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Weierstrass formula and zero-finding methods

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Summary. Classical Weierstrass' formula [29] has been often the subject of investigation of many authors. In this paper we give some further applications of this formula for finding the zeros of polynomials and analytic functions. We are concerned with the problems of localization of polynomial zeros and the construction of iterative methods for the simultaneous approximation and inclusion of these zeros. Conditions for the safe convergence of Weierstrass' method, depending only on initial approximations, are given. In particular, we study polynomials with interval coefficients. Using an interval version of Weierstrass' method enclosures in the form of disks for the complex-valued set containing all zeros of a polynomial with varying coefficients are obtained. We also present Weierstrass-like algorithm for approximating, simultaneously, all zeros of a class of analytic functions in a given closed region. To demonstrate the proposed algorithms, three numerical examples are included.

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1. Introduction

Consider a monic polynomial of degree $n \geq 3$

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n}(z - \zeta_j) \quad (a_i \in \mathbb{C})$$

with simple complex zeros $\zeta_1, ..., \zeta_n$. Since

$$P(z) = (z - \zeta_i)\prod_{\substack{j=1 \\ j \neq i}}^{n}(z - \zeta_j),$$

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we obtain the fixed point relations

$$\zeta_i = z - \frac{P(z)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z-\zeta_j)} \quad (i = 1, \dots, n). \tag{1.1}$$

Assume that distinct complex numbers $z_1, \dots, z_n$ are reasonably good approximations to the zeros $\zeta_1, \dots, \zeta_n$ of P. Putting $z = z_i$ and substituting the zeros ζ_j by their approximations z_j $(j \neq i)$ in (1.1), we obtain

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)} \quad (i = 1, \dots, n). \tag{1.2}$$

Here $\hat{z}_i$ appears to be a new approximation to the zero ζ_i. In fact, this formula is a classical result due to Weierstrass [29,p. 258] connected with a constructive proof of the fundamental theorem of algebra. For this reason, the formula (1.2) is often called Weierstrass' formula although Weierstrass was not using it for a numerical calculation of polynomial zeros. The quotient

$$W(z_i) = \frac{P(z_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)}$$

will be called *Weierstrass' correction.* Sometimes, we will write W_i instead of $W(z_i)$.

According to (1.2) the following iterative method can be formulated for approximating, simultaneously, all zeros of the polynomial P:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{P(z_i^{(m)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}\big(z_i^{(m)}-z_j^{(m)}\big)} \quad (i = 1, \dots, n;\ m = 0, 1, \dots). \tag{1.3}$$

Algorithm (1.3) has been rediscovered several times (see, e.g. Durand [9], Dochev [8], Börsch-Supan [3], Kerner [14], S. Prešić [25]) and it has been derived in various ways. Dochev [8] was the first who proved the quadratic convergence of this algorithm. A more economical realization from a computational point of view was given by Werner [30]. Finally, let us note that, starting from the fixed point relation (1.1) and disjoint initial disks $Z_1^{(0)}, \dots, Z_n^{(0)}$ which contain the zeros $\zeta_1, \dots, \zeta_n$ respectively, Alefeld and Herzberger [1,Ch.8] constructed the interval version of Weierstrass' formula in the form

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{P(z_i^{(m)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}\big(z_i^{(m)}-Z_j^{(m)}\big)} \quad (i = 1, \dots, n;\ m = 0, 1, \dots), \tag{1.4}$$

where $z_i^{(m)}$ is the center of the disk $Z_i^{(m)}$. The main advantage of the interval method (1.4) is that $\zeta_i \in Z_i^{(m)}$ for all $i = 1, ..., n$ and $m = 1, 2, ...,$ which provides a control of the accuracy in each iteration step.

In this paper we will give some further applications of Weierstrass' formula concerned with iterative methods for finding zeros and related topics. In Sect. 2 we give a new result concerning localization of polynomial zeros. This result, based on Weierstrass' corrections, is used for the construction of inclusion disks which are necessary for the application of inclusion methods.

