ON CONVERGENCE RATES OF APPROXIMATE SOLUTIONS OF STOCHASTIC EQUATIONS

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Abstract: Continuity properties of the mappings $\mu \to \mu F^{-1}$ and $\mu \to \int h d\mu$ with respect to bounded Lipschitz distance on probability measures are investigated. The results are applied to the case where $x=F(z)$ is the solution of the differential equation $dx(s) = f(x(s))ds+g(x(s))dz(s)$ and $h(z)$ is some functional of x.

1. Introduction

Consider the mapping S_1 which turns $z \in C^1[0,1]$ into the solution $x=S_1(z)$ of the (scalar) integral equation

$$x(t) = x_0 + {}_0\int^t f(x(s))ds + {}_0\int^t g(x(s))dz(s) \qquad (1.1.)$$

Under certain conditions on f and g, S_1 extends to a mapping S defined on all bounded, measurable $z:[0,1]\to R$, which is continuous with respect to several metrics [1, 16, 11, 14]. Under the mapping S, the distribution μ of a random input z is carried into an output distribution μS^{-1}, and the mapping $\mu \to \mu S^{-1}$ inherits certain continuity properties with respect to suitable metrics on the space of probability distributions, which serve to obtain convergence rates, e. g. if Wiener measure is approximated by a sequence of "simpler" measures. Another natural question is the continuous dependence of certain moments like $\int \|x\|_\infty(\mu S^{-1})(dx)$ on the input distribution μ.

In Section 2, problems of this type are treated in the general framework of a mapping F between two separable metric spaces (Z, d_Z) and (X, d_X), following the line of research in [17, 18, 19, 5, 13]. As a metric on the space of probability distributions we will consider the bounded Lipschitz distance $\beta_Z(\mu,\nu)$ (cf. [2, 4]) defined by

$$\beta_Z(\mu,\nu) := \sup\{ |\int\psi d(\mu-\nu)| : \psi:Z\to R, \|\psi\|_{BL} \le 1\} \qquad (1.2.)$$

where

$$\|\psi\|_{BL} := \|\psi\|_\infty + \sup \{ |\psi(z_1)-\psi(z_2)|/d_Z(z_1,z_2) : z_1, z_2 \in Z,\ z_1 \neq z_2 \} \qquad (1.3.)$$

In Section 3, these results are applied to the mapping S, thus obtaining, in particular, convergence rates of the output distributions resp. certain moments of these, if the input distribution μ is approximated by a sequence of distributions μ_n. Similar results may be obtained in higher dimensions under additional restrictions on the coefficient function g in (1.1.) (see [14]; for related results which do not hinge on these restrictions but consider more or less special approximations of semimartingale inputs,see, e. g., [7, 9, 12, 10].

2. General results

Let (Z, d_Z) be a separable metric space, and $\mathit{0}$ be a fixed element of Z. For any locally Lipschitz continuous mapping G from (Z, d_Z) into some other metric space (Y, d_Y) we put

$$\begin{aligned} L_G(r) &:= \|G|_{K_r}\|_L \\ &:= \sup\{ d_Y(G(z_1),G(z_2))/d_Z(z_1,z_2) : z_1,z_2 \in K_r,\ z_1 \neq z_2 \} \end{aligned} \tag{2.1.}$$

where $K_r := \{z \in Z : d_Z(z, \mathit{0}) \le r\}$.

For any real valued and locally Lipschitz continuous mapping h defined on Z we put

$$B_h(r) := \|h|_{K_r}\|_\infty$$

$$BL_h(r) := \|h|_{K_r}\|_{BL} = L_h(r) + B_h(r)$$

Note that L_G as well as BL_h are nondecreasing and left continuous.

For any probability measure μ on the σ-algebra $\mathcal{B}(Z)$ of Borel subsets on Z we put

$$\varepsilon_\mu(r) := \mu(Z - K_r) \tag{2.2.}$$

noting that ε_μ is nonincreasing, right continuous and tends to zero for $r \to \infty$.

The following theorem improves Theorem 2 in [5]. There, one can find also a similar result for the Prokhorov metric instead of the bounded Lipschitz metric.

