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ON PERTURBATION BEHAVIOUR IN NON-LINEAR DYNAMICS

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SUMMARY

The static bifurcation criterion, i.e. singularity of the tangential stiffness matrix, is discussed in the sense of non-linear dynamics since it is sometimes used in the engineering literature. In this paper the connection between static stability, i.e. that the tangential stiffness matrix is positive-definite, and insensitivity, i.e. local damping of small perturbations, is proved. Since, in general, there is no connection between insensitivity and asymptotic stability, the concept of sensitivity cannot replace the classical stability theory of motion

1. INTRODUCTION

Many engineering structures exhibit loss of stability under static and dynamic loading. Contrary to the static case, where the determinant of the tangent stiffness matrix indicates loss of stability, we have to consider different cases in dynamic problems which are treated in detail in the literature.

Within the autonomous case we can use the so-called Ljapunov first approximation, where still the tangent stiffness matrix determines the stability behaviour; this is well known. For cases of periodic and non-periodic excitement Ljaponov's general criterion is valid, which leads to very complicated and time-consuming investigations to 'detect instability'; this is also well known; see, for example, Kreuzer,[1] and Parker and Chua.[2]

These considerations yield information about the asymptotic behaviour (i.e. the situation for $t \to \infty$; see Figure 1). However, up to continuity and flow-box diffeomorphy, the local behaviour is not investigated–see the situation depicted in the frame in Figure 1.

Engineering structures are in general very complicated, and simulations are time-consuming. Thus it is of interest to have an indicator for dynamical instability phenomena which is computationally efficient. This may be reason why many engineering stability problems which exhibit dynamical effects like buckling or snap-through of shells are treated as static cases; see Burmeister and Ramm,[3] and Krätzig.[4] In Kleiber *et al.*[5] the loss of 'stability' (detected by the singularity of the stiffness matrix) is called 'quasibifurcation' (see Figure 1). There we consider the situation within the box, where small perturbations have global consequences. Clearly the behaviour of a perturbed and unperturbed solution in the box of Figure 1 says nothing about the global stability behaviour.

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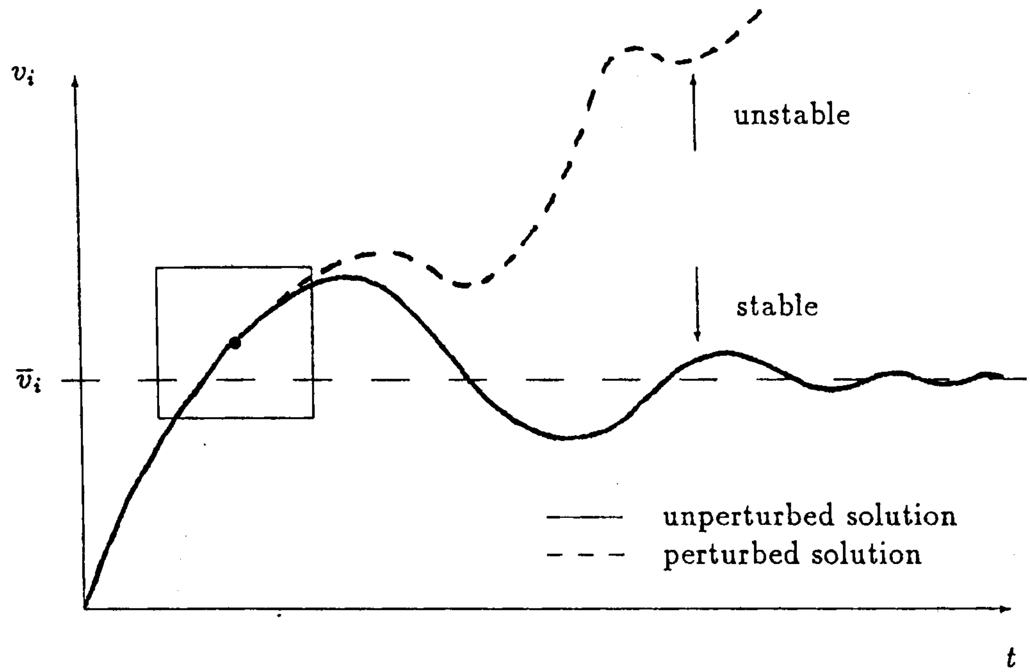

Figure 1.

However, as explained in this paper, static stability has a consequence regarding the stability behaviour in non-linear dynamics. Indeed, we prove an insensitivity interpretation of static stability, i.e. we prove that small perturbations are damped during a small period of time.

The paper is organized as follows. In Section 2 we discuss the question, 'What is the information obtained by the static stability criterion in non-linear dynamics?' for a simple model problem and motivate the use of the complicated definition of insensitivity introduced in Section 3. It is proved that the loss of regularity of the local linearization gives exactly the change from insensitivity to sensitivity. In Section 4 we specify the results to the particular case of the equations of motion. We conclude with a discussion of the static stability criterion applied to non-linear dynamics.

