y with probabilities p_j, $j = 1, \ldots, n$. Then the second order dominance constraints (i.e. $k = 2$) in the split variable formulation can be expressed as

$$\sum_{j=1}^{n} p_j[\eta - X_j]_+ \leq \sum_{j=1}^{n} p_j[\eta - Y_j]_+ \quad (\forall \eta \in I).$$

The latter condition can be shown to be equivalent to

$$\sum_{j=1}^{n} p_j[Y_k - X_j)]_+ \leq \sum_{j=1}^{n} p_j[Y_k - Y_j]_+ \quad (\forall k = 1, \ldots, n).$$

if $Y_k \in I$, $k = 1, \ldots, n$. Here, $[\,\cdot\,]_+ = \max\{0, \cdot\,\}$.
Hence, the second order dominance constraints may be reformulated as linear constraints for the X_j, $j = 1, \ldots, n$, in

$$G(x, \xi_j) \geq X_j \quad (j = 1, \ldots, n).$$

D. Dentcheva, A. Ruszczyński: Optimality and duality theory for stochastic optimization problems with nonlinear dominance constraints, *Math. Progr.* 99 (2004), 329–350.
J. Luedtke: New formulations for optimization under stochastic dominance constraints, *SIAM J. Optim.* 19 (2008), 1433–1450.

Metrics associated to $\succeq_{(k)}$

Rachev metrics on $\mathcal{L}_{k-1}$:

$$\mathbb{D}_{k,p}(X,Y) := \begin{cases} \left(\int_{\mathbb{R}} \left| F_X^{(k)}(\eta) - F_Y^{(k)}(\eta) \right|^p d\eta \right)^{\frac{1}{p}} & , 1 \le p < \infty \\ \sup_{\eta \in \mathbb{R}} \left| F_X^{(k)}(\eta) - F_Y^{(k)}(\eta) \right| & , p = \infty \end{cases}$$

Proposition: It holds for any $X, Y \in \mathcal{L}_{k-1}$

$$\mathbb{D}_{k,p}(X,Y) = \zeta_{k,p}(X,Y) := \sup_{f \in \mathcal{D}_{k,p}} \left| \int_{\mathbb{R}} f(x) P_X(dx) - \int_{\mathbb{R}} f(x) P_Y(dx) \right|$$

if $\mathbb{E}(X^i) = \mathbb{E}(Y^i)$, $i = 1, \ldots, k-1$.
Here, $\mathcal{D}_{k,p}$ denotes the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that have measurable kth order derivatives $f^{(k)}$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |f^{(k)}(x)|^{\frac{p}{p-1}} dx \le 1 \quad (p > 1) \quad \text{or} \quad \operatorname{ess\,sup}_{x \in \mathbb{R}} |f^{(k)}(x)| \le 1 \quad (p = 1).$$

Note that the condition $\mathbb{E}(X^i) = \mathbb{E}(Y^i)$, $i = 1, \ldots, k-1$, is implied by the finiteness of $\zeta_{k,p}(X, Y)$, since $\mathcal{D}_{k,p}$ contains all polynomials of degree $k-1$. Conversely, if X and Y belong to $\mathcal{L}_{k-1}$ and $\mathbb{E}(X^i) = \mathbb{E}(Y^i)$, $i = 1, \ldots, k-1$, holds, then the distance $\mathbb{D}_{k,p}(X, Y)$ is finite.

Proposition:
There exists $c_k > 0$ (only depending on k) such that

$$\zeta_{k,\infty}(X,Y) \leq \zeta_{1,\infty}(X,Y) \leq c_k \zeta_{k,\infty}(X,Y)^{\frac{1}{k}} \quad (\forall X, Y \in \mathcal{L}_{k-1}).$$

$\zeta_{1,\infty}$ is the Kolmogorov metric and $\zeta_{1,1}$ the first order Fourier-Mourier or Wasserstein metric.

S. T. Rachev: *Probability Metrics and the Stability of Stochastic Models*, Wiley, 1991.

Structure and stability

We consider the kth order SD constrained optimization model

$$\min\left\{f(x) : x \in D,\ F^{(k)}_{G(x,\xi)}(\eta) \leq F^{(k)}_{Y}(\eta),\ \forall\eta \in \mathbb{R}\right\}$$

as semi-infinite program.

Relaxation: Replace $\mathbb{R}$ by some compact inverval $I = [a, b]$.