Theorem 1. Let F be a locally Lipschitz continuous mapping from (Z, d_Z) into some other separable metric space (X, d_X).Then there holds for any two probability measures μ, ν on $\mathcal{B}(Z)$:

$$\beta_X(\mu F^{-1}, \nu F^{-1}) \le \inf\{\beta_Z(\mu,\nu)[4+\max\{1, L_F(r)\}] + 4\varepsilon_\mu(r-1) \ : r > 1\} \tag{2.3.}$$

and

$$\beta_X(\mu F^{-1}, \nu F^{-1}) \le \beta_Z(\mu,\nu)[8+\max\{1, L_F(1+\varepsilon_\mu^{-1}(\beta_Z(\mu,\nu)))\}] \tag{2.4.}$$

where $\varepsilon_\mu^{-1}(t) := \inf\{r>0 : \varepsilon_\mu(r) < t\} \quad (t > 0)$.

The proof of Theorem 1 (and also that of Theorem 2 below) is based on the following key

Lemma 1.a) Let $h: Z \to R$ be locally Lipschitz continuous. Then there holds for any two probability measures μ, ν on $\mathcal{B}(Z)$ and and $r>0$:

$$\left|\int_Z h\, d(\mu-\nu)\right| \le \int_{Z-K_r} (|h|+B_h(r))d(\mu+\nu) + BL_h(r)\, \beta_Z(\mu,\nu). \tag{2.5.}$$

b) If, in addition, h is bounded, then there holds for any $r>1$:

$$\left|\int_Z h\, d(\mu-\nu)\right| \le \beta_Z(\mu,\nu)[4\|h\|_\infty + BL_h(r)] + 4\|h\|_\infty \varepsilon_\mu(r-1) \tag{2.6.}$$

Proof: According to [2, Lecture 7] there exists, for any $r>0$, a bounded, Lipschitz continuous extension h_r of $h|_{K_r}$ to the whole of Z, having the properties

$$\|h_r\|_{BL} = BL_h(r),\ \|h_r\|_\infty = B_h(r) \tag{2.7}$$

For any fixed $r>1$ we thus obtain the following chain of inequalities:

$$\left|\int_Z h\, d(\mu-\nu)\right|$$
$$\le \left|\int_Z (h-h_r)d\mu\right| + \left|\int_Z h_r\, d(\mu-\nu)\right| + \left|\int_Z (h-h_r)d\nu\right|$$
$$\le \int_{Z-K_r} (|h|+|h_r|)d(\mu+\nu) + \|h_r\|_{BL}\, \beta_Z(\mu,\nu)$$
$$\le \int_{Z-K_r} (|h|+ B_h(r))d(\mu+\nu) + BL_h(r)\, \beta_Z(\mu,\nu),$$

showing the validity of (2.5.). Under the assumption of b), we may proceed in our estimate by

$$\le 2\, \|h\|_\infty(\mu(Z-K_r) + \nu\, (Z-K_r)) + BL_h(r)\, \beta_Z(\mu,\nu).$$

The mapping $\phi:Z\to R_+$ defined by $\phi(z) := \min\{1, d_Z(z, K_{r-1})\}$ obeys

$$\|\phi\|_{BL} \le 2 \text{ and } 1_{Z-K_r} \le \phi \le 1_{Z-K_{r-1}}.$$

This leads to

$$\nu\, (Z-K_r) \le \int_Z \phi d\nu \le \int_Z \phi d\mu + \|\phi\|_{BL}\, \beta_Z(\mu,\nu) \le \mu(Z-K_{r-1}) + 2\, \beta_Z(\mu,\nu).$$

Thus we get

$$\left|\int_Z h\, d(\mu-\nu)\right| \le 4\, \|h\|_\infty\, \mu(Z-K_{r-1}) + \beta_Z(\mu,\nu)[4\|h\|_\infty + BL_h(r)],$$

which is (2.6.). ♦

Proof of Theorem 1: For all $\psi:X\to R$ with the property $\|\psi\|_{BL} \le 1$ there holds according to Lemma 1b)

$$\left|\int_X \psi\, d(\mu F^{-1}-\nu F^{-1})\right| = \left|\int_Z \psi\circ F\, d(\mu-\nu)\right|$$
$$\le \inf\{\beta_Z(\mu,\nu)[4+ BL_{\psi\circ F}(r)] + 4\varepsilon_\mu(r-1) : r > 1\}$$