2. A MODEL EXAMPLE

In this Section we analyse a simple mechanical example to study the implication of the static stability criterion in dynamics. This will lead us to the concept of sensitivity which is formally treated in Section 3. To explain the notion of sensitivity and to motivate the investigation in Section 3 we consider a non-linear one-dimensional model problem.

The mechanical system is shown in Figure 2. ϕ denotes the angle of the rotation which yields the moment of the spring: $(c_M\phi)$, of the constant load F: $(-Fl\sin\phi)$ and of the time-dependent force $f(t)$: $(-f(t)l\cos\phi)$. The moment of inertia is given by $\Theta_M\ddot{\phi}$ and viscous damping by $\mathbf{D}\dot{\phi}$. We obtain from equilibrium

$$\Theta_M\ddot{\phi} + \mathbf{D}\dot{\phi} + c_M\phi = Fl\sin\phi + f(t)l\cos\phi$$

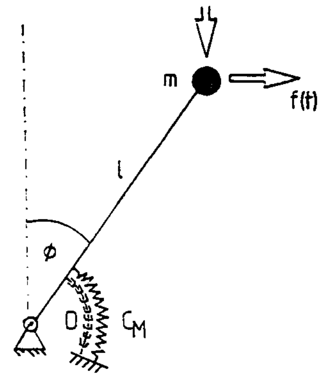

Figure 2.

This leads to the formulation of a model equation

$$\ddot{\phi} + \eta\dot{\phi} + \omega^2\phi - \lambda \sin \phi = f(t) \cos \phi \tag{1}$$

with $\eta, \omega^2, \lambda > 0$ for which we want to investigate the effect of small perturbations in the initial data $\phi_0 = 0$ and $\dot{\phi}_0 = 0$ at time $t_0 = 0$.

Note that for $f(0) \neq 0$ there is no equilibrium. f can be an arbitrary function (non-vanishing, non-periodic) and is possibly unknown at later times. For instance, f may represent external forces resulting from an earthquake. When we consider the response of the system in a fixed time period the asymptotic behaviour is not of interest. Instead we discuss whether the system's response depends heavily on small perturbations or not. Moreover we ask whether the response of the perturbed system differs significantly from the unperturbed one in the next time period.

Mathematically the situation is clear. Since we have no equilibrium the response of the perturbed and unperturbed system can be viewed withing the notion of the flow box theorem. In this paper we derive a quantitative measure which yields information about possible divergence of the streamlines or not.

The static version of (1) reduces to

$$\omega^2\phi - \lambda \sin \phi = f(t) \cos \phi$$

thus a linearization in $\phi = 0$ yields the stability condition

$$\omega^2 - \lambda = 0$$

The question is, in what sense are the two situations $\omega^2 > \lambda$ and $\omega^2 < \lambda$ 'stable' and 'unstable', respectively?

In the following we discuss the effects of a perturbation (α, β) of $(\phi_0, \dot{\phi}_0) = (0, 0)$ in the model problem (1).

Let $\Phi_{(\alpha,\beta)} = (\phi_{(\alpha,\beta)}, \dot{\phi}_{(\alpha,\beta)})^t$ denote the solution of (1) related to the initial conditions

$$\Phi_{(\alpha,\beta)}(t_0) = (\alpha, \beta)^t$$

Furthermore we define the matrix

$$\mathbf{B} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{2}$$

and assume that B is positive-definite, i.e. $a, c, ac - b^2 > 0$. Matrix $\mathbf{B}$ provides a distance measure for $\Phi_{(\alpha,\beta)} - \Phi_{(0,0)}$.

The distance $\Phi_{(\alpha,\beta)} - \Phi_{(0,0)}$ in the so-called $\mathbf{B}$-norm $\mathbf{e}_{\mathbf{B},(\alpha,\beta)}$ is defined as

$$\mathbf{e}^2_{\mathbf{B},(\alpha,\beta)}(t) := (\Phi_{(\alpha,\beta)} - \Phi_{(0,0)})^t \mathbf{B} (\Phi_{(\alpha,\beta)} - \Phi_{(0,0)})$$

Note that it is necessary and fundamental to employ different $\mathbf{B}$-norms for different distance measures–see below.

Now we examine the change of $\mathbf{e}_{\mathbf{B},(\alpha,\beta)}$. Recall that solutions of ordinary differential equations are related to the initial conditions by continuity

$$\forall t > 0 \qquad \forall \mathbf{B} \text{ positive-definite} \qquad \lim_{(\alpha,\beta)\to 0} \qquad \mathbf{e}_{\mathbf{B},(\alpha,\beta)}(t) = 0$$

while (asymptotic) stability is described by

$$\exists \mathbf{B} \text{ positive-definite} \qquad \lim_{(\alpha,\beta)\to 0} \qquad \lim_{t\to\infty} \qquad \mathbf{e}_{\mathbf{B},(\alpha,\beta)}(t) = 0$$

When we consider the local perturbation behaviour of the system, then we ask whether $\mathbf{e}_{\mathbf{B},(\alpha,\beta)}$ is monotonously increasing or decreasing in a neighbourhood of $t_0 = 0$.