Proposition:
Under the general assumptions the feasible set

$$\mathcal{X}(\xi, Y) = \left\{x \in D : F^{(k)}_{G(x,\xi)}(\eta) \leq F^{(k)}_{Y}(\eta),\ \forall\eta \in I\right\}$$

is closed and convex in $\mathbb{R}^m$.

kth order uniform dominance condition ($k$udc) at (ξ, Y):
There exists $\bar{x} \in D$ such that

$$\min_{\eta\in I}\left(F^{(k)}_{Y}(\eta) - F^{(k)}_{G(\bar{x},\xi)}(\eta)\right) > 0\,.$$

Metrics on $\mathcal{L}^s_{k-1} \times \mathcal{L}_{k-1}$:

$$d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})) = \ell_{k-1}(\xi, \tilde{\xi}) + \mathbb{D}_{k,\infty}(Y, \tilde{Y}),$$

where $k \in \mathbb{N}$, $k \geq 2$ is the degree of the SD constraint and ℓ_{k-1} is the L_{k-1}-minimal distance or $(k-1)$th order Wasserstein distance defined by

$$\ell_{k-1}(\xi, \tilde{\xi}) := \inf \left\{ \int_{\Xi \times \Xi} \|x - \tilde{x}\|^{k-1} \eta(dx, d\tilde{x}) \right\}^{\frac{1}{k-1}},$$

where the infimum is taken w.r.t. all probability measures η on $\Xi \times \Xi$ with marginal P_ξ and $P_{\tilde{\xi}}$, respectively.

Proposition:

Let D be compact and assume that the function G satisfies

$$|G(x,u) - G(x,\tilde{u})| \leq L_G \|u - \tilde{u}\|$$

for all $x \in D$, $u, \tilde{u} \in \Xi$ and some constant $L_G > 0$. Assume that the kth order uniform dominance condition is satisfied at (ξ, Y).

Then there exist constants $L(k) > 0$ and $\delta > 0$ such that

$$d_{\mathrm{H}}(\mathcal{X}(\xi, Y), \mathcal{X}(\tilde{\xi}, \tilde{Y})) \leq L(k)\, d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})),$$

whenever the pair $(\tilde{\xi}, \tilde{Y})$ is chosen such that $d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})) < \delta$. Here, d_{H} denotes the Pompeiu-Hausdorff distance on bounded closed subsets of $\mathbb{R}^m$.

Note that $L(k)$ gets smaller with increasing $k \in \mathbb{N}$ if $\|\xi\|_{k-1}$ grows at most exponentially with k. Hence, higher order stochastic dominance constraints may have improved stability properties.

Let $v(\xi, Y)$ denote the optimal value and $S(\xi, Y)$ the solution set of

$$\min \left\{ f(x) : x \in D,\ x \in \mathcal{X}(\xi, Y) \right\}.$$

We consider the growth function

$$\psi_{(\xi,Y)}(\tau) := \inf \left\{ f(x) - v(\xi, Y) : d(x, S(\xi, Y)) \geq \tau,\ x \in \mathcal{X}(\xi, Y) \right\}$$

and

$$\Psi_{(\xi,Y)}(\theta) := \theta + \psi^{-1}_{(\xi,Y)}(2\theta) \quad (\theta \in \mathbb{R}_+),$$

where we set $\psi^{-1}_{(\xi,Y)}(t) = \sup\{\tau \in \mathbb{R}_+ : \psi_{(\xi,Y)}(\tau) \leq t\}$.

Note that $\Psi_{(\xi,Y)}$ is increasing, lower semicontinuous and vanishes at $\theta = 0$.

Main stability result

Theorem:

Let D be compact and assume that the function G satisfies

$$|G(x,u) - G(x,\tilde{u})| \leq L_G \|u - \tilde{u}\|$$

for all $x \in D$, $u, \tilde{u} \in \Xi$ and some constant $L_G > 0$. Assume that the kth order uniform dominance condition is satisfied at (ξ, Y).

Then there exist positive constants $L(k)$ and δ such that

$$\begin{aligned} |v(\xi,Y) - v(\tilde{\xi},\tilde{Y})| &\leq L(k)\, d_k((\xi,Y),(\tilde{\xi},\tilde{Y})) \\ \sup_{x\in S(\tilde{\xi},\tilde{Y})} d(x, S(\xi,Y)) &\leq \Psi_{(\xi,Y)}(L(k)\, d_k((\xi,Y),(\tilde{\xi},\tilde{Y}))) \end{aligned}$$

whenever $d_k((\xi,Y),(\tilde{\xi},\tilde{Y})) < \delta$.