But

$$BL_{\psi\circ F}(r) \le \|\psi\|_\infty + \|\psi\|_L \cdot L_F(r) \le \max\{1, L_F(r)\},$$

which yields (2.3.). (2.4.) follows immediately by putting $r := 1+\varepsilon_\mu^{-1}(\beta_Z(\mu,\nu))$ in (2.3.), noting that $\varepsilon_\mu(\varepsilon_\mu^{-1}(\beta_Z(\mu,\nu))) \le \beta_Z(\mu,\nu)$ by the right continuity of ε_μ. ♦

Now we are going to deal with quantitative continuity of generalized moments with respect to bounded Lipschitz distance. The following theorem is a slight improvement of [13, Thm.2.1]:

Theorem 2. Let $h:Z\to R$ be an unbounded locally Lipschitz continuous mapping. Then there holds for any two probability measures μ,ν on $\mathcal{B}(Z)$ and all $p\le 1$:

$$\left|\int_Z h\, d(\mu-\nu)\right| \le \beta_Z(\mu,\nu)^{1-(1/p)} \cdot [|h(0)| + 3(\|\underline{BL}_h(|z|)\|_{p,\mu} + \|\underline{BL}_h(|z|)\|_{p,\nu})] \qquad (2.8.)$$

where we use the abbreviations

$$|z| := d_Z(z,0) \quad \text{and} \quad \|\underline{BL}_h(|z|)\|_{p,\sigma} := \left(\int_Z \underline{BL}_h(|z|)^p\, \sigma(dz)\right)^{1/p} \quad (\sigma=\mu,\nu)$$

and $\underline{BL}_h$ denotes the right continuous modification of the function BL_h (note that $BL_h \le \underline{BL}_h$).

Proof: Using Lemma 1a) we obtain, for any $r>0$, the following chain of inequalities:

$$\left|\int_Z h\, d(\mu-\nu)\right|$$
$$\le \int_{Z-K_r} (|h|+B_h(r))(\mu+\nu)(dz) + BL_r(h)\, \beta_Z(\mu,\nu)$$
$$\le 2\int_{Z-K_r} (BL_h(|z|))(\mu+\nu)(dz) + BL_r(h)\, \beta_Z(\mu,\nu)$$

The first summand may be estimated as follows:

$$\int_{Z-K_r} (BL_h(|z|))(\mu+\nu)(dz)$$
$$\le \underline{BL}_h(r)^{1-p} \int_Z (\underline{BL}_h(|z|)^p(\mu+\nu)\, (dz),$$

hence results

$$\left|\int_Z h\, d(\mu-\nu)\right|$$
$$\le 2\, \underline{BL}_h(r)^{1-p}[(\|\underline{BL}_h(|z|)\|_{p,\mu})^p + \|(\underline{BL}_h(|z|)\|_{p,\nu})^p] + BL_h(r)\,\beta_Z(\mu,\nu) \qquad (2.9.)$$

Now we put

$$r := \sup\{s \ge 0 : BL_h(s) \le [\,\|\underline{BL}_h(|z|)\|_{p,\mu} + \|\underline{BL}_h(|z|)\|_{p,\nu} + |h(0)|]\,\beta_Z(\mu,\nu)^{-1/p}\}$$

In view of the left continuity of BL_h and the right continuity of $\underline{BL}_h$ we get

$$BL_h(r) \le [\,\|\underline{BL}_h(|z|)\|_{p,\mu} + \|\underline{BL}_h(|z|)\|_{p,\nu} + |h(0)|]\,\beta_Z(\mu,\nu)^{-1/p} \qquad (2.10.)$$

and

$$\underline{BL}_h(r) \ge [\,\|\underline{BL}_h(|z|)\|_{p,\mu} + \|\underline{BL}_h(|z|)\|_{p,\nu} + |h(0)|]\,\beta_Z(\mu,\nu)^{-1/p},$$

the latter inequality implying

$$\underline{BL}_h(r)^{1-p} \le \min\{(\|\underline{BL}_h(|z|)\|_{p,\mu})^{1-p}, (\|\underline{BL}_h(|z|)\|_{p,\nu})^{1-p}\}\,\beta_Z(\mu,\nu)^{1-(1/p)} \qquad (2.11.)$$