For this purpose, we investigate the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}^2_{\mathbf{B},(\alpha,\beta)} = 2(\phi_{(0,0)} - \phi_{(\alpha,\beta)}, \dot\phi_{(0,0)} - \dot\phi_{(\alpha,\beta)})\mathbf{B}\begin{pmatrix} \dot\phi_{(0,0)} - \dot\phi_{(\alpha,\beta)} \\ \ddot\phi_{(0,0)} - \ddot\phi_{(\alpha,\beta)} \end{pmatrix}$$

and obtain by (1) and the initial conditions

$$\begin{aligned} 1/2\frac{\mathrm{d}}{\mathrm{d}t} e^2_{\mathbf{B},(\alpha,\beta)}(0) &= (\alpha,\beta)\mathbf{B}\begin{pmatrix} \beta \\ \lambda\sin\alpha - \omega^2\alpha - \eta\beta - f(t)(1-\cos\alpha) \end{pmatrix} \\ &= (\alpha,\beta)\mathbf{B}_1\binom{\alpha}{\beta} \end{aligned}$$

Standard computations yield

$2\mathbf{B}_1 =$

$$\begin{pmatrix} 2b\left\{\lambda\dfrac{\sin\alpha}{\alpha} - \omega^2 - f(0)\dfrac{1-\cos\alpha}{\alpha}\right\} & a - b\eta + c\left\{\lambda\dfrac{\sin\alpha}{\alpha} - \omega^2 - f(0)\dfrac{1-\cos\alpha}{\alpha}\right\} \\ a - b\eta + c\left\{\lambda\dfrac{\sin\alpha}{\alpha} - \omega^2 - f(0)\dfrac{1-\cos\alpha}{\alpha}\right\} & 2(b - c\eta) \end{pmatrix}$$

Clearly, if $\alpha = 0$ then $\sin\alpha/\alpha$ or $1 - \cos\alpha/\alpha$ has to be replaced by 1 or 0, respectively. If α and β are small we have to consider $\lim_{(\alpha,\beta)\to 0}\mathbf{B}_1 = \mathbf{B}_2$ with

$$\mathbf{B}_2 = \tfrac{1}{2}\begin{pmatrix} 2b\{\lambda - \omega^2\} & a - b\eta + c\{\lambda - \omega^2\} \\ a - b\eta + c\{\lambda - \omega^2\} & 2(b - c\eta) \end{pmatrix}$$

Then $\mathbf{e}_{\mathbf{B},(\alpha,\beta)}$ is monotonically decreasing (increasing) at $t_0 = 0$ for small perturbations in the direction of (α,β) if

$$R_2(\alpha,\beta) := (\alpha,\beta)\mathbf{B}_2\binom{\alpha}{\beta}$$

is negative (positive). Hence we have to inspect the sign of $\mathbf{R}_2(\alpha,\beta)$. Obviously, the sign of $\mathbf{R}_2(\alpha,\beta)$ depends on (α,β) and $\mathbf{B}$. For example, if $(\alpha,\beta) = (0,1)$, then $\mathbf{R}_2(0,1) = b - c\eta$ is negative for $b \leqslant 0$ and we must choose, for example, $a = c = 1, b = 0$ such that $\mathbf{B}$ is positive-definite. $R_2(0,1)$ is positive for $b > c\eta$ and we have to choose, for example $c = 1/\eta, b = 2$ and $a = 5\eta$ such that $\mathbf{B}$ is positive-definite.

We conclude that the monotonic behaviour of $e_{\mathbf{B},(\alpha,\beta)}$ depends heavily heavily on the distance measure used, i.e. on $\mathbf{B}$, which is the reason for introducing $\mathbf{B}$.

The example shows that we cannot hope to obtain

$$\forall \mathbf{B} \text{ positive-definite} \qquad \forall(\alpha,\beta) \neq 0, \qquad R_2(\alpha,\beta) < 0$$

Thus we can only expect to reach

$$\exists \mathbf{B} \text{ positive-definite} \qquad \forall(\alpha,\beta) \neq 0, \qquad R_2(\alpha,\beta) < 0$$

It is easy to see that this is equivalent to constructing $\mathbf{B}$ positive-definite such that $\mathbf{B}_2$ is negative-definite, i.e. the diagonal entries of $\mathbf{B}_2$ are negative while det $\mathbf{B}_2$ is positive.