In the literature, initial conditions for the safe convergence of simultaneous method for polynomial zeros most frequently involve unattainable data (for instance, minimal distance of zeros), which is not of sufficient practical importance. In Sect. 3 we adopt the result from [24] in order to state initial conditions for the convergence of Weierstrass' method, which depend only on the initial approximations and the degree of a polynomial. According to these results two combined methods for the inclusion of polynomial zeros are constructed.

Polynomials whose coefficients are uncertain numbers or lie in some intervals appear in mathematical models of scientific or engineering disciplines. Their zeros are contained in some closed complex-valued sets, called *zero-sets.* In Sect. 4 we give a procedure for finding circular enclosures of zero-sets, based on the result from Sect. 2. Furthermore, we give a version of Weierstrass' interval method for the contraction of these inclusion disks.

Section 5 is devoted to an iterative method of Weierstrass' type for the simultaneous finding of the zeros of a class of analytic functions. A convergence theorem and an analysis of numerical stability of this method are included.

For a practical demonstration, the presented algorithms of Weierstrass' type have been illustrated on numerical examples within Sects. 3, 4 and 5. These examples were realized in FORTRAN 77 in quadruple-precision arithmetic (about 33 significant decimal digits) on the Micro VAX II computer.

2. Localization of zeros

Weierstrass' correction $W(z_i)$ has been often used for a posteriori error estimates for a given set of approximate zeros. Braess and Hadeler [4] have proved that the disk given by

$$|z - z_i| \leq n|W(z_i)| \tag{2.1}$$

contains at least one zero of the polynomial P. Smith [27] has improved slightly this result; namely, he has shown that the disk

$$|z - (z_i - W(z_i))| \leq (n-1)|W(z_i)| \tag{2.2}$$

also contains at least one zero of P. The purpose of this section is to present some new inclusion disks based on Weierstrass' corrections.

It is known (see [10] [6]) that the characteristic polynomial of the $n \times n$-matrix

$$B := \mathrm{diag}(z_1, \ldots, z_n) - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \cdot (W_1, \ldots, W_n)$$

is equal to $(-1)^n P(z)$. Hence, via Gerschgorin's inclusion theorem applied to B we can get locations for the zeros of P. Before doing this, we may transform the matrix B into $T^{-1}BT$ having the same eigenvalues for any regular matrix T. The question "which T gives the best inclusion disks?" is solved (in some sense) if T belongs to the class of diagonal matrices. It turns out that the best "Gerschgorin's disks" lead to the following estimate, proved in [10] and [6].

Theorem A. *For $p \in \{1, 2, \ldots, n\}$ and $\xi \in \mathbb{C}$ let r be a positive number bounded by*

$$\max_{j=1,\ldots,p} (|z_j - W_j - \xi| - |W_j|) < r < \min_{j=p+1,\ldots,n} (|z_j - W_j - \xi| + |W_j|) \tag{2.3}$$

such that

$$1 > h(r) := \sum_{j=1}^{p} \frac{|W_j|}{r - |z_j - W_j - \xi| + |W_j|} + \sum_{j=p+1}^{n} \frac{|W_j|}{|z_j - W_j - \xi| + |W_j| - r} \geq 0.$$

Then there are exactly p zeros in the open disk with center ξ and radius r.

Remark 1. In the case $p = n$ the conditions on the upper bound of r and the last sum must be neglected. A reordering leads to more flexibility in Theorem A.

Remark 2. Adopting notations from Theorem A, it follows from $h(r) \leq 1$ by continuity that at least p zeros of P lies in the closed disk with center ξ and radius r.

In the case $p = 1$ Theorem A can be specified giving the following simpler estimate proved in [5, Satz 3] (cf. also e.g. [3, 4, 6] for similar results) and used in this paper in Sect. 4.

