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On a general ρ-algorithm

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Abstract: In this note a generalisation of the ρ-algorithm is proved using the Schur-complement-method.

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1. Introduction

In numerical analysis various lozenge algorithms are in use (see [1,3,12]). For instance, Wynn's (classical) ϵ-algorithm [13], the first and second generalisation of the ϵ-algorithm by Brezinski [1], the ρ- and θ-algorithm [1] are tools for convergence acceleration. Most of them are obtained as particular cases of quasi-linear extrapolation (see [3] for instance). Then the E-algorithm can also be used, see [9,7,3]. But the latter has a complexity $O(n^3)$ instead of $O(n^2)$ for the lozenge algorithm. The question is: Why does the complexity reduce?

This is of great interest for two reasons. The first is to get an abstract understanding of the algorithm which clearifies the relations between different lozenge rules and leads to a systematic treatment of similar particular cases. The second is to derive new algorithms—if possible.

This topic was discussed by the author in [5] for the ϵ-algorithm, see 2.5. In this paper the same methods are applied to the ρ-algorithm and lead to an identity for some determinant quotients. It is called general ρ-algorithm and is stated and proved in Section 2 using the Schur-complement-method [10,11].

In Section 3 the common ρ-algorithm is obtained as a particular case of the general ρ-algorithm, while Section 4 deals with another rational particular case.

A rigorous discussion of all particular cases of the general ρ-algorithm is an important but open question. Under further regularity assumptions any lozenge rule could be a particular case. This is shown in Section 5.

2. A general ρ-algorithm

2.1. Notation. Let $\mathbb{N}$ or $\mathbb{N}_0$ denote the set of positive or nonnegative integers, respectively, and let $\mathbb{C}$ be the complex field.

Throughout this section $(s_n \mid n \in \mathbb{N}_0) \in \mathbb{C}^{\mathbb{N}_0}$, $(\alpha_n \mid n \in \mathbb{N}_0) \in \mathbb{C}^{\mathbb{N}_0}$ and $(g_i(n) \mid i \in \mathbb{N},\ n \in \mathbb{N}_0) \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}_0}$ are complex sequences. For each $i \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we use the abbreviations $\Delta s_n := s_{n+1} - s_n$, $\Delta\alpha_n := \alpha_{n+1} - \alpha_n$ and $\Delta g_i(n) := g_i(n+1) - g_i(n)$.

For nonvanishing denominators we define for any $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\rho_{2k}^n := \frac{\begin{vmatrix} s_n & \cdots & s_{n+2k} \\ g_1(n) & \cdots & g_i(n+2k) \\ \vdots & & \vdots \\ g_{2k}(n) & \cdots & g_{2k}(n+2k) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_1(n+2k-1) \\ \vdots & & \vdots \\ \Delta g_{2k}(n) & \cdots & \Delta g_{2k}(n+2k-1) \end{vmatrix}}, \tag{1}$$

$$\rho_{2k+1}^n := \frac{\begin{vmatrix} \Delta\alpha_n & \cdots & \Delta\alpha_{n+2k} \\ \Delta g_1(n) & \cdots & \Delta g_1(n+2k) \\ \vdots & & \vdots \\ \Delta g_{2k}(n) & \cdots & \Delta g_{2k}(n+2k) \end{vmatrix}}{\begin{vmatrix} \Delta s_n & \cdots & \Delta s_{n+2k} \\ \Delta g_1(n) & \cdots & \Delta g_1(n+2k) \\ \vdots & & \vdots \\ \Delta g_{2k}(n) & \cdots & \Delta g_{2k}(n+2k) \end{vmatrix}}, \tag{2}$$

$$\lambda_{2k}^n := \frac{\begin{vmatrix} \Delta\alpha_n & \cdots & \Delta\alpha_{n+2k-1} \\ \Delta s_n & \cdots & \Delta s_{n+2k-1} \\ \Delta g_1(n) & \cdots & \Delta g_1(n+2k-1) \\ \vdots & & \vdots \\ \Delta g_{2k-2}(n) & \cdots & \Delta g_{2k-2}(n+2k-1) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_1(n+2k-1) \\ \vdots & & \vdots \\ \Delta g_{2k}(n) & \cdots & \Delta g_{2k}(n+2k-1) \end{vmatrix}}, \tag{3}$$

$$\lambda_{2k+1}^n := \frac{\begin{vmatrix} g_1(n) & \cdots & g_1(n+2k-1) \\ \vdots & & \vdots \\ g_{2k}(n) & \cdots & g_{2k}(n+2k-1) \end{vmatrix}}{\begin{vmatrix} \Delta s_n & \cdots & \Delta s_{n+2k-2} \\ \Delta g_1(n) & \cdots & \Delta g_1(n+2k-2) \\ \vdots & & \vdots \\ \Delta g_{2k-2}(n) & \cdots & \Delta g_{2k-2}(n+2k-2) \end{vmatrix}} \tag{4}$$

In addition we initialize $\rho_0^n := s_n$ and $\rho_1^n := \Delta\alpha_n/\Delta s_n$ and use the convention that empty determinants are equal to one.