If this construction is possible, then we will say that (1) is insensitive in $t_0 = 0, \phi_0 = 0, \dot{\phi}_0 = 0$ while otherwise it is sensitive. Again, insensitivity means that there exists a certain measure such that any error obtained by a sufficient small perturbation will be damped in a neighbourhood of t_0. Sensitivity means that perturbations (which can be chosen arbitrarily small) exist for any measure such that the obtained errors increase in a neighbourhood of $t_0 = 0$.

Suppose $\mathbf{B}$ is positive- and $\mathbf{B}_2$ is negative-definite. Then it is not hard to see that $\lambda \geqslant \omega^2$ contradicts det $\mathbf{B}_2$ and det $\mathbf{B}$ positive. Thus insensitivity implies static stability in this case.

Suppose $\lambda < \omega^2$ and define

$$a := \eta^2/2 + \omega^2 - \lambda, \qquad b := \eta/2, \qquad c := 1 \tag{3}$$

Then $\mathbf{B}$ is positive- and $\mathbf{B}_1$ is negative-definite. Here the static stability condition implies insensitivity.

Summing up, we are led to the conjecture that the static stability condition just determines sensitivity within this example. In a general context this will be proved in Section 3 and in a more interesting particular case in Section 4.

In Section 5 the significance of insensitivity is discussed critically. In the course of this we will return to the model example of this Section.

3. SENSITIVITY

In this Section we generalize the observations made in Section 2 to ordinary differential equations and introduce sensitivity and insensitivity. Moreover we prove equivalence of insensitivity and static stability. For this purpose we need further notation.

Let f: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map and consider the unique solution $\mathbf{x}$: $(-a, +a) \to \mathbb{R}^n$, $a > 0$, of the differential equation $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t)), t \in (-a, +a)$, satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. It is well known that $\mathbf{x}$ depends continuously on $\mathbf{x}_0$ (cf. Hirsch and Smale [6] and Willems[7]), but we are less interested in this qualititative result than in a quantitative one.

If $a = +\infty$ (i.e. $\mathbf{x}$ exists for any time), then asymptotic stability of $\mathbf{x}$ in the sense of Liapunov is well known. The following example from Willems[7], pp. 49, 113, gives a sufficient condition for asymptotic stability.

Theorem I

If $f(t,\mathbf{x}) = (\mathbf{A} + \mathbf{B}(t))\mathbf{x}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B}$: $\mathbb{R} \to \mathbb{R}^{n \times n}$ is a C^1 map satisfying $\lim_{t\to\infty} \mathbf{B}(t) = 0$, then the trivial solution $\mathbf{x} = 0$ (for $\mathbf{x}_0 = 0$) is asymptotically stable if all eigenvalues of $\mathbf{A}$ have negative real parts. □

Remark. Note that, if $\mathbf{A}$ depends on t (as $\mathbf{B}$ does), then the stability result above is false even if all eigenvalues of $\mathbf{A}(t)$ have real parts smaller than a fixed negative real number $\delta < 0$ (see Willems[7] p. 120) for the counterexample of Vinogradov). Indeed it is necessary to observe the (real) eigenvalues of the symmetric part $\mathbf{H} := (\mathbf{A} + \mathbf{A}^t)/2$ of $\mathbf{A}$ (see Willems[7] p. 117).

We will modify the theorem for f being a non-linear function and $t - t_0$ being small (in contrast to $t \to \infty$). Then the non-linearity has only a minor influence (see below).

Let $t_0 \in \mathbb{R}, \mathbf{x}_0 \in \mathbb{R}^n$ and $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map (where f can be only piece-wise C^1 in the first variable). Let $\mathbf{x}: (t_0 - a, t_0 + a) \to \mathbb{R}^n$ denote the solution of the initial-value problem $\dot{\mathbf{x}} = f(\cdot, \mathbf{x}), \mathbf{x}(t_0) = \mathbf{x}_0$, which exists on $(t_0 - a, t_0 + a)$. Since f is C^1 the theorem of Picard and Lindelöf yields the (unique) existence of a solution $\mathbf{x}$ and gives a lower estimate for $a > 0$. Moreover it implies the existence of an universal $a > 0$ and a neighbourhood U of $\mathbf{x}_0$ in $\mathbb{R}^n$ such that for all $\xi \in U$ the initial-value problem

$$\forall t \in (t_0 - a, t_0 + a) \qquad \dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t)), \qquad \mathbf{x}(t_0) = \xi \tag{4}$$

has a unique solution denoted by $\phi_\xi: (t_0 - a, t_0 + a) \to \mathbb{R}^n$, which exists on $(t_0 - a, t_0 + a)$. Clearly $\mathbf{x} = \phi_{\mathbf{x}_0}$.