(Klatte 94, Rockafelar-Wets 98)

Proof: Let the pair $(\tilde{\xi}, \tilde{Y}) \in \mathcal{L}^2_{k-1}$ be such that $\hat{\delta} := d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})) < \delta$, where $\delta > 0$ is the corresponding constant from the Proposition. Now, let $x \in S(\xi, Y)$ and $\tilde{x} \in S(\tilde{\xi}, \tilde{Y})$. Then there exists $\hat{x} \in \mathcal{X}(\xi, Y)$ such that

$$\|\hat{x} - \tilde{x}\| \leq L_H \hat{\delta}\,,$$

where L_H is the Lipschitz constant of the feasible set mapping. We obtain

$$\begin{aligned} v(\xi, Y) - v(\tilde{\xi}, \tilde{Y}) &= f(x) - f(\tilde{x}) \\ &\leq f(x) - f(\hat{x}) + f(\hat{x}) - f(\tilde{x}) \leq f(\hat{x}) - f(\tilde{x}) \\ &\leq L_f \|\hat{x} - \tilde{x}\| \leq L_f L_H \hat{\delta}\,, \end{aligned}$$

where L_f is the Lipschitz modulus of the function f on the compact set D. Analogously, we obtain the same estimate for $v(\tilde{\xi}, \tilde{Y}) - v(\xi, Y)$. Hence, we may set $L := L_f L_H$.
To derive the second estimate, let the pair $(\tilde{\xi}, \tilde{Y}) \in \mathcal{L}^2_{k-1}$ be selected as above and let $\tilde{x} \in S(\tilde{\xi}, \tilde{Y})$. Then there exists $x \in \mathcal{X}(\xi, Y)$ such that $\|\tilde{x} - x\| \leq L_H \hat{\delta}$. According to the definition of the growth function $\psi_{(\xi,Y)}$ we have

$$f(x) - v(\xi, Y) \geq \psi_{(\xi,Y)}(d(x, S(\xi, Y))).$$

Furthermore, we obtain the following chain of estimates

$$\begin{aligned} 2L\hat{\delta} &\geq L_f \|\tilde{x} - x\| + L\hat{\delta} \geq f(x) - f(\tilde{x}) + v(\tilde{\xi}, \tilde{Y}) - v(\xi, Y) \\ &= f(x) - v(\xi, Y) \geq \psi_{(\xi,Y)}(d(x, S(\xi, Y))), \end{aligned}$$

Finally, we conclude

$$\begin{aligned} d(\tilde{x}, S(\xi, Y)) &\leq L_H \hat{\delta} + d(x, S(\xi, Y)) \\ &\leq L_H \hat{\delta} + \psi^{-1}_{(\xi,Y)}(2L\hat{\delta}) \\ &\leq \Psi_{(\xi,Y)}(\max\{L_H, L\}\hat{\delta}). \end{aligned}$$

This completes the proof.

Dual multipliers and utilities

Let $\mathcal{Y} = C(I)$ and $\mathcal{Y}^*$ its dual which is isometrically isomorph to the space $\mathbf{rca}(I)$ of regular countably additive measures μ on I having finite total variation $|\mu|(I)$. The dual pairing is given by

$$\langle \mu, y \rangle = \int_I y(\eta)\mu(d\eta) \quad (\forall y \in \mathcal{Y},\ \mu \in \mathbf{rca}(I)).$$

We consider the closed convex cone

$$K = \{y \in \mathcal{Y} : y(\eta) \geq 0,\ \forall \eta \in I\}$$

and its polar cone K^-

$$K^- = \{\mu \in \mathbf{rca}(I) : \langle \mu, y \rangle \leq 0, \forall y \in K\}.$$