The main result of this note is given in the following theorem and proved in 2.4.

2.2. Theorem. *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

(a) *If the denominators of ρ_{2k-2}^{n+1}, ρ_{2k-1}^{n}, ρ_{2k-1}^{n+1} and λ_{2k}^n do not vanish, then the denominator of ρ_{2k}^n is nonzero and*

$$\left(\rho_{2k}^n - \rho_{2k-2}^{n+1}\right)\cdot\left(\rho_{2k-1}^{n+1} - \rho_{2k-1}^{n}\right) = \lambda_{2k}^n\cdot\left(\lambda_{2k+1}^{n+1} - \lambda_{2k+1}^{n}\right). \tag{5}$$

(b) *If the denominators of ρ_{2k-1}^{n+1}, ρ_{2k}^n and ρ_{2k}^{n+1} do not vanish, then*

$$\rho_{2k}^n \neq \rho_{2k}^{n+1}$$

implies that the denominator of ρ_{2k+1}^n is nonzero and

$$\left(\rho_{2k+1}^n - \rho_{2k-1}^{n+1}\right)\cdot\left(\rho_{2k}^{n+1} - \rho_{2k}^{n}\right) = \lambda_{2k+1}^{n+1}\cdot\left(\lambda_{2k}^{n} - \lambda_{2k}^{n+1}\right). \tag{6}$$

Moreover, 2.4 will show the following:

2.3. Additional Remarks. (i) The nonvanishing of all denominators used on the left-hand side of (5) or (6), respectively, implies that all denominators used on the right-hand side are nonzero.

(ii) In (a) $\rho_{2k-1}^n \neq \rho_{2k-1}^{n+1}$ implies that the numerator of λ_{2k}^n is nonzero.

2.4. Proofs. Consider the matrix A,

$$A := \begin{pmatrix} \alpha_n & \Delta\alpha_n & \cdots & \Delta\alpha_{n+2k} \\ s_n & \Delta s_n & \cdots & \Delta s_{n+2k} \\ g_1(n) & \Delta g_1(n) & \cdots & \Delta g_1(n+2k) \\ \vdots & \vdots & & \vdots \\ g_{2k}(n) & \Delta g_{2k}(n) & \cdots & \Delta g_{2k}(n+2k) \end{pmatrix} \in \mathbb{C}^{(2k+2)\times(2k+2)}. \tag{7}$$

For convenience of notation $A\binom{i_1 \cdots i_m}{j_1 \cdots j_m}$ or $\left|\begin{smallmatrix} i_1 \cdots i_m \\ j_1 \cdots j_m \end{smallmatrix}\right|$ is referred to as the submatrix of A consisting of the rows $j_1 \ldots j_m$ and the columns $i_1 \ldots i_m$ of A or its determinant, respectively. Therefore

$$\rho_{2k-2}^{n+1} = \frac{\begin{vmatrix} 1, 3\ldots 2k \\ 2\ldots 2k \end{vmatrix} + \begin{vmatrix} 2\ldots 2k \\ 2\ldots 2k \end{vmatrix}}{\begin{vmatrix} 3\ldots 2k \\ 3\ldots 2k \end{vmatrix}}, \qquad \rho_{2k-1}^{n} = \frac{\begin{vmatrix} 2\ldots 2k \\ 1, 3\ldots 2k \end{vmatrix}}{\begin{vmatrix} 2\ldots 2k \\ 2\ldots 2k \end{vmatrix}},$$

$$\rho_{2k-1}^{n+1} = \frac{\begin{vmatrix} 3\ldots 2k+1 \\ 1, 3\ldots 2k \end{vmatrix}}{\begin{vmatrix} 3\ldots 2k+1 \\ 2\ldots 2k \end{vmatrix}}, \qquad \rho_{2k}^{n} = \frac{\begin{vmatrix} 1\ldots 2k+1 \\ 2\ldots 2k+2 \end{vmatrix}}{\begin{vmatrix} 2\ldots 2k+1 \\ 3\ldots 2k+2 \end{vmatrix}},$$