Furthermore, let $\mathbf{B}$ be a positive-definite matrix which yields an inner product. The related norm, called, the $\mathbf{B}$-norm, is given by $\|\mathbf{y}\|_{\mathbf{B}} := \sqrt{(\mathbf{y}^t\mathbf{B}\mathbf{y})}, \mathbf{y} \in \mathbb{R}^n$, and denoted by $\|\cdot\|_{\mathbf{B}}$. For $\mathbf{B} \in \mathbb{R}^{n \times n}$ positive-definite and $\xi \in U$ let

$$e_{\mathbf{B},\xi}: (t_0 - a, t_0 + a) \to \mathbb{R}, \qquad t \mapsto \|\mathbf{x}(t) - \phi_\xi(t)\|_{\mathbf{B}} \tag{5}$$

be the $\mathbf{B}$-norm of the difference of the solutions of (4) for the initial value $\mathbf{x}_0$ and the perturbed one ξ.

We are now in the position to define sensitivity and insensitivity formally.

Definition

$\mathbf{x}$ is called insensitive in t_0 iff

$$\begin{gathered}\exists \delta > 0 \qquad \exists \mathbf{B} \in \mathbb{R}^{n \times n}, \mathbf{B} \text{ positive-definite} \qquad \exists \alpha \in (0, a) \forall \xi \in \mathbb{R}^n, \|\mathbf{x}_0 - \xi\|_{\mathbf{B}} < \delta \\ e_{\mathbf{B},\xi} \text{ is monotone decreasing on } (0, \alpha)\end{gathered} \tag{6}$$

On the other hand $\mathbf{x}$ is called sensitive in t_0 iff

$$\begin{gathered}\forall \delta > 0 \qquad \forall \mathbf{B} \in \mathbb{R}^{n \times n}, \mathbf{B} \text{ positive-definite} \qquad \exists \alpha \in (0, a) \qquad \exists \xi \in \mathbb{R}^n, \|\mathbf{x}_0 - \xi\|_{\mathbf{B}} < \delta \\ e_{\mathbf{B},\xi} \text{ is monotone increasing on } (0, \alpha)\end{gathered} \tag{7}$$

If $\mathbf{x}$ is sensitive (insensitive) in t_0 for any $t_0 \in J \subseteq \mathbb{R}$, then $\mathbf{x}$ is said to be sensitive (insensitive) on J.

Note that $\mathbf{x}$ sensitive is not equivalent to $\mathbf{x}$ not insensitive (even though $\mathbf{x}$ sensitive implies $\mathbf{x}$ insensitive), but in view of the next theorem we find the difference less interesting.

Theorem II

Let $\mathbf{D}_2 f(t_0, \mathbf{x}(t_0))$ denote the Jacobian of a C^2 map f with respect to the second variable. If all eigenvalues of $\mathbf{D}_2 f(t_0, \mathbf{x}(t_0))$ have negative real parts, then $\mathbf{x}$ is insensitive in t_0. If conversely at least one eigenvalue of $\mathbf{D}_2 f(t_0, \mathbf{x}(t_0))$ has a positive real part, then $\mathbf{x}$ is sensitive in t_0.

Remark. Theorem II states the equivalence of insensitivity and the static stability criterion.

Proof. Let $\mathbf{A} := \mathbf{D}_2 f(t_0, \mathbf{x}(t_0))$. First we assume the existence of $\beta > 0$ such that all eigenvalues of $\mathbf{A}$ have real parts smaller than $-\beta$. By the Lemma from Hirsch and Smale,[6] p. 145f, we found a positive-definite matrix $\mathbf{B}$ such that, for all $\mathbf{z} \in \mathbb{R}^n, \mathbf{z}^t\mathbf{B}\mathbf{A}\mathbf{z} \leqslant -\beta\mathbf{z}^t\mathbf{B}\mathbf{z}$. By continuity of $\mathbf{D}_2 f$ and $\alpha \in (a, 0)$ exists such that $\mathbf{z}^t\mathbf{B}\mathbf{D}_2 f(t, \mathbf{x}(t))\mathbf{z} \leqslant -\beta/2\mathbf{z}^t\mathbf{B}\mathbf{z}$ for all $t \in (t_0 - \alpha, t_0 + \alpha)$ and $\mathbf{z} \in \mathbb{R}^n$. Since f is a C^2 map we may write

$$\| f(t, \mathbf{x}(t) + \mathbf{z}) - f(t, \mathbf{x}(t)) - \mathbf{D}_2 f(t, \mathbf{x}(t))\mathbf{z} \|_{\mathbf{B}} \leqslant M \| \mathbf{z} \|_{\mathbf{B}}^2$$

for all $t \in (t_0 - \alpha, t_0 + \alpha), \mathbf{z} \in \mathbb{R}^n, \| \mathbf{z} \|_{\mathbf{B}} \leqslant 1$ and a constant $M > 0$. Since ϕ_ξ depends continuously on ξ and $(t_0 - \alpha, t_0 + \alpha)$ is bounded, we can choose $\delta > 0$ such that for all $\xi \in \mathbb{R}^n$ with $\| \xi - x_0 \|_{\mathbf{B}} < \delta$ and all $t \in (t_0 - \alpha, t_0 + \alpha) e_{\mathbf{B},\xi}(t) \leqslant \min\{1, 0{\cdot}25\beta/M\}$ holds.