Theorem B. *Let $\xi := z_i - W_i \in \mathbb{C} \setminus \{z_1, \ldots, z_n\}$ and set*

$$\delta_i := |W_i| \cdot \max_{j=1,\ldots,n, j\neq i} |z_j - \xi|^{-1}, \quad \sigma_i := \sum_{j=1, j\neq i}^{n} \frac{|W_j|}{|z_j - \xi|},$$

$i \in \{1, \ldots, n\}$. If $\sqrt{1 + \delta_i} > \sqrt{\delta_i} + \sqrt{\sigma_i}$ then there is exactly one zero of P in the disk with center ξ and radius

$$|W_i| \cdot \left(1 - \frac{2(1 - 2\sigma_i - \delta_i)}{1 - \sigma_i - 2\delta_i + \sqrt{(1 - \sigma_i - 2\delta_i)^2 + 4\delta(1 - 2\sigma_i - \delta_i)^2)}}\right). \tag{2.4}$$

If $\sqrt{1 + \delta_i} > \sqrt{\delta_i} + \sqrt{\sigma_i}$ and $\delta_i + 2\sigma_i < 1$, then there is exactly one zero of P in the disk with center ξ and radius

$$|W_i| \frac{\delta_i + \sigma_i}{1 - \sigma_i}.$$

In the sequel we apply Theorem A to the sequences $\{z_1^{(m)}\}, \ldots, \{z_n^{(m)}\}$ of approximations of zeros $\zeta_1, \ldots, \zeta_n$ generated from Weierstrass method (1.3). Let $r_j^{(m)} := |z_j^{(m)} - \zeta_j|$, $r^{(m)} := \max r_j^{(m)}$ and let $\delta := \min_{j\neq k} |\zeta_j - \zeta_k| > 0$ (since P has simple zeros).

Theorem 2.1. *If $r^{(1)} \leq r^{(0)} \leq \frac{\delta}{4n}$ then, for any $i = 1, \ldots, n$, the closed disk $\{z_i^{(1)}; |z_i^{(1)} - z_i^{(0)}|\}$ contains ζ_i.*

Proof. We use Theorem A with $z_i := z_i^{(0)}$, $\hat{z}_i := z_i^{(1)}$, $W_i := z_i - \hat{z}_i$ (Weierstrass' corrections), $r_i := r_i^{(0)}$, $\hat{r_i} := r_i^{(1)}$ for $i \in \{1, \ldots, n\}$. According to the triangle inequality we have

$$(2.5) \qquad |W_i| \leq r_i + \hat{r_i} \leq r^{(0)} + r^{(1)} \qquad (i = 1, \ldots, n).$$

In order to prove Theorem 2.1 for i let, without loss of generality, $i = 1 = p$, $\xi := \hat{z_i} := z_i - W_i$, and $r := |W_1|$ in Theorem A. Due to (2.5) and $r^{(1)} \leq r^{(0)} \leq \frac{\delta}{4n}$

$$r < \delta - r^{(0)} - r^{(1)}.$$

Hence r satisfies (2.3). Therefore we are allowed to take $r = |W_1|$ in Theorem A so that it remains to prove $h(|W_1|) \leq 1$ (h defined in Theorem A), which is equivalent to

$$\sum_{k=2}^{n} \frac{|W_k|}{|\hat{z_k} - \hat{z_1}| + |W_k| - |W_1|} \leq \frac{1}{2}.$$

Using (2.5),

$$\begin{aligned}\sum_{k=2}^{n} \frac{|W_k|}{|\hat{z_k} - \hat{z_1}| + |W_k| - |W_1|} &\leq \sum_{k=2}^{n} \frac{r_k + \hat{r_k}}{|\hat{z_k} - \hat{z_1}| + r_k + \hat{r_k} - |W_1|} \\ &\leq \sum_{k=2}^{n} \frac{2r^{(0)}}{\delta - \hat{r_1} - r} \leq \frac{2r^{(0)}(n-1)}{\delta - 3r^{(0)}},\end{aligned}$$

which is $\leq 1/2$ if $4nr^{(0)} \leq \delta$. $\square$

Remark 3. In Theorem A the optimal bound, i.e. the smallest r in some interval (given in (2.3)) with $h(r) = 1$, can be easily calculated, e.g. with Newton-Raphson method or regula-falsi since h is convex.