The semi-infinite constraint may be written as

$$\mathcal{G}(x; P_\xi, P_Y) := F_Y^{(k)} - F_{G(x,\xi)}^{(k)} \in K$$

and the semi-infinite program is

$$\min \left\{ f(x) : x \in D,\ \mathcal{G}(x; P_\xi, P_Y) \in K \right\}.$$

Lemma: (Dentcheva-Ruszczyǹski 03)
Let $k \geq 2$, $I = [a, b]$, $\mu \in -K^-$. There exists $u \in \mathcal{U}_{k-1}$ such that

$$\langle \mu, F_X^{(k)} \rangle = \int_I F_X^{(k)}(\eta)\mu(d\eta) = -\mathbb{E}[u(X)]$$

holds for every $X \in \mathcal{L}_{k-1}$. Here, $\mathcal{U}_{k-1}$ denotes the set of all functions $u \in C^{k-1}(\mathbb{R})$, for which there exists a nonnegative, nonincreasing, left-continuous, bounded function $\varphi : I \to \mathbb{R}$ such that

$$\begin{aligned} u^{(k-1)}(t) &= (-1)^k \varphi(t) &&, \mu\text{-a.e. } t \in [a, b], \\ u^{(k-1)}(t) &= (-1)^k \varphi(a) &&, t < a, \\ u(t) &= 0 &&, t \geq b, \\ u^{(i)}(b) &= 0 &&, i = 1, \ldots, k-2, \end{aligned}$$

where the symbol $u^{(i)}$ denotes the ith derivative of u. In particular, the utilities $u \in \mathcal{U}_{k-1}$ are nondecreasing and concave on $\mathbb{R}$.

Proof: Let $\mu \in \mathbf{rca}(I)$, $\mu \geq 0$. Then μ is extended to $\mathcal{B}(\mathbb{R})$ by assigning measure 0 to Borel sets not intersecting I. The function $u \in \mathcal{U}_{k-1}$ is then defined by putting $u(t) = 0$, $t \geq b$, $u^{(k-1)}(t) = (-1)^k \mu([t, b])$, μ-a.e. $t \leq b$, $u^{(i)}(b) = 0$, $= 1, \ldots, k-2$.
One obtains by repeated integration by parts for any $X \in \mathcal{L}_{k-1}$

$$\langle \mu, F_X^{(k)} \rangle = (-1)^k \int_{-\infty}^{b} F_X^{(k)}(\eta) du^{(k-1)}(t) = -\int_{-\infty}^{b} u(t) dF_X(t) = -\mathbb{E}[u(X)].$$

Optimality and duality

Define the Lagrange-like function $\mathfrak{L} : \mathbb{R}^m \times \mathcal{U}_{k-1} \to \mathbb{R}$ as

$$\mathfrak{L}(x, u; P_\xi, P_Y) := f(x) - \int_\Xi u(G(x, z))P_\xi(dz) + \int_\mathbb{R} u(t)P_Y(dt).$$

Theorem:

Let $k \geq 2$ and $I = [a, b]$. Assume the kth order uniform dominance condition at (ξ, Y). If a feasible point $\hat{x} \in \mathbb{D}$ is an optimal solution, then a function $\hat{u} \in \mathcal{U}_{k-1}$ exists so that

$$\mathfrak{L}(\hat{x}, \hat{u}; P_\xi, P_Y) = \min_{x \in D} \mathfrak{L}(x, \hat{u}, P_\xi, P_Y)$$

$$\int_\Xi \hat{u}(G(\bar{x}, z))P_\xi(dz) = \int_\mathbb{R} \hat{u}(t)P_Y(dt)$$

If $\hat{x}$ satisfies the dominance constraint and the above conditions for some $\hat{u} \in \mathcal{U}_{k-1}$, then $\hat{x}$ is an optimal solution. Furthermore, the dual problem is

$$\max_{u \in \mathcal{U}_{k-1}} \left[\inf_{x \in D} \left[f(x) + \mathbb{E}\left[u(G(x;\xi))\right] - \mathbb{E}\left[u(Y)\right]\right]\right]$$

and the duality relation holds.

Proof: The Lagrangian Λ associated with the primal program can be formulated as follows:

$$\Lambda(x, \mu; P_\xi, P_Y) = \begin{cases} f(x) + \langle \mu, \mathcal{G}(x; P_\xi, P_Y) \rangle & \text{if} \quad x \in D,\ \mu \in K^-, \\ -\infty & \text{if} \quad x \in D,\ \mu \notin K^-, \\ +\infty & \text{if} \quad x \notin D. \end{cases}$$