Let $t \in (t_0 - \alpha, t_0 + \alpha), \xi \in \mathbb{R}^n, \|\mathbf{x}_0 - \xi \|_{\mathbf{B}} < \delta$ and $\mathbf{z} := \phi_\xi(t) - \mathbf{x}(t), e_{\mathbf{B},\xi} = \| \mathbf{z} \|_{\mathbf{B}}$. Then we obtain the result

$$\begin{aligned} 1/2(e^2_{\mathbf{B},\xi})'(t) &= \mathbf{z}^t\mathbf{B}\dot{\mathbf{z}} = \mathbf{z}^t\mathbf{B}\mathbf{D}_2 f(t, \mathbf{x}(t))\mathbf{z} \\ &\quad + \mathbf{z}^t\mathbf{B}(f(t, \mathbf{x}(t) + \mathbf{z}) - f(t, \mathbf{x}(t)) - \mathbf{D}_2\mathbf{f}(t, \mathbf{x}(t))\mathbf{z}) \\ &\leqslant -\beta/2 \| \mathbf{z} \|_{\mathbf{B}}^2 + M \| \mathbf{z} \|_{\mathbf{B}}^3 \leqslant -1/4\beta e^2_{\mathbf{B},\xi}(t) \leqslant 0 \end{aligned}$$

This implies the assertion in the first part. In the second one we assume the existence of an eigenvalue λ of $\mathbf{A}$ with positive real part. Let $\mathbf{B}$ be positive-definite and $\delta' > 0$. Consider an eigenvector $\mathbf{x} + \mathrm{i}\mathbf{y}$ of $\mathbf{A}$ related to $\lambda = \beta + \mathrm{i}\gamma, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \beta, \gamma \in \mathbb{R}, \mathrm{Re}\lambda = \beta > 0$. Since $\mathbf{A}(\mathbf{x} + \mathrm{i}\mathbf{y}) = (\beta + \mathrm{i}\gamma)(\mathbf{x} + \mathrm{i}\mathbf{y})$ implies

$$\mathbf{x}^t\mathbf{B}\mathbf{A}\mathbf{x} = \beta \| \mathbf{x} \|_{\mathbf{B}}^2 - \gamma\mathbf{x}^t\mathbf{B}\mathbf{y} \text{ and } \mathbf{y}^t\mathbf{B}\mathbf{A}\mathbf{y} = \beta \| \mathbf{y} \|_{\mathbf{B}}^2 + \gamma\mathbf{y}^t\mathbf{B}\mathbf{x}$$

Here a real vector $\mathbf{z}$ exists which satisfies $\mathbf{z}^t\mathbf{B}\mathbf{A}\mathbf{z} \geqslant \beta \| \mathbf{z} \|_{\mathbf{B}}^2 > 0$. Next we choose M and $\delta < \delta'$ as above and $\xi := \mathbf{x}_0 + \tau\mathbf{z}$, where $\tau \in \mathbb{R}$ and $0 < \| \tau\mathbf{z} \|_{\mathbf{B}} < \delta$. Then, as above,

$$1/2(e^2_{\mathbf{B},\xi})'(t_0) \geqslant \beta \| \tau\mathbf{z} \|_{\mathbf{B}}^2 - M \| \tau\mathbf{z} \|_{\mathbf{B}}^3 \geqslant 3/4\beta e^2_{\mathbf{B},\xi}(t_0) > 0$$

By continuity of $\mathbf{x}$ and ϕ_ξ an $\alpha > 0$ exists such that $(e^2_{\mathbf{B},\xi})' \geqslant \beta e^2_{\mathbf{B},\xi}$ holds on $(t_0 - \alpha, t_0 + \alpha)$. This completes the proof. □

Remarks.

(i) Looking at the proof of the theorem, insensitivity and sensitivity can be described. For this purpose we will distinguish between two cases. In the first case there exists a certain **B**-norm such that any small perturbation of displacement and velocity at that time t_0 will be reduced exponentially (with respect to the **B**-norm) during a following small period of time. This situation seems 'stable' and t_0 is called insensitive–see Figure 3. In the second case (using any **B**-norm) there exists a certain direction (possibly depending on the used **B**-norm) such that any small perturbation in this direction will increase exponentially during a following small period of time. This situation seems 'unstable' and t_0 is called sensitive–see Figure 4. In the second case we look at a real (and hence positive) eigenvalue. Then any real eigenvector $\mathbf{z}$ determines the direction which yields an exponential increase of perturbation (independently of any **B**-norm)–see Figure 4. Both cases are discussed also within the model example in Section 2.