Combining (2.10.) and (2.11.) with (2.9.), we arrive at (2.8.). ♦

Remark 1.a) In virtue of the estimate

$$BL_h(r) \le L_h(r)(r+1) + |h(0)| \quad (r\ge 0), \qquad (2.12)$$

Theorem 2 is better than Thm. 2.1. in [13] in the sense that the finite moment condition $\int_Z (L_h(|z|)|z|)^p \mu(dz) < \infty$ required there guarantees finiteness of $\|\underline{BL}_h(|z|)\|_{p,\mu}$, but not vice versa. Indeed, consider the example $Z:=\mathbb{R}$, $h:=\sin(z^2)$, $\mu(dz):=(z^4+1)^{-1}dz$, $p:=2$. Then $BL_h(r)\le 2r+1$, hence $\|\underline{BL}_h(|z|)\|_{2,\mu} < \infty$, whereas $L_h(r)\ge 2(r-\pi)$ and thus $\int_Z (L_h(|z|)|z|)^2\mu(dz) = \infty$

b) Obviously, the inequalities in Theorems 1 and 2 remain valid if ε_μ, L_F and BL_h, respectively, are replaced by upper estimates. If, e. g., μ is "of Gaussian type", i.e. obeys an estimate

$$\varepsilon_\mu(r) \le c_1 \exp(-c_2 r^2) \quad (r>0) \qquad (2.13.)$$

and F is "of exponential type", i.e. obeys

$$L_F(r) \le k_1 \exp(k_2 r) \qquad (2.14.)$$

then (2.4.) yields

$$\beta_X(\mu F^{-1}, \nu F^{-1}) \le \gamma_1 \exp(\gamma_2 |\log \beta_Z(\mu,\nu)|^{1/2})\, \beta_Z(\mu,\nu) \qquad (2.15.)$$

(Note that, for all $\delta>0$, the r.h.s. of (2.15.) is $o(\beta_Z(\mu,\nu)^{1-\delta})$ for small $\beta_Z(\mu,\nu)$.)

If F has property (2.14.), then $h(z) := (d_X(F(z),x_o))^k$ (where x_o is some fixed element of X and $k\in\mathbb{N}$) admits an estimate

$$BL_h(r) \le \alpha_1 k \exp(\alpha_2 k r) \qquad (2.16.)$$

If, in addition to (2.16.), μ and ν are of "Gaussian type" (2.13.), then Theorem 2 yields, for all $\delta>0$:

$$\left|\int_Z h\, d(\mu-\nu)\right| = o(\beta_Z(\mu,\nu)^{1-\delta}) \text{ for small } \beta_Z(\mu,\nu) \qquad (2.17.)$$

We conclude this section by giving examples that at least the order of the estimates in Theorems 1 and 2 is optimal for small $\beta_Z(\mu,\nu)$):

Example 1. $X = Z = \mathbb{R}_+$, $k > 1$, $F(z) := z^k$. For $0<\alpha<2^{-1}$ we put

$$\mu_\alpha := 2^{-1}(\delta_0+\delta_{\alpha^{1/(1-k)}}),\ \nu_\alpha := 2^{-1}(\delta_0+\delta_{(\alpha^{1/(1-k)}+\alpha)}).$$

Then $\beta_Z(\mu_\alpha,\nu_\alpha) \le \alpha$, $\varepsilon_\mu^{-1}(\beta_Z(\mu_\alpha,\nu_\alpha))) = \alpha^{1/(1-k)}$, $L_F(r) = kr^{k-1}$, hence the r.h.s. of (2.4.) is bounded from above by a constant. On the other hand one has $\beta_X(\mu_\alpha F^{-1}, \nu_\alpha F^{-1}) \ge 2^{-1}$.

Example 2. $X = Z = \mathbb{R}_+$, $k,p > 1$, $h(z) := z^k$. For $0<\alpha<1$ we put

$$\mu_\alpha := \delta_0,\ \nu_\alpha := (1-\alpha)\,\delta_0 + \alpha\delta_{\alpha^{-1/kp}}.$$

Then $\beta_Z(\mu_\alpha,\nu_\alpha) \le \alpha$ and $BL_h(r) \le (k+1)r^k$, leading to

$$\|\underline{BL}_h(|z|)\|_{p,\mu_\alpha}=0 \text{ and } \|\underline{BL}_h(|z|)\|_{p,\nu_\alpha} \le (\alpha(k+1)^p(\alpha^{-1/kp})^{kp})^{1/p} = k+1.$$

Hence the r.h.s. of (2.8.) is bounded from above by const$\cdot\alpha^{1-1/p}$. But on the other hand there holds $\left|\int_Z h\, d(\mu_\alpha-\nu_\alpha)\right| = \alpha^{1-1/p}$.