The optimality conditions for the primal problem state that if a feasible point $\bar{x}$ is an optimal solution, then a measure $\bar{\mu} \in K^-$ exists, so that

$$\begin{aligned} \Lambda(\bar{x}, \bar{\mu}; P_\xi, P_Y) &= \min_{x \in D} \Lambda(x, \bar{\mu}; P_\xi, P_Y) \\ \langle \bar{\mu}, \mathcal{G}(\bar{x}; P_\xi, P_Y) \rangle &= 0. \end{aligned}$$

The dual problem has the form (Rockafellar 74)

$$\max \Big\{ \inf_{x \in D} \{ f(x) + \langle \mu, \mathcal{G}(x; P_\xi, P_Y) \rangle \} : \mu \in K^- \Big\},$$

Using the Lemma, we associate a function $\bar{u} \in \mathcal{U}_{k-1}$ with the measure $\bar{\mu}$ and reformulate the Lagrangian Λ to the following form:

$$\Lambda(x, \bar{\mu}; P_\xi, P_Y) = \mathfrak{L}(x, \bar{u}; P_\xi, P_Y) = f(x) + \int_\Xi \bar{u}(G(x,z)) P_\xi(dz) - \int_\mathbb{R} \bar{u}(t) P_Y(dt)$$

whenever $x \in D$. The optimality conditions and the dual problem are reformulated using $\bar{u}$ and the new Lagrangian has the desired form. The duality relation holds due to the convexity of the problem and the uniform dominance condition.

Sensitivity of the optimal value function

Let the infimal function $v : C(D) \to \mathbb{R}$ be given by

$$v(g) = \inf_{x \in D} g(x).$$

If D is compact, v is finite and concave on $C(D)$, and Lipschitz continuous with respect to the supremum norm $\|\cdot\|_\infty$ on $C(D)$. Hence, it is Hadamard directionally differentiable on $C(D)$ and

$$v'(g; d) = \min \left\{ d(x) : x \in \arg\min_{x \in D} g(x) \right\}.$$

Let $\mathcal{U}^*_{k-1}$ denote the solution set of the dual problem. Any $\bar{u} \in \mathcal{U}^*_{k-1}$ is called shadow utility. For some shadow utility $\bar{u}$ and $g_{\bar{u}} = \mathfrak{L}(\cdot\,, \bar{u}; P_\xi, P_Y)$, the duality theorem implies $v(g_{\bar{u}}) = v(P_\xi, P_Y)$.

Corollary: Let D be compact and the assumptions of the duality theorem be satisfied. Then the optimal value function $v(P_\xi, P_Y)$ is Hadamard directionally differentiable on $C(D)$ and the directional derivative into any direction $d \in C(D)$ is

$$v'(g_{\bar{u}}; d) = v'(P_\xi, P_Y; d)) = \min \left\{ d(x) : x \in S(P_\xi, P_Y) \right\}.$$

Limit theorems for empirical approximations

Let (ξ_n, Y_n), $n \in \mathbb{N}$, be a sequence of i.i.d. (independent, and identically distributed) random vectors on some probability space. Let $P_\xi^{(n)}$ and $P_Y^{(n)}$ denote the corresponding empirical measures.

Empirical approximation:

$$\min\left\{f(x) : x \in D, \sum_{i=1}^{n} [\eta - G(x,\xi_i)]_+^{k-1} \leq \sum_{i=1}^{n} [\eta - Y_i]_+^{k-1}, \eta \in I\right\}$$

Optimal value:

$$\begin{aligned} v(P_\xi, P_Y) &= \inf_{x\in D} \mathfrak{L}(x, \bar{u}; P_\xi, P_Y) \\ &= \inf_{x\in D} \mathbb{E}\big[f(x) + \bar{u}(G(x,\xi)) - \bar{u}(Y)\big] \\ &= \inf_{x\in D} P(f(x) + \bar{u}(G(x,z)) - \bar{u}(t)), \end{aligned}$$

where $\bar{u}$ is a shadow utility and $P := P_\xi \times P_Y$.

Proposition:

Let the assumptions of the main stability theorem be satisfied. Let D and the supports $\Xi = \mathrm{supp}(P_\xi)$ and $\Upsilon = \mathrm{supp}(P_Y)$ be compact.