(ii) Mathematically, sensitivity as well as static stability is described only by a linearization of f in time and space. Hence sensitivity can state only a local behaviour and is in general unsuited for a 'global stability message'.

(iii) Clearly, all **B**-norms are equivalent in $\mathbb{R}^n$ and hence the topological statements '$\mathbf{z}$ grows to infinity' or '$\mathbf{z}$ decreases to zero' in (t_0, ∞) are independent of these norms in contrast

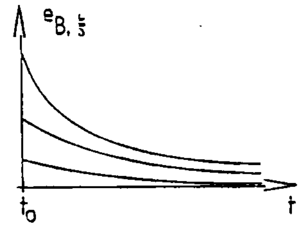

Figure 3.

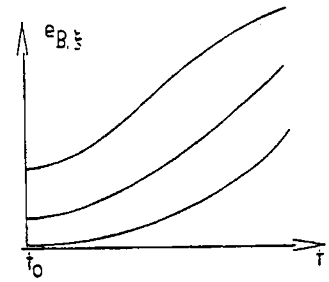

Figure 4.

to '**z** increases' or '**z** decreases' in $(t_0, t_0 + \alpha)$. For example, let $\delta > 0$ be a parameter and let $\mathbf{B} \in \mathbb{R}^2$ be given in (2) with diagonal ($b = 0$) and the entries $a = \delta$ and $c = 1$. Consider the mapping h: $(0, \pi/2) \to \mathbb{R}, t \mapsto (\sin t, \cos t)^t$. Then $\| \mathbf{h} \|_{\mathbf{B}}$ is constant for $\delta = 1$, strictly increasing for $\delta > 1$ and strictly decreasing for $\delta < 1$. Therefore sensitivity has essentially to respect the norms used in $\mathbb{R}^n$ in contrast to asymptotic stability which is independent of norms.

(iv) In the theorem the non-linearity of f is ignored by considering only $\mathbf{D}_2 f(t_0, \mathbf{x}_0)$. On the other hand the critical case in which all eigenvalues of $\mathbf{D}_2 f(t_0, \mathbf{x}_0)$ have negative real parts but at least one eigenvalue with zero real part depends essentially on the non-linear behaviour of f.

To explain this we return to the model example in Section 2 in the critical case $\lambda = \omega^2 > 0$. Then $\mathbf{B}_1$ becomes

$$\mathbf{B}_1 = \begin{pmatrix} b\xi & \dfrac{a - b\eta + c\xi}{2} \\ \dfrac{a - b\eta + c\xi}{2} & b - c\eta \end{pmatrix} \quad \text{with} \quad \xi := \omega^2 \left\{ \frac{\sin \alpha}{\alpha} - 1 \right\} - f(0) \frac{1 - \cos \alpha}{\alpha}$$

It is not hard to see that

$$\mathbf{B} \text{ positive-definite, } \mathbf{B}_1 \text{ negative-definite} \Rightarrow \xi < 0$$

Therefore the negative definiteness of $\mathbf{B}_1$ depends on $f(0)$ too (since $\xi \approx -f(0)\alpha/2$ if $\alpha \approx 0$) and further considerations are needed.

Indeed, if $f(0) \neq 0$ and $|\alpha|$ is sufficiently small, then $\xi < 0$ implies $\pm f(0)|\alpha| > 0$, a contradiction; $t_0 = 0$ is sensitive. If, conversely, $f(0) = 0$ and $|\alpha|$ is sufficiently small, then $\xi < 0$. Moreover, by (3) we find that $\mathbf{B}_1$ is negative-definite, and thus $t_0 = 0$ is insensitive.

(v) The time t_0 in which $\mathbf{x}$ changes the sensitivity might be interpreted as a critical point like a bifurcation point in quasistatic considerations (see Kleiber *et al.*[5]). But this comparison seems not to be allowed for several reasons. First, the influence of perturbations acts only in a period of time and not at that time t_0. Secondly, we should stress that we are (in general) not dealing with equilibria since $\dot{\mathbf{x}}(t_0) \neq 0$. Hence the flow-box theorem (see Hirsch and Smale[6]) shows that the neighbourhood of (t_0, x_0) looks qualitatively identical to neighbouring points, which is in contrast to static bifurcation. Thirdly, there is no bifurcation–all initial-value problems under consideration have unique solutions.

On the other hand, numerical computations as well as imperfections yield small perturbations which will be reduced during sensitivity and can increase exponentially just after the insensitivity has changed to sensitivity. Hence time t_0 indicates a qualitative change of the solutions. In general, however, perturbation occurring in sensitive situations could yield large quantitative changes–see Kleiber *et al.*[5]

4. PARTICULAR CASE

In the following we discuss insensitivity in the particular situation of the equations of motion. This yields the main theorem on insensitivity in applications.