3. An application to approximate solutions of stochastic differential equations

Let us return to the integral equation (1.1.). Under the conditions

f is locally Lipschitz continuous and satisfies a linear growth condition

g has a bounded and locally Lipschitz continuous derivative (3.1.)

the solution $x = S_1(z)$ of (1.1.) has the following representation [1, 16]

$$\begin{aligned} &x(t) = \phi(\xi_z(t), z(t)-z(0)) \\ &\xi_z(t) = x_0 + {}_0\!\int^t \eta(\xi_z(s), z(s)-z(0))ds \\ &\eta(\alpha,\beta) = (\delta\phi/\partial\alpha)(\alpha,\beta)^{-1} f(\phi(\alpha,\beta)) \\ &(\delta\phi/\partial\beta)(\alpha,\beta) = g(\phi(\alpha,\beta))\ ;\ \phi(\alpha,0) = \alpha \end{aligned} \tag{3.2.}$$

Defining x as in (3.2.) even for any bounded , measurable $z:[0,1]\to R$, one gets a mapping $S:z\to x$ which can be shown [14, Thm.1] to be continuous with respect to the norm

$$\|z\|_1 := |z(0)| + {}_0\!\int^1 |z(s)|ds$$

on $\{z : \|z\|_\infty \le R\}$ for each $R > 0$, hence is a continuous extension of S_1 in this sense.

Obviously S maps the space C[0,1] (of continuous functions) and the space D[0,1] (of right continuous functions with left limits), respectively, into itself, and it can be shown [16, 14] that S is locally Lipschitz continuous w.r. to the sup-norm on C[0,1] and w.r. to the modified Skorokhod metric d_0 on D[0,1] defined by

$$d_0(z_1,z_2) := \inf_{\lambda\in\Lambda} \max\{ \|z_1-z_2\circ\lambda\|_\infty , \sup_{0\le s<t\le 1} |\log|\lambda(t)-\lambda(s)|\,(t-s)^{-1}| \} \tag{3.3.}$$

where Λ is the set of all mappings λ from [0,1] onto [0,1] which are strictly monotonically increasing.

A mapping $z\in D[0,1]$ is said to have finite quadratic variation along some fixed sequence of partitions τ_n of [0,1] with mesh size tending to zero, if the weak limit ζ of the measures

$$\zeta_n := \sum_{t_i\in\tau_n} (z(t_{i+1})-z(t_i))^2\, \delta_{t_i}$$

exists (cf. [6]); the distribution function of ζ is denoted by $t\to\langle z\rangle(t)$. For $z\in D[0,1]$ of finite quadratic variation, $x=S(z)$ obeys the integral equation [14, Prop.1]

$$\begin{aligned} x(t) = x_0 &+ {}_0\!\int^t f(x(s))ds + {}_0\!\int^t g(x(s-))dz(s) + 1/2\cdot {}_0\!\int^t (gg')(x(s-))d\langle z\rangle^c(s) \\ &+ \sum_{s\le t}[\phi(x(s-),\Delta z(s))-x(s-) - g(x(s-)\Delta z(s)] \end{aligned} \tag{3.4.}$$

where

$${}_0\!\int^t g(x(s-))dz(s) := \lim_{n\to\infty} \sum_{t_i\in\tau_n, t_i<t} g(t_i)(z(t_{i+1}) - z(t_i))$$

and

$$\langle z\rangle(t) = \langle z\rangle^c(t) + \sum_{s\le t} \Delta z(s)^2$$

is the decomposition of $\langle z\rangle$ into its continuous and jump part.