$$\rho_{2k}^{n+1} = \frac{\begin{vmatrix} 1, 3\ldots 2k+2 \\ 2\ldots 2k+2 \end{vmatrix} + \begin{vmatrix} 2\ldots 2k+2 \\ 2\ldots 2k+2 \end{vmatrix}}{\begin{vmatrix} 3\ldots 2k+2 \\ 3\ldots 2k+2 \end{vmatrix}}, \qquad \rho_{2k+1}^{n} = \frac{\begin{vmatrix} 2\ldots 2k+2 \\ 1, 3\ldots 2k+2 \end{vmatrix}}{\begin{vmatrix} 2\ldots 2k+2 \\ 2\ldots 2k+2 \end{vmatrix}},$$

$$\lambda_{2k}^{n}=\frac{\begin{vmatrix}2\ldots 2k+1\\ 1\ldots 2k\end{vmatrix}}{\begin{vmatrix}2\ldots 2k+1\\ 3\cdots 2k+2\end{vmatrix}},\qquad \lambda_{2k}^{n+1}=\frac{\begin{vmatrix}3\ldots 2k+2\\ 1\ldots 2k\end{vmatrix}}{\begin{vmatrix}3\ldots 2k+2\\ 3\ldots 2k+2\end{vmatrix}},\qquad \lambda_{2k+1}^{n}=\frac{\begin{vmatrix}1\ldots 2k\\ 3\ldots 2k+2\end{vmatrix}}{\begin{vmatrix}2\ldots 2k\\ 2\ldots 2k\end{vmatrix}},$$

$$\lambda_{2k+1}^{n+1}=\frac{\begin{vmatrix}1,3\ldots 2k+1\\ 3\ldots 2k+2\end{vmatrix}+\begin{vmatrix}2\ldots 2k+1\\ 3\ldots 2k+2\end{vmatrix}}{\begin{vmatrix}3\ldots 2k+1\\ 2\ldots 2k\end{vmatrix}},$$

where we have occasionally used linearity in the first column and then replaced every other column by its difference with the preceding one. All matrices occurring in these determinants contain $P := A\binom{3\ldots 2k}{3\ldots 2k}$ as a (possibly empty) submatrix. In the proof of (a) we have $\left|\begin{smallmatrix}3\ldots 2k\\ 3\ldots 2k\end{smallmatrix}\right| \neq 0$. Hence the Schur complement [8] A/P can be considered. It is computed by transforming A into

$$B=\begin{pmatrix} a & b & 0 & c & d\\ e & f & 0 & g & h\\ * & * & P & * & *\\ i & j & 0 & q & r\\ s & t & 0 & u & v\end{pmatrix}$$

by block-elimination taking P as a pivot block [4] (where $*$ represents unimportant entries of B). Then

$$A/P=B\begin{pmatrix}1,2,2k+1,2k+2\\ 1,2,2k+1,2k+2\end{pmatrix}=\begin{pmatrix} a & b & c & d\\ e & f & g & h\\ i & j & q & r\\ s & t & u & v\end{pmatrix}.$$

Exploiting the quotient property of Schur complements one easily computes

$$\rho_{2k-2}^{n+1}=e+f,\qquad \rho_{2k-1}^{n}=b/f,\qquad \rho_{2k-1}^{n+1}=c/g,$$

$$\rho_{2k}^{n}=\frac{\begin{vmatrix}e & f & g\\ i & j & q\\ s & t & u\end{vmatrix}}{\begin{vmatrix}j & q\\ t & u\end{vmatrix}},\qquad \rho_{2k}^{n+1}=\frac{\begin{vmatrix}e & g & h\\ i & q & r\\ s & u & v\end{vmatrix}+\begin{vmatrix}f & g & h\\ j & q & r\\ t & u & v\end{vmatrix}}{\begin{vmatrix}q & r\\ u & v\end{vmatrix}},$$

$$\rho_{2k+1}^{n}=\frac{\begin{vmatrix}b & c & d\\ j & q & r\\ t & u & v\end{vmatrix}}{\begin{vmatrix}f & g & h\\ j & q & r\\ t & u & v\end{vmatrix}},\qquad \lambda_{2k}^{n}=\frac{\begin{vmatrix}b & c\\ f & g\end{vmatrix}}{\begin{vmatrix}j & q\\ t & u\end{vmatrix}},\qquad \lambda_{2k}^{n+1}=\frac{\begin{vmatrix}c & d\\ g & h\end{vmatrix}}{\begin{vmatrix}q & r\\ u & v\end{vmatrix}},$$

$$\lambda_{2k+1}^{n}=\frac{\begin{vmatrix}i & j\\ s & t\end{vmatrix}}{f},\qquad \lambda_{2k+1}^{n+1}=\frac{\begin{vmatrix}i & q\\ s & u\end{vmatrix}+\begin{vmatrix}j & q\\ t & u\end{vmatrix}}{g}.$$

We first note that the denominator of λ_{2k}^n equals that of ρ_{2k}^n etc. Secondly we note that $\rho_{2k-1}^n \neq \rho_{2k-1}^{n+1}$ implies $b \cdot g - c \cdot f \neq 0$ and hence $\lambda_{2k}^n \neq 0$. This proves Remarks 2.3(i) and (ii).