The non-linear initial-value problem

$$\mathbf{M}\ddot{u} + \mathbf{C}\dot{u} + \mathbf{R}(u) = \mathbf{P}(t), \qquad u(0) = u_0, \qquad \dot{u}(0) = \dot{u}_0 \tag{8}$$

with $\mathbf{M}, \mathbf{C} \in \mathbb{R}^{n \times n}$, $\mathbf{M}$ positive-definite, and the C^2 maps $\mathbf{R}$: $\mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{P}$: $\mathbb{R}^n \to \mathbb{R}^n$ will be transformed via

$$\mathbf{x} = \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \in \mathbb{R}^{2n}$$

into the initial-value problem $\dot{\mathbf{x}} = f(t, \mathbf{x}(t))$,

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{u} \\ -\mathbf{M}^{-1}[\mathbf{R}(u) + \mathbf{C}\dot{u} - \mathbf{P}(t)] \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} \tag{9}$$

Problems (8) and (9) are equivalent. The definition of sensitivity and the theorem is applicable for (9). Let $\mathbf{x}$: $(-a, +a) \to \mathbb{R}^{2n}$ be the solution of (9), $a > 0$, and let f: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by the first right-hand side of (9).

Lemma.

(i) For all $t \in (-a, +a)$ we have

$$\mathbf{D}_2 f(t, \mathbf{x}(t)) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}(t) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}$$

where $\mathbf{K}(t) := \mathbf{D}_2\mathbf{R}(u(t))$ is the Jacobian of $\mathbf{R}$ at $u(t)$, i.e. the tangent stiffness matrix.

(ii) The eigenvalues of $\mathbf{D}_2 f$ are just the eigenvalues of the λ-matrix,

$$\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \tag{10}$$

(iii) Let $\mathbf{C} := d_1\mathbf{M}$ for a fixed positive number d_1. Then, any eigenvalue λ of (10) is given by

$$\lambda = -\frac{d_1}{2} \pm \sqrt{\left(\frac{d_1^2}{4} + \mu\right)} \tag{11}$$

with an eigenvalue μ of $(\mu\mathbf{M} + \mathbf{K})$.

Proof.

(i) is obvious. Little calculation shows that λ is an eigenvalue with a related eigenvector $(\phi, \eta)^t \in \mathbb{R}^{2n}$ iff λ is an eigenvalue of (10) with a related eigenvector ϕ, $\eta = \lambda\phi$. This implies (ii) and leads to (iii). □

Summing up, we conclude this Section with the following main result which is directly implied by the Lemma and Theorem II.

Theorem III

Let the solution **u** of (8) exist on $[0, a)$ and let $t_0 \in (0, a)$. In addition, let $\mathbf{K} := \mathbf{DR}(u)$ be symmetric and $\mathbf{C} := d_1 \cdot \mathbf{M}$ with $\mathbf{M}$ positive-definite and $d_1 > 0$. If $\mathbf{K}(t_0) := \mathbf{DR}(u(t_0))$ is positive-definite, then the solution u is insensitive in t_0. If $\mathbf{K}(t_0)$ has some negative eigenvalue then u is sensitive in t_0. □

Remark. It seems interesting that, in the situation of the theorem, sensitivity behaviour is independent of $d_1 > 0$.

In practical applications the damping matrix $\mathbf{C}$ is small and can be neglected in the numerical computation of short processes like the beginning of vibrations etc. Then the theorem can be applied only if the computation is assumed to be an approximation for the true solution of (8) with $\mathbf{C} = d_1\mathbf{M}$, $d_1 > 0$ but d_1 small. It is notable that the static stability criterion is independent of the damping constant d_1, while for $d_1 = 0$ equation (11) gives imaginary eigenvalues of $\mathbf{D}_2 f$ so that Theorem II cannot be applied.

5. DISCUSSION

We recall that the static stability is equivalent to insensitivity. But there are several reasons below showing that the static stability criterion is of minor significance in classical non-linear dynamics. Conversely, insensitivity is important if the perturbation behaviour of a mechanical system is assessed at time t_0.

Insensitivity does not imply asymptotic stability (see the counter-example of Vinodradov in Reference 7, p. 120). On the other hand, asymptotic stability does not imply global insensitivity.

A connection is only given in the linear case of Theorem I. In general, sensitivity is local in time and space and therefore (in general) of no relevance for (global) stability.

Whereas classical asymptotic stability is important (if it can be proved), we stress insensitivity as an indication of a safe engineering construction. Conversely, sensitivity characterizes the possiblity that small perturbations of the current situation can increase and thus can lead to a change of the asymptotic behaviour (see Figure 1). Consequently, sensitive situations should be considered with great attention in both the engineering design and numerical simulations.

This holds true even if we have a non-autonomous and non-equilibrium problem and no other criterion can be used.

As explained in remark (v) in Section 3, this paper confirms the observations in Burmeister and Ramm,[3] Krätzig[4] and Kleiber *et al.*,[5] that the static stability criterion is of significance in non-linear dynamics.

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