3. Hybrid algorithms and initial conditions

Most of the initial conditions for the convergence of iterative methods treated in literature are not of sufficient practical interest since they depend on unattainable data (for instance, on the minimal distance between (unknown) zeros). In this section we give practicable conditions for the convergence of Weierstrass' iterative methods (1.3) and (1.4) in complex and circular complex arithmetic which depend only on a set of initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$. These conditions are stated by simplifying the initial conditions for Weierstrass' method (1.3) given by M. Prešić [24].

Let $\{z_1^{(m)}\}, \ldots, \{z_n^{(m)}\}$ be the sequences generated by (1.3) and let $d^{(m)} = \min \{|z_i^{(m)} - z_j^{(m)}|\}$ $(i \neq j)$. Then we have

Theorem 3.1. *Let* $z_1^{(0)}, \ldots, z_n^{(0)}$ $(n \geq 3)$ *be distinct approximations to the zeros* $\zeta_1, \ldots, \zeta_n$ *of the polynomial* P *and let* $d^{(0)} = \min\limits_{\substack{i,j \\ i \neq j}} \{|z_i^{(0)} - z_j^{(0)}|\}$. *If*

$$(3.1) \qquad \max_{1 \leq i \leq n} |W(z_i^{(0)})| \leq \frac{d^{(0)}}{5n},$$

then for all $m = 0, 1, \ldots$ *we have*

(i) $\displaystyle\max_{1\le i\le n} |W(z_i^{(m)})| \le \frac{d^{(m)}}{5n}$;

(ii) $d^{(m+1)} \ge \frac{5n-2}{5n} d^{(m)}$;

(iii) $|z_i^{(m+2)} - z_i^{(m+1)}| \le \frac{1}{2}|z_i^{(m+1)} - z_i^{(m)}| = \frac{1}{2}|W(z_i^{(m)})|$.

Theorem 3.1 can be proved in the similar way as in [24] and, for this reason, we omit the proof.

Using the assertions of Theorem 3.1 we are in the possibility to state the following assertions.

Theorem 3.2. *If the condition (3.1) is satisfied, then for Weierstrass' method (1.3) the following is valid:*

(i) the sequences $\{z_1^{(m)}\}, \ldots, \{z_n^{(m)}\}$ *converge to the zeros* $\zeta_1, \ldots, \zeta_n$, *respectively;*

(ii) $\zeta_i \in D_i^{(m-1)} := \{z_i^{(m)}; |W(z_i^{(m-1)})|\}$ $\quad (i = 1, \ldots, n;\ m = 1, 2, \ldots)$

(iii) $D_i^{(m-1)} \cap D_j^{(m-1)} = \emptyset\ (i \ne j) \quad (m = 1, 2, \ldots)$.

Proof. We give an outline of the proof of Theorem 3.2. Let the condition (3.1) be satisfied. Then, according to the assertion (iii) of Theorem 3.1 we can construct the sequences of disks $\{D_1^{(m)}\}, \ldots, \{D_n^{(m)}\}$ by $D_i^{(m)} := \{z : |z - z_i^{(m+1)}| \le |W(z_i^{(m)})|\}$ such that

$$|W(z_i^{(m)})| \le \frac{1}{2}|W(z_i^{(m-1)})| \quad \text{and} \quad D_i^{(0)} \supset D_i^{(1)} \supset D_i^{(2)} \supset \cdots \quad (i = 1, \ldots, n).$$

Hence $|W(z_i^{(m-1)})| \to 0$ when $m \to \infty$. Since the metric subspace $D_i^{(0)}$ is complete (as a closed set in $\mathbb{C}$), there exists a unique point $z_i^* \in D_i^{(0)}$ so that

$$z_i^{(m)} \to z_i^* \text{ and } z_i^* \in \bigcap_{m=0}^{\infty} D_i^{(m)} \subset D_i^{(k)} \quad \text{for each } k = 0, 1, \ldots .$$

In this limit case Weierstrass' formula (1.3) yields $P(z_i^*) = 0$ whence $\zeta_i = z_i^*$ $(i = 1, \ldots, n)$. Hence, the assertions (i) and (ii) follow.