Using the above identities for the coefficients occurring in (5) then with

$$\begin{vmatrix} e & f & g \\ i & j & q \\ s & t & u \end{vmatrix} - (e+f) \cdot \begin{vmatrix} j & q \\ t & u \end{vmatrix} = g \cdot \begin{vmatrix} i & j \\ s & t \end{vmatrix} - f \cdot \left(\begin{vmatrix} j & q \\ t & u \end{vmatrix} - \begin{vmatrix} i & q \\ s & u \end{vmatrix} \right)$$

one easily verifies (a).

To prove (b) we firstly assume $\left|{}^{3\ldots 2k}_{3\ldots 2k}\right| \neq 0$. Hence the Schur complement A/P can be considered and leads to the identities given above. Using these representations some simple computations show

$$\rho_{2k}^{n+1} - \rho_{2k}^n = \begin{vmatrix} f & g & h \\ j & q & r \\ t & u & v \end{vmatrix} \cdot \frac{\begin{vmatrix} i & q \\ s & u \end{vmatrix} + \begin{vmatrix} j & q \\ t & u \end{vmatrix}}{\begin{vmatrix} q & r \\ u & v \end{vmatrix} \cdot \begin{vmatrix} j & q \\ t & u \end{vmatrix}}. \tag{8}$$

Hence $\rho_{2k}^{n+1} \neq \rho_{2k}^n$ implies that the denominator of ρ_{2k+1}^n is nonzero. Again, using the above identities for the coefficients occurring in (6) and (8), then with

$$\begin{vmatrix} b & c & d \\ j & q & r \\ t & u & v \end{vmatrix} \cdot g - \begin{vmatrix} f & g & h \\ j & q & r \\ t & u & v \end{vmatrix} \cdot c = \begin{vmatrix} q & r \\ u & v \end{vmatrix} \cdot \begin{vmatrix} b & c \\ f & g \end{vmatrix} - \begin{vmatrix} c & d \\ g & h \end{vmatrix} \cdot \begin{vmatrix} j & q \\ t & u \end{vmatrix}$$

one easily obtains (b).

Finally we drop the additional assumption that $\left|{}^{3\ldots 2k}_{3\ldots 2k}\right| \neq 0$ using continuity arguments. Indeed, let R denote the set of all matrices $A \in \mathbb{C}^{(2k+2)\times(2k+2)}$ having regular submatrices

$$A\begin{pmatrix} 3\ldots 2k+1 \\ 2\ldots 2k \end{pmatrix}, \qquad A\begin{pmatrix} 2\ldots 2k+1 \\ 3\ldots 2k+2 \end{pmatrix}, \qquad A\begin{pmatrix} 3\ldots 2k+2 \\ 3\ldots 2k+2 \end{pmatrix}$$

and satisfying $\rho_{2k}^{n+1} \neq \rho_{2k}^n$. Then, the subset

$$R_0 := \left\{ A \in R \,\middle|\, \begin{vmatrix} 3\ldots 2k \\ 3\ldots 2k \end{vmatrix} \neq 0 \right\}$$

is dense in R (endowed with the sup-norm for instance). On the other hand, the identities of (b) are continuous in R and proved in R_0. Hence (b) also holds in R. □

2.5. Connection with a general ϵ-algorithm. In [5] the author presented a general ϵ-algorithm which can be written as

$$e_{2k}^n = e_{2k-2}^{n+1} + \frac{\mu_k^n}{e_{2k-1}^{n+1} - e_{2k-1}^n}, \qquad e_{2k+1}^n = e_{2k-1}^{n+1} + \frac{\mu_k^n}{e_{2k}^{n+1} - e_{2k}^n}. \tag{9}$$

This algorithm reduces to Wynn's ϵ-algorithm [13] in the case of $g_k(n) := \Delta^k s_n$.

It seems quite interesting that the e_k^n have the same representation as the ρ_k^n, indeed

$$E_{2k}^n = e_{4k}^n = \rho_{2k}^n \quad \text{and} \quad e_{4k+1}^n = \rho_{2k+1}^n \tag{10}$$

for $k, n \in \mathbb{N}_0$ with E_k^n from the E-algorithm [3,(3)].