For $z \in C[0,1]$ and $\langle z\rangle(t) \equiv t$ (which is a property shared by almost every Wiener path), (3.4.) specializes to

$$x(t) = x_0 + {}_0\int^t (f+(1/2)gg')(x(s))ds + {}_0\int^t g(x(s-))dz(s) \qquad (3.5.)$$

(which, for a Wiener input z, is an Itô stochastic differential equation with Stratonovich correction).
For a piecewise constant function z, (3.4.) specializes to

$$x(t) = x_0 + {}_0\int^t f(x(s))ds + \sum_{s\le t} [\phi(x(s-),\Delta z(s)) - x(s-)] \qquad (3.6.)$$

For certain coefficient functions f and g obeying (3.1.), the growth of S (and hence also that of L_S) may be larger than exponential, as the following example shows:

Example 3. Put $f(\alpha) := \alpha$, $g(\alpha) := \sin(\alpha\pi)$ $(\alpha \in R)$. Then the function ϕ occuring in (3.2.) is given by

$$\phi(\alpha,\beta) = (2/\pi)\arctan[\tan(\alpha\pi/2)\exp(\pi\beta)] \qquad (3.7.)$$

Put $x_0 := 1$, and define, for any $n \in N$, a sequence (t_m) by

$$\exp(t_1) = 1+1/n$$
$$(m-1/n)\exp(t_m) = m+1/n \quad (m\ge 2)$$

It is easily checked that

$$t_1 < 1/n \ ; \ 1/(nm) < t_m < 2/[n(m-1)] \ \text{ for } m\ge 2 \qquad (3.8.)$$

Choose $C_n \in R$ such that

$$\phi(1/n, C_n) = 1 - 1/n$$

The following chain of implications

$$\arctan[\tan(\pi/2n)\exp(\pi C_n)] = (\pi/2)(1-1/n)$$
$$\Rightarrow \tan(\pi/2n)\exp(\pi C_n) = \tan[(\pi/2)(1-1/n)]$$
$$\Rightarrow \exp(\pi C_n) = \cot^2(\pi/2n) \le n^2$$

shows that

$$C_n \le \log n \qquad (3.9.)$$

Put

$$T_m := t_1 + \ldots + t_m \ (m\ge 1), \quad T_0 := 0,$$

define

$$\underline{z}_n(t) := \begin{cases} 0 & \text{for } T_{2(m-1)} \le t < T_{2m-1} \\ C_n & \text{for } T_{2m-1} \le t < T_{2m} \end{cases}$$

and let z_n be the restriction of $\underline{z}_n$ to [0,1]. By (3.9.) there holds

$$\|z_n\|_\infty \le \log n \qquad (3.10.)$$

The solution $x_n := S(z_n)$ of equation (3.6.) with input z_n increases exponentially on any interval $[T_{m-1}, T_m)$ from $m-1/n$ to $m+1/n$, and jumps at any time point T_m from $m+1/n$ to $m+1-1/n$. In paricular, x_n is increasing and obeys

$$x_n(T_m) \ge m. \qquad (3.11.)$$

In virtue of (3.8.) we get the estimate

$$T_m \le (3/n)\log m \qquad (3.12.)$$

Combining (3.11.) and (3.12.), one arrives at

$$x_n(1) \ge \exp(n/3) \qquad (3.13.)$$

which together with (3.10.) yields

$$\|S(z_n)\|_\infty \geq \exp((1/3)\exp(\|z_n\|_\infty))$$ ♦

Under the following conditions, however, L_S has only exponential growth:

Theorem 3.[14, Thm.3] Assume, in addition to (3.1.), that f is globally Lipschitz continuous and that $0 < m \leq |g| \leq M < \infty$ for some real constants m,M. Then there holds for suitable k_1, k_2

$$L_S(r) \leq k_1 \exp(k_2 r) \qquad (r>0) \tag{3.14.}$$

in any of the following cases:

a) $(Z, d_Z) = (X, d_X) = (C[0,1], \text{sup-distance})$

b) $(Z, d_Z) = (X, d_X) = (D[0,1], d_o)$

c) $(Z, d_Z) = (M[0,1], d_s)$; $(X, d_X) = (D[0,1], d_s)$

where $M[0,1] := \{z \in D[0,1] : z \text{ is nondecreasing}\}$, and d_s is the Skorokhod distance defined by

$$d_s(z_1,z_2) := \inf_{\lambda\in\Lambda} \max\{ \|z_1-z_2 \circ \lambda\|_\infty, \|\lambda-id\|_\infty\} \tag{3.15.}$$