Due to (i) and (ii) we find

$$\begin{aligned}|z_i^{(m)} - z_j^{(m)}| &\ge d^{(m)} \ge \frac{5n-2}{5n} d^{(m-1)} \ge (5n-2) \max_{1\le i\le n} |W(z_i^{(m-1)})| \\ &> |W(z_i^{(m-1)})| + |W(z_i^{(m-1)})|\end{aligned}$$

which proves (iii). □

Remark 4. Assuming that the condition (3.1) holds, by the assertion (i) of Theorem 3.1 it is easy to derive the estimates

$$\delta_i \le \frac{1}{5n-1}, \quad \sigma_i \le \frac{n-1}{5n-1}.$$

Hence, applying Theorem B for $\xi = z_i^{(m-1)} - W_i^{(m-1)}$ and (ii), (iii) of Theorem 3.2, after simple calculations we obtain

$$\zeta_i \in \left\{ z_i^{(m)}; \frac{1}{4}|W_i^{(m-1)}| \right\} \quad (i = 1, ..., n).$$

Let us note the above inclusion disks are disjoint (according to (iii) of Theorem 3.2).

The assertions of Theorems 3.1 and 3.2 can be applied to the construction of inclusion methods which provide upper error bounds of the obtained approximations to the zeros. We will consider two versions of these methods which, in fact, have a “hybrid” structure. Speaking about hybrid methods we assume a combination of algorithms in ordinary complex arithmetic and circular complex arithmetic. For more details on hybrid (combined) methods see [20].

Hybrid method (I)

In order to decrease the computational costs of interval methods, it is preferable to apply a combined procedure. In the case of Weierstrass-like algorithms this procedure consists of the following:

1. Using Weierstrass’ method (1.3) in ordinary (complex) arithmetic, calculate the complex approximations $z_i^{(M)}$ $(i = 1, ..., n)$ to any wanted accuracy (after M iteration steps);

2. In the final step *provide the enclosure* of zeros; for instance, using *either* an a posteriori error estimate procedure like (1.5), (1.6), or (ii) in Theorem 3.2, *or* Weierstrass’ interval method (1.4).

Assume that we have found the approximations $z_1^{(M)}, \dots, z_n^{(M)}$ to the desired accuracy applying the step 1. under the condition (3.1) for Weierstrass’ method (1.3). As mentioned above, the step 2. can be realized in several ways:

Procedure (Ia): For $i = 1, ..., n$ calculate $W(z_i^{(M)}), ..., W(z_i^{(M)})$ and find Braess-Hadeler’s disks $K_i^{(M)} := \{z_i^{(M)}; n|W(z_i^{(M)})|\}$ (according to (1.5)). As it was proved by Braess and Hadeler [4], each of these disks contains at least one zero and their union contains all zeros of the considered polynomial. But using the assertion (i) of Theorem 3.1 we have for arbitrary $i, j \in \{1, ..., n\}$ $(i \neq j)$

$$|z_i^{(M)} - z_j^{(M)}| \geq d^{(M)} \geq 5n \max_{1 \leq i \leq n} |W(z_i^{(M)})| > n|W(z_i^{(M)})| + n|W(z_j^{(M)})|,$$

which proves that all disks $K_1^{(M)}, \dots, K_n^{(M)}$ are mutually disjoint. This implies that each disk contains one and only one zero of P, that is, $\zeta_i \in K_i^{(M)}$ $(i = 1, ..., n)$.

Procedure (Ib): By virtue of the assertion (ii) of Theorem 3.2 and Remark 4, we can choose the disks $D_i^{(M-1)} = \{z_i^{(M)}; \frac{1}{4}|W(z_i^{(M-1)})|\}$ $(M \geq 1)$ to be the inclusion disks for the zeros ζ_i. According to (iii) of Theorem 3.2 these disks are nonoverlapping.