Using the notation of 2.4 the coefficients μ_k^n are given by

$$\mu_{2k-1}^n = \begin{vmatrix} 2\ldots 2k+1 \\ 1\ldots 2k \end{vmatrix} \cdot \frac{\begin{vmatrix} 1, 3\ldots 2k \\ 3\ldots 2k+1 \end{vmatrix} + \begin{vmatrix} 2\ldots 2k \\ 3\ldots 2k+1 \end{vmatrix}}{\begin{vmatrix} 3\ldots 2k+1 \\ 2\ldots 2k \end{vmatrix} \cdot \begin{vmatrix} 2\ldots 2k \\ 3\ldots 2k+1 \end{vmatrix}} \tag{11}$$

which can easily be transformed to

$$\mu_{2k-1}^n = \lambda_{2k}^n \cdot \lambda_{2k+1}^{n+1} \cdot \frac{\begin{vmatrix} j & q \\ t & u \end{vmatrix} \cdot (i+j)/j}{\begin{vmatrix} i & q \\ s & u \end{vmatrix} + \begin{vmatrix} j & q \\ t & u \end{vmatrix}} = \lambda_{2k}^n \cdot \lambda_{2k+1}^{n+1} \cdot \frac{g_{2k-1,2k}^{n+1} - g_{2k-1,2k}^n}{g_{2k-1,2k}^{n+1}}, \tag{12}$$

where

$$g_{2k-1,2k}^{n+1} = -\frac{\begin{vmatrix} 1, 3\ldots 2k+1 \\ 3\ldots 2k+2 \end{vmatrix} + \begin{vmatrix} 2\ldots 2k+1 \\ 3\ldots 2k+2 \end{vmatrix}}{\begin{vmatrix} 3\ldots 2k+1 \\ 3\ldots 2k+1 \end{vmatrix}} = -\frac{\begin{vmatrix} i & q \\ s & u \end{vmatrix} + \begin{vmatrix} j & q \\ t & u \end{vmatrix}}{q}$$

and

$$g_{2k-1,2k}^n = -\frac{\begin{vmatrix} 1\ldots 2k \\ 3\ldots 2k+2 \end{vmatrix}}{\begin{vmatrix} 2\ldots 2k \\ 3\ldots 2k+1 \end{vmatrix}} = -\frac{\begin{vmatrix} i & j \\ s & t \end{vmatrix}}{j}$$

are well-known coefficients of the E-algorithm [3].

2.6. A general cross rule. If one is not interested in the entries in the ρ-table with odd lower indices they can be eliminated. This was introduced by Wynn [14] for the ϵ-algorithm and can be done for any lozenge rule. For the ρ-algorithm the result is a general cross rule

$$\begin{aligned} &\frac{\lambda_{2k+1}^{n+1} \cdot (\lambda_{2k}^n - \lambda_{2k}^{n+1})}{\rho_{2k}^n - \rho_{2k}^{n+1}} + \frac{\lambda_{2k+1}^{n+2} \cdot (\lambda_{2k}^{n+1} - \lambda_{2k}^{n+2})}{\rho_{2k}^{n+2} - \rho_{2k}^{n+1}} \\ &= \frac{\lambda_{2k}^{n+1} \cdot (\lambda_{2k+1}^{n+2} - \lambda_{2k+1}^{n+1})}{\rho_{2k-2}^{n+2} - \rho_{2k}^{n+1}} + \frac{\lambda_{2k+2}^n \cdot (\lambda_{2k+3}^{n+1} - \lambda_{2k+3}^n)}{\rho_{2k+2}^n - \rho_{2k}^{n+1}} \end{aligned} \tag{13}$$

for $k \in \mathbb{N}$, $n \in \mathbb{N}_0$. (For a proof use (5) and (6) to express any summand in (13) in terms of ρ with an odd lower index.)

In [5, Theorem 4.3] the author has chosen a particular sequence (α_n) to obtain the E-algorithm from a general ϵ-algorithm. The same sequence can be applied here and gives a similar result which seems less interesting and will be omitted.

3. The classical ρ-algorithm as a particular case

3.1. Notation. Let $(s_n \mid n \in \mathbb{N}_0)$ and $(x_n \mid n \in \mathbb{N}_0)$ be fixed complex sequences, where the knots $x_0, x_1, \ldots$ are nonzero and pairwise distinct. For any $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ define

$$\alpha_n := x_n, \quad g_{2k-1}(n) := \frac{s_n}{x_n^k} \quad \text{and} \quad g_{2k}(n) := \frac{1}{x_n^k}.$$

For nonvanishing denominators 2.1 gives the definitions of coefficients ρ_{2k}^n and ρ_{2k+1}^n which are usually written slightly different as follows: let $p_i : \mathbb{C} \to \mathbb{C}$ denote the ith monomial, $p_i(z) = z^i$, and define

$$f : \{x_i \mid i \in \mathbb{N}_0\} \to \mathbb{C}, \qquad x_i \mapsto s_i$$

for any $i \in \mathbb{N}_0$. Furthermore introduce (generalised) Vandermonde determinants by writing

$$V\begin{pmatrix} g_0, \ldots, g_k \\ x_0, \ldots, x_k \end{pmatrix} := \begin{vmatrix} g_0(x_0) & \cdots & g_k(x_0) \\ \vdots & & \vdots \\ g_0(x_k) & \cdots & g_k(x_k) \end{vmatrix}$$

for a list of functions $g_0, \ldots, g_k$ and a list of knots $x_0, \ldots, x_k$.