If μ_W is Wiener measure on C[0,1], then ε_{μ_W} obeys (2.13.) ; hence follows by (2.15.) that, under the assumptions of Theorem 3a) there exist constants γ_1, γ_2 such that for all probability measures ν on C[0,1] there holds

$$\begin{aligned}&\beta_{C[0,1]}(\mu_W S^{-1}, \nu S^{-1})\\ &\leq \gamma_1 \exp(\gamma_2 |\log \beta_{C[0,1]}(\mu_W,\nu)|^{1/2})\, \beta_{C[0,1]}(\mu_W,\nu)\end{aligned} \tag{3.16}$$

If μ_P is (unit mean) Poisson measure on M[0,1], then a simple estimate shows that

$$\varepsilon_{\mu_P}(r) \leq 1/\Gamma(r) \tag{3.17.}$$

Hence follows by (2.4.) that, under the assumptions of Theorem 3c) there exists, for any δ>0, a constant c such that for all probability measures ν on M[0,1] there holds

$$\beta_{D[0,1]}(\mu_P S^{-1}, \nu S^{-1}) \leq c \cdot \beta_{M[0,1]}(\mu_W,\nu)^{1-\delta} \tag{3.18.}$$

Finally we mention convergence rates with respect to bounded Lipschitz distance of some approximations to Wiener resp. Poisson distribution:

Example 4. Let, for $n\in N$, $(Y_{n,j})_{j=1,\dots,n}$ be a sequence of independent random variables, with

$$P[Y_{n,j}=1] = 1/n = 1-P[Y_{n,j}=0] \quad (j=1,\dots,n)$$

Put $\qquad z_n(t) := \sum_{1\leq i\leq j} Y_{n,i}$ for $j/n \leq t < (j+1)/n$; $0 \leq j \leq n$.

Let μ_n be the distribution of z_n, and μ_P be standard Poisson measure on M[0,1].Then there holds according to [4, Thm.6.1.]

$$\beta_{M[0,1]}(\mu_P, \mu_n) = O(n^{-1}) \tag{3.19.}$$

Combining (3.18.) and (3.19.) one gets

$$\beta_{D[0,1]}(\mu_P S^{-1}, \mu_n S^{-1}) = O(n^{-1+\delta}) \quad \text{for all } \delta>0. \tag{3.20.}$$

Example 5.a) Let, for $n\in N$, $(Y_{n,j})_{j=1,\dots,n}$ be a sequence of independent random variables, with

$$P[Y_{n,j}=n^{-1/2}] = P[Y_{n,j}=-n^{-1/2}] = 1/2 \quad (j=1,\dots,n)$$

Put $\qquad z_n(t) := \sum_{1\leq i\leq j} Y_{n,i} + (t-j/n)Y_{n,j+1}$ for $j/n \leq t \leq (j+1)/n$, $0\leq j\leq n$.

Let μ_n be the distribution of z_n, and μ_W be standard Wiener measure on C[0,1]. Then one derives from [8, Thm.1]:

$$\beta_{C[0,1]}(\mu_W, \mu_n) = O(n^{-1/2}\log n) \qquad (3.21.)$$

Combining (3.16.) and (3.21.) one gets, for suitable $\gamma>0$,

$$\beta_{C[0,1]}(\mu_W S^{-1}, \mu_n S^{-1}) = O(\, n^{-1/2}\exp(\gamma(\log n)^{1/2})) \qquad (3.22.)$$

b) If w(t) is a standard Wiener process and $w_n(t)$ is a "polygonal approximation" of w(t) (coinciding with w in t = 0, 1/n, 2/n,...,1 and piecewise linear between these points), then it can be shown (cf.[15, Remark 2b)]) that

$$E[\|w-w_n\|_\infty{}^p]^{1/p} = O(n^{-1/2}(\log n)^{1/2}) \qquad (3.23.)$$

holds for all $p\geq 1$.

c) The convergence rate (3.23.) even holds true if w_n is the conditional expectation of w with respect to a certain discrete σ-algebra. More precisely, let $I_{n,1},\ldots, I_{n,m(n)}$ be disjoint subintervals of R, each having standard normal probability $m(n)^{-1}$. Put

$$A_n := \sigma(\{\underline{w}_j \in I_{n,i}\} : j=1,\ldots,n;\ i=1,\ldots,m(n)\})$$

where $\underline{w}_j := n^{1/2}(w(j/n)-w((j-1)/n)$. In [15, Thm.1] it is proved that $w_n := E[w \mid A_n]$ (which is a "polygonal approximation of w with finitely many relizations") has the convergence rate (3.23.), provided that $\sup\{n/m(n) : n\in N\}$ is finite.

d) If - in either of the cases b) and c) - μ_n denotes the distribution of w_n, then (3.23.) (with p=1) implies immediately that

$$\beta_{C[0,1]}(\mu_W,\mu_n) = O(n^{-1/2}(\log n)^{1/2}) \qquad (3.24.)$$

which is a slightly better convergence rate than (3.21.).