Procedure (Ic): Choosing initial disks $D_1^{(M)}, \dots, D_n^{(M)}$ as in Procedure (Ib), apply Weierstrass’ interval method (1.4). The constructed combined method of Weierstrass’ type has the form

$$Z_i^{(M,1)} = z_i^{(M)} - \frac{P(z_i^{(M)})}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \left(z_i^{(M)} - D_j^{(M-1)}\right)}. \tag{3.2}$$

In this way the enclosure of the zeros is provided using only one interval iteration, which enables a high computational efficiency of the stated combined method (for more details about the efficiency of combined methods see [20]). The upper "index" $(M,1)$ indicates that the inclusion disk $Z_i^{(M,1)}$ is obtained by M "point" iterations and one interval iteration.

The following lemma guarantees that (3.2) is feasible, i.e. that zero does not belong to the numerator set in (3.2).

Lemma 3.1. *If (3.1) holds we have*

$$0 \notin \prod_{j \neq i} \left(z_i^{(M)} - D_j^{(M-1)} \right).$$

Proof. Because of Theorem 3.1 (i) we have

$$\begin{aligned} \prod_{j \neq i} \left(z_i^{(M)} - D_j^{(M-1)} \right) &\subseteq \prod_{j \neq i} \left\{ z_i^{(M)} - z_j^{(M)}; \frac{1}{4} |W(z_j^{(M-1)})| \right\} \\ &\subseteq \left(\prod_{j \neq i} (z_i^{(M)} - z_j^{(M)}) \right) \cdot \left\{ 1; \frac{d^{(M)}}{4(5n-2)} \cdot \sum_{j \neq i} |z_i^{(M)} - z_j^{(m)}|^{-1} \right\} \\ &\subseteq \left(\prod_{j \neq i} (z_i^{(M)} - z_j^{(M)}) \right) \cdot \left\{ 1; \frac{n-1}{4(5n-2)} \right\}. \end{aligned}$$

Since $0 \notin \{1; 1/20\}$ and because of Theorem 3.2 (iii) this concludes the proof. $\square$

Remark 5. The procedure (Ia) and (Ib) are of the same type. The error bound obtained by Procedure (Ia) is considerably sharper in reference to that produced by Procedure (Ib). Indeed, we have

$$\text{rad } D_i^{(M-1)} = O(|z_i^{(M-1)} - \zeta_i|), \quad \text{rad } K_i^{(M)} = nO(|z_i^{(M-1)} - \zeta_i|^2).$$

Although this improvement requires additional computational effort (extra calculation of $W(z_i^{(M)})$), the produced (very precise) bounds justify this cost. On the other hand, once $W(z_i^{(M)})$ is known, we may apply one step of (1.3) which is cheap then and gives better bounds via (Ia). However, Procedure (Ic) yields the sharpest bounds but, compared in Procedure (Ia), it requires the calculations of the intervals $\prod_{j \neq i} (z_i^{(M)} - D_j^{(M-1)})$ instead of the complex numbers $\prod_{j \neq i} (z_i^{(M)} - z_j^{(M)})$. The presented comparisons are evident from the numerical example given below.

Hybrid method (II)

The construction of this combined method is similar to Procedure (Ic). Namely, we first apply *one* iteration in complex arithmetic using the iteration method (1.3) assuming that the condition (3.1) is fulfilled and obtain the approximations $z_1^{(0)}, \dots, z_n^{(n)}$. After that we take the disks $D_1^{(0)}, \dots, D_n^{(0)}$ as initial inclusion disks and apply Weierstrass' interval method (1.4). In this case the question of the convergence of the interval method (1.4) arises. The answer can be given applying (3.1) and the following convergence condition for the method (1.4) presented in [19, Theorem 3.2]:

Theorem 3.3. *Let $Z_1^{(0)}, \ldots, Z_n^{(0)}$ be the initial disks containing the zeros $\zeta_1, \ldots, \zeta_n$ respectively, and let the interval sequences $\{Z_i^{(m)}\}, \ldots, \{Z_i^{(m)}\}$ be produced by (1.4). Furthermore, let*

$$\eta^{(0)} := \min_{\substack{i,j \\ i \neq j}} \{|z_i^{(0)} - z_j^{(0)}| - r_j^{(0)}\}, \quad r^{(0)} = \max_{1 \le j \le n} r_j^{(0)},$$