Then

$$\rho_{2k}^n = \frac{V\begin{pmatrix} fp_k,\ fp_{k-1},\ p_{k-1},\ fp_{k-2}, \ldots, fp_0,\ p_0 \\ x_n, \ldots, x_{n+2k} \end{pmatrix}}{V\begin{pmatrix} p_k,\ fp_{k-1},\ p_{k-1},\ fp_{k-2}, \ldots, fp_0,\ p_0 \\ x_n, \ldots, x_{n+2k} \end{pmatrix}} \tag{14}$$

and

$$\rho_{2k+1}^n = \frac{V\begin{pmatrix} p_{k+1},\ p_k,\ fp_{k-1}, \ldots, fp_0,\ p_0 \\ x_n, \ldots, x_{n+2k+1} \end{pmatrix}}{V\begin{pmatrix} fp_k,\ p_k,\ fp_{k-1}, \ldots, fp_0,\ p_0 \\ x_n, \ldots, x_{n+2k+1} \end{pmatrix}}. \tag{15}$$

(To see this it is only necessary to multiply the ith row in (1) and (2) with x_i^k.)

As a particular case of Theorem 2.2 we have the following theorem.

3.2. Theorem. *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. If the denominators of ρ_{k-1}^{n+1}, ρ_k^n, ρ_k^{n+1} do not vanish, then*

$$\rho_k^{n+1} \neq \rho_k^n$$

implies that the denominator of ρ_{k+1}^n is nonzero and

$$\rho_{k+1}^n = \rho_{k-1}^{n+1} + \frac{x_{n+k+1} - x_n}{\rho_k^{n+1} - \rho_k^n}. \tag{16}$$

Proof. Firstly replace k by $2k-1$. By Remark 2.3(ii) the nominator of λ_{2k}^n is nonzero. Since in the particular case of this section

$$\lambda_{2k}^n = \frac{V\begin{pmatrix} p_k, p_{k+1}, fp_k, fp_{k-1}, p_{k-1}, \dots, fp_1, p_1 \\ x_n, \dots, x_{n+2k} \end{pmatrix}}{V\begin{pmatrix} p_k, fp_{k-1}, p_{k-1}, \dots, fp_0, p_0 \\ x_n, \dots, x_{n+2k} \end{pmatrix}} = x_n \cdot \ldots \cdot x_{n+2k} \tag{17}$$

the denominators of λ_{2k}^n and ρ_{2k}^n are nonzero too. Since

$$\lambda_{2k+1}^n = \frac{V\begin{pmatrix} fp_{k-1}, p_{k-1}, \dots, fp_0, p_0 \\ x_n, \dots, x_{n+2k-1} \end{pmatrix}}{V\begin{pmatrix} p_k, fp_k, fp_{k-1}, p_{k-1}, \dots, fp_1, p_1 \\ x_n, \dots, x_{n+2k-1} \end{pmatrix}} = -\frac{1}{x_n \cdot \ldots \cdot x_{n+2k-1}} \tag{18}$$

Theorem 2.2(a) implies (16) for $2k-1$ instead of k.

Finally replace k by $2k$. Then Theorem 2.2(b), (17) and (18) imply (16). □

3.3. Remark. Roughly speaking the general ϵ-algorithm specializes in the classical one because

$$\Delta g_i(n) = g_{i+1}(n), \qquad g_0 := \Delta s_n.$$

Similarly, the general ρ-algorithm specializes in the classical one because

$$g_i(n) = x_n \cdot g_{i+2}(n), \qquad g_1(n) = s_n, \qquad g_0(n) = 1.$$

4. Another rational particular case

We use the notation from 3.1 except that for each $n \in \mathbb{N}_0$

$$\alpha_n := f(x_n) \cdot x_n.$$

Then ρ_{2k}^n and λ_{2k+1}^n are given by (14) and (18), respectively. Therefore, as in the classical ρ-algorithm in the particular case of this section ρ_{2k}^n can be understood as certain value of rational extrapolation.