If ψ is a real valued mapping on C[0,1] and J is a convex majorant of $BL_{\psi \circ S}$ such that $\int J(\|z\|_\infty)^p \mu_W(dz)$ is finite for all p>1 (a function J with these properties exists, e. g., for $\psi(x) = \|x\|_\infty{}^k$ $(k \in N)$ under the assumptions of Theorem 3, cf. (2.16.)), then Jensen's inequality guarantees that

$$\int J(\|z\|_\infty)^p\, \mu_n(dz) \leq \int J(\|z\|_\infty)^p\, \mu_W(dz) < \infty \quad (n\in N)\,;$$

together with Theorem 2 then follows for all $\delta>0$

$$\left|\int \psi(x)\, \mu_n S^{-1}(dx) \;-\; \int \psi(x)\, \mu_W S^{-1}(dx)\right| = O(n^{-(1/2)+\delta}).$$

References:

[1] Doss, H.: Liens entre équations différentielles stochastiques et ordinaires. Ann. Inst. Henri Poincaré, Sect. B.XIII, 99-125 (1977).

[2] Dudley, R.M.: Probabilities and Metrics. Lecture Notes Series No. 45. Aarhus Universitet 1976.

[3] Dudley, R.M.: Speeds of metric probability convergence. Z. Wahrscheinlichkeitsth. verw. Geb. 22, 323-332 (1972).

[4] Dudley, R.M.: Distances of probability measures and random variables. Ann. Math. Statist. 39, 1563-1572 (1968).

[5] Engl, H.W., Wakolbinger, A.: Continuity properties of the extension of a locally Lipschitz continuous map to the space of probability measures. Mh. Math. 100, 85-103 (1985).

[6] Föllmer, H.: Calcul d'Itô sans probabilités, in: Séminaire de probabilités XV. Lecture Notes Math. 850, 143-150. Berlin-Heidelberg-New York: Springer 1981.

[7] Ikeda, N., Watanabe,S.: Stochastic differential equations and diffusion processes. Amsterdam, Tokyo: North Holland/Kodansha 1981.

[8] Komlos, J., Major, P., Tusnády, G.: An approximation of partial sums of independent random variables, and the sample distribution function, II. Z. Wahrscheinlichkeitsth. verw. Geb. 34, 33-58 (1976).

[9] Konecny, F.: On Wong-Zakai approximation of stochastic differential equations. J. Multivariate Anal. 13, 605-611 (1983).

[10] Kubilius, K., Mikulevicius, R.; On the rate of convergence of semimartingales. Preprint, to appear.

[11] Marcus, S.I.: Modeling and approximation of stochastic differential equations driven by semimartingales. Stochastics 4, 223-245 (1981).

[12] Protter, Ph.: Approximations of solutions of stochastic differential equations driven by semimartingales. Ann. Probab. 13, 716-743 (1985).

[13] Römisch, W., Wakolbinger, A.: Obtaining convergence rates for approximations in Stochastic Programming. To appear in: J. Guddat (ed.), Parametric Optimization and Related Topics, Berlin: Akademie-Verlag.

[14] Römisch, W., Wakolbinger, A.: On Lipschitz dependence in systems with differentiated inputs. Math. Ann. 272, 237-248 (1985).

[15] Römisch, W., Wakolbinger, A.: Convergence rates of approximate solutions of stochastic differential equations. Technical report, Linz 1983.

[16] Sussmann, H. On the gap between deterministic and stochastic ordinary differential equations. Ann. Probab. 6, 19-44 (1978).

[17] Topsœ, F.: Preservation of weak convergence under mappings. Ann. Math. Statist. 38, 1661-1665 (1967).

[18] Whitt, W.: Preservation of rates of convergence under mappings. Z. Wahrscheinlichkeitsth. verw. Geb. 29, 39-44 (1974).

[19] Zolotarev, V.M.: Metric distances in spaces of random variables and their distributions, Math. USSR Sbornik 30, 373-401 (1976).