Then Γ_k is a Donsker class, i.e., the empirical process $\mathcal{E}_n g$ indexed by $g \in \Gamma_k$

$$\mathcal{E}_n g = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n} g(\xi_i, Y_i) - \mathbb{E}(g(\xi, Y))\Big) \xrightarrow{d} \mathbb{G}(g) \quad (g \in \Gamma_k)$$

converges in distribution to a Gaussian limit process $\mathbb{G}$ on the space $\ell^\infty(\Gamma_k)$ (of bounded functions on Γ_k) equipped with supremum norm, where

$$\Gamma_k = \big\{g_x : g_x(z,t) = f(x) + \bar{u}(G(x,z)) - \bar{u}(t), (z,t) \in \Xi \times \Upsilon, x \in D\big\}.$$

The Gaussian process $\mathbb{G}$ has zero mean and covariances $\mathbb{E}[\mathbb{G}(x)\,\mathbb{G}(\tilde{x})] = \mathbb{E}_P[g_x g_{\tilde{x}}] - \mathbb{E}_P[g_x]\mathbb{E}_P[g_{\tilde{x}}]$ for $x, \tilde{x} \in D$.

Γ_k is a parametric family of functions having a uniform Lipschitz modulus with bracketing number $\leq C\varepsilon^{-m}$ and, hence, a Donsker class.

Proposition: (functional delta method)
Let B_1 and B_2 be Banach spaces equipped with their Borel σ-fields and B_1 be separable. Let (X_n) be random elements of B_1, $h : B_1 \to B_2$ be a mapping and (τ_n) be a sequence of positive numbers tending to infinity as $n \to \infty$. If

$$\tau_n(X_n - \theta) \xrightarrow{d} X$$

for some $\theta \in B_1$ and some random element X of B_1 and h is Hadamard directionally differentiable at θ, it holds

$$\tau_n(h(X_n) - h(\theta)) \xrightarrow{d} h'(\theta; X),$$

where $\xrightarrow{d}$ means convergence in distribution.

Application:
$B_1 = C(D)$, $B_2 = \mathbb{R}$, $h(g) = \inf_{x \in D} g(x)$, h is concave and Lipschitz w.r.t. $\|\cdot\|_\infty$, and $h'(g; d) = \min\{d(y) : y \in \arg\min_{x \in D} g(x)\}$.

Theorem:

Let the assumptions of the Donsker class Proposition be satisfied. Then the optimal values $v(P_\xi^{(n)}, P_Y^{(n)})$, $n \in \mathbb{N}$, satisfy the limit theorem

$$\sqrt{n}\big(v(P_\xi^{(n)}, P_Y^{(n)}) - v(P_\xi, P_Y)\big) \xrightarrow{d} \min\{\mathbb{G}(x) : x \in S(P_\xi, P_Y)\}$$

where $\mathbb{G}$ is a Gaussian process with zero mean and covariances $\mathbb{E}[\mathbb{G}(x)\,\mathbb{G}(\tilde{x})] = \mathbb{E}_P[g_x g_{\tilde{x}}] - \mathbb{E}_P[g_x]\mathbb{E}_P[g_{\tilde{x}}]$ for $x, \tilde{x} \in S(P_\xi, P_Y)$.
If $S(P_\xi, P_Y)$ is a singleton, i.e., $S(P_\xi, P_Y) = \{\bar{x}\}$, the limit $\mathbb{G}(\bar{x})$ is normal with zero mean and variance $\mathbb{E}_P[g_{\bar{x}}^2] - (\mathbb{E}_P[g_{\bar{x}}])^2$.

The result allows the application of resampling techniques to determine asymptotic confidence intervals for the optimal value $v(P_\xi, P_Y)$, in particular, bootstrapping if $S(P_\xi, P_Y)$ is a singleton and subsampling in the general case.

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Conclusions

- Quantitative continuity properties for optimal values and solution sets in terms of a suitable distance of probability distributions have been obtained.
- A limit theorem for empirical optimal values is proved which allows to derive confidence intervals.
- **Extensions** to multivariate dominance constraints are desirable, e.g., for the concept

$$X \succeq_{(m,k)} Y \quad \text{iff} \quad v^\top X \succeq_{(k)} v^\top Y, \quad \forall v \in \mathcal{V},$$

where $\mathcal{V}$ is convex in $\mathbb{R}^m_+$ and $X, Y \in L^m_{k-1}$.
For example, $\mathcal{V} = \{v \in \mathbb{R}^m_+ : \|v\|_1 = 1\}$ is studied in (Dentcheva-Ruszczyński 09) and $\mathcal{V} \subseteq \{v \in \mathbb{R}^m : \|v\|_1 \leq 1\}$ in (Hu/Homem-de-Mello/Mehrotra 11).

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