Instead of (15) or (17) we have

$$\rho_{2k+1}^n = \frac{V\begin{pmatrix} fp_{k+1}, p_k, fp_{k-1}, \dots, fp_0, p_0 \\ x_n, \dots, x_{n+2k+1} \end{pmatrix}}{V\begin{pmatrix} fp_k, p_k, fp_{k-1}, \dots, fp_0, p_0 \\ x_n, \dots, x_{n+2k+1} \end{pmatrix}} \tag{19}$$

or

$$\lambda_{2k}^n = \frac{V\begin{pmatrix} p_k, fp_{k+1}, fp_k, fp_{k-1}, p_{k-1}, \dots, fp_1, p_1 \\ x_n, \dots, x_{n+2k} \end{pmatrix}}{V\begin{pmatrix} p_k, fp_{k-1}, p_{k-1}, \dots, fp_0, p_0 \\ x_n, \dots, x_{n+2k} \end{pmatrix}} = x_n \cdot \ldots \cdot x_{n+2k} \cdot \rho_{2k}^n, \tag{20}$$

respectively.

Hence Theorem 2.2 gives

$$\left(\rho_{2k}^{n}-\rho_{2k-2}^{n+1}\right)\cdot\left(\rho_{2k-1}^{n+1}-\rho_{2k-1}^{n}\right)=\left(x_{n+2k}-x_{n}\right)\cdot\rho_{2k}^{n}, \tag{21}$$

$$\left(\rho_{2k+1}^{n}-\rho_{2k-1}^{n+1}\right)\cdot\left(\rho_{2k}^{n+1}-\rho_{2k}^{n}\right)=x_{n+2k+1}\cdot\rho_{2k}^{n+1}-x_{n}\cdot\rho_{2k}^{n}. \tag{22}$$

These identities facilitate the computation of the entries in the two-dimensional ρ-array with an even upper index. Since the coefficients in the ρ-algorithm are simpler, the algorithm based on (21) and (22) seems less interesting than the classical ρ-algorithm.

It is remarkable that the cross rule (13)—which corresponds to (21) and (22)—can easily be transformed into the cross rule of the classical ρ-algorithm.

5. Prescribing the coefficients in a lozenge rule

We consider a lozenge rule which is given by coefficients $(\eta_k^n \mid k\in\mathbb{N},\ n\in\mathbb{N}_0)$; i.e., a sequence $(s_n \mid n\in\mathbb{N}_0)$ is mapped onto a two-dimensional array $(r_k^n \mid k,\ n\in\mathbb{N}_0)$ with the initialisations $r_0^n := s_n$ and $r_{-1}^n := 0$ and the rule

$$r_k^n := r_{k-2}^{n+1}+\frac{\eta_k^n}{r_{k-1}^{n+1}-r_{k-1}^{n}} \tag{23}$$

for $k=1, 2, 3,\ldots$ and $n=0, 1, 2,\ldots$.

We will always assume that every denominator in (23) does not vanish. Then the coefficients $(\eta_k^n \mid k\in\mathbb{N},\ n\in\mathbb{N}_0)$ in the lozenge rule define a mapping $(s_n)\mapsto(r_k^n)$.

We are interested in whether algorithm (23) is a particular case of Theorem 2.2 or not. This question will be solved by construction of $g_k(n)$ and α_n such that $r_k^n=\rho_k^n$ where the latter are defined by the determinant quotients of Notation 2.1.

By Theorem 2.2 it suffices to determine $g_k(n)$ and α_n such that

$$\eta_{2k}^{n}=\lambda_{2k}^{n}\cdot\left(\lambda_{2k+1}^{n+1}-\lambda_{2k+1}^{n}\right)\quad\text{and}\quad\eta_{2k+1}^{n}=\lambda_{2k+1}^{n+1}\cdot\left(\lambda_{2k}^{n}-\lambda_{2k}^{n+1}\right). \tag{24}$$

We note that (23) directly implies $\Delta\alpha_n=\eta_1^n$. Then, we determine $(g_k(n))$ by induction on $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$ in three steps.

(i) First assume that $g_1(n),\ldots, g_{2k-2}(n)$ are chosen for any $n\in\mathbb{N}_0$. Choose some coefficients $g_{2k-1}(0),\ldots, g_{2k-1}(2k-1)$ and $g_{2k}(0),\ldots, g_{2k}(2k-1)\in\mathbb{C}$. Then (for nonvanishing denominator) $\hat\lambda_{2k+1}^0:=\lambda_{2k+1}^0$ is uniquely determined by (4). Naturally (λ_j^n) and therefore $(g_j(n))$ depends on (s_n). Choose $\hat\lambda_{2k}^0\in\mathbb{C}\setminus\{0\}$.

(ii) Compute $\hat\lambda_{2k}^{n+1}$ and $\hat\lambda_{2k+1}^{n+1}$ for any $n\in\mathbb{N}_0$ by

$$\hat\lambda_{2k+1}^{n+1}=\hat\lambda_{2k+1}^{n}+\frac{\eta_{2k}^{n}}{\hat\lambda_{2k}^{n}}, \tag{25}$$

$$\hat\lambda_{2k}^{n+1}=\hat\lambda_{2k}^{n}-\frac{\eta_{2k+1}^{n}}{\hat\lambda_{2k+1}^{n+1}}. \tag{26}$$

Therefore $\hat\lambda_{2k}^n$ and $\hat\lambda_{2k+1}^n$ satisfy (24) with $\hat\lambda_j^n$ instead of λ_j^n.

(iii) Finally determine $g_{2k-1}(2k+n)$ and $g_{2k}(2k+n)$ by induction on $n\in\mathbb{N}_0$ such that

$$\hat\lambda_{2k}^{n}=\lambda_{2k}^{n}\quad\text{and}\quad\hat\lambda_{2k+1}^{n+1}=\lambda_{2k+1}^{n+1},$$

where λ_{2k}^n or λ_{2k+1}^{n+1} are given by (3) or (4), respectively. Provided $\lambda_{2k+1}^n \neq 0$, this is always possible.

Proof. We write $\lambda_{2k+1}^{n+1} = a/b$ and $\lambda_{2k}^n = c/d$ with a and d being the numerator or denominator determinant of (4) or (3), respectively. Expanding a and d with respect to the last column leads to

$$a = g_{2k}(n+2k)\cdot a_1 - g_{2k-1}(n+2k)\cdot a_2 - a_3, \tag{27}$$

$$d = \Delta g_{2k}(n+2k-1)\cdot d_1 - \Delta g_{2k-1}(n+2k-1)\cdot d_2 - d_3, \tag{28}$$

where (using the notation of 2.4)

$$a_1 = \begin{vmatrix} 1, 3\dots 2k \\ 3\dots 2k+1 \end{vmatrix} + \begin{vmatrix} 2\dots 2k \\ 3\dots 2k+1 \end{vmatrix} = i + j,$$

$$a_2 = \begin{vmatrix} 1, 3\dots 2k \\ 3\dots 2k,\ 2k+2 \end{vmatrix} + \begin{vmatrix} 2\dots 2k \\ 3\dots 2k,\ 2k+2 \end{vmatrix} = s + t,$$

$$d_1 = \begin{vmatrix} 2\dots 2k \\ 3\dots 2k+1 \end{vmatrix} = j, \qquad d_2 = \begin{vmatrix} 2\dots 2k \\ 3\dots 2k,\ 2k+2 \end{vmatrix} = t.$$

Then, the condition for $g_{2k-1}(n+2k)$ and $g_{2k}(n+2k)$ can be written as

$$\begin{pmatrix} a_1 & -a_2 \\ d_1 & -d_2 \end{pmatrix} \cdot \begin{pmatrix} g_{2k}(n+2k) \\ g_{2k-1}(n+2k) \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_{2k+1}^{n+1}\cdot b + a_3 \\ \dfrac{c}{\hat{\lambda}_{2k}^n} + d_3' \end{pmatrix}. \tag{29}$$

This is a linear system of equations for $g_{2k-1}(n+2k)$ and $g_{2k}(n+2k)$ in which all other coefficients are known. It is uniquely soluble because

$$\begin{vmatrix} a_1 & -a_2 \\ d_1 & -d_2 \end{vmatrix} = -\begin{vmatrix} i & s \\ j & t \end{vmatrix} = -\begin{vmatrix} 1\dots 2k \\ 3\dots 2k+2 \end{vmatrix} \neq 0$$

since $\lambda_{2k+1}^n \neq 0$ (see (4)). □

If the computations above are possible, i.e., any denominator appearing there is nonzero, Theorem 2.2 states that the lozenge rule considered here is a particular case of the general ρ-algorithm presented in this paper.

Remarks. (i) The assumptions on the denominators appearing in the computation above are needed only for a finite subtable. It leads to a certain regularity of some submatrices of $(g_j(n))$. Since $(g_j(n))$ depends on (s_n) this is a condition for the sequence (s_n) as well as for the (η_j^n).

(ii) The coefficients (η_j^n) may also depend on the sequence (s_n). In this case (η_j^n) is regarded as a function defined for certain sequences (s_n).

(iii) An important application of the results in this section could be the construction of $g_j(n)$ for such lozenge rules in which a representation of the form (1) is actually not known—as for the θ-algorithm for instance. Though this application may be very difficult (the dependence of $g_j(n)$ on (s_n) and $(\eta(s_n)_j^n)$ has to be discussed explicitely) it seems in principle to be possible.

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