where $z_i^{(m)} = \mathrm{mid}\ Z_i^{(m)}$, $r_i^{(m)} = \mathrm{rad}\ Z_i^{(m)}$ $(m = 0, 1, \ldots)$. Then, under the condition

$$\eta^{(0)} > \frac{7(n-1)}{2} r^{(0)}, \tag{3.3}$$

for each $i = 1, \ldots, n$ and $m = 0, 1, \ldots$ we have

1. $\zeta_i \in Z_i^{(m)}$;
2. $r^{(m+1)} < \frac{7(n-1)}{4(\eta^{(0)} - 5r^{(0)})} (r^{(m)})^2$. $\square$

Taking $Z_i^{(0)} = D_i^{(0)}$ $(i = 1, \ldots, n)$ we will have

$$r^{(0)} := \max_i |W(z_i^{(0)})|, \quad \eta^{(0)} = \min_{\substack{i,j \\ i \neq j}} (|z_i^{(1)} - z_j^{(1)}| - r_j^{(0)}).$$

Using the above notations and the assertions (i) and (ii) of Theorem 3.2 we obtain

$$\begin{aligned} \eta^{(0)} &\ge d^{(1)} - r^{(0)} > \frac{5n-2}{5n} d^{(0)} - r^{(0)} \ge (5n-2) \max_{\substack{i,j \\ i \neq j}} |W(z_i^{(0)})| - r^{(0)} \\ &= (5n-3) r^{(0)} > \frac{7(n-1)}{2} r^{(0)}. \end{aligned}$$

Therefore, the condition (3.1) for the initial complex approximations implies the inequality (3.3), which means that the assertions 1. and 2. of Theorem 3.3 are valid for the mentioned choice of disks.

Example 1. For demonstration, we apply the hybrid methods (I) and (II) for the inclusion of the eigenvalues of Hessenberg's matrix $H = [h_{ij}]$. As a concrete example we consider the matrix

$$H = \begin{bmatrix} 8+12i & 1 & 0 & 0 \\ 0 & 6+9i & 1 & 0 \\ 0 & 0 & 4+6i & 1 \\ 1 & 0 & 0 & 2+3i \end{bmatrix}$$

The characteristic polynomial of the above matrix is

$$P(\lambda) = \lambda^4 - (20+30i)\lambda^3 + (-175+420i)\lambda^2 + (2300-450i)\lambda - 2857 - 2880i. \tag{3.4}$$

Gerschgorin's disks containing all eigenvalues of H are of the form $\{h_{ii}; 1\}$, where h_{ii} are the diagonal elements of the matrix H. To start our methods we take the centers of these disks to be initial approximations, that is,

$$z_1^{(0)} = 8 + 12\mathrm{i},\ z_2^{(0)} = 6 + 9\mathrm{i},\ z_3^{(0)} = 4 + 6\mathrm{i},\ z_4^{(0)} = 2 + 3\mathrm{i}\ .$$

We note that for such a choice of data the condition (3.1) is fulfilled.

Applying the hybrid method (I) with two iterations in the step 1. ($M = 2$) one obtains

$$z_1^{(2)} = 7.996505070225 + \mathrm{i}\ 11.99932088107,$$
$$z_2^{(2)} = 6.010455791121 + \mathrm{i}\ 9.002056973200,$$
$$z_3^{(2)} = 3.989544208879 + \mathrm{i}\ 5.997943026799,$$
$$z_4^{(2)} = 2.003494929774 + \mathrm{i}\ 3.000679118928.$$

With these approximations we calculate the values $n|W(z_i^{(2)})|$ ($i = 1, 2, 3, 4$) getting the radii of the inclusion disks $K_i^{(2)}$ required in Procedure (Ia)

$$\mathrm{rad}\ K_1^{(2)} = 4.10 \times 10^{-11}, \quad \mathrm{rad}\ K_2^{(2)} = 4.40 \times 10^{-10},$$
$$\mathrm{rad}\ K_3^{(2)} = 4.40 \times 10^{-10}, \quad \mathrm{rad}\ K_4^{(2)} = 4.10 \times 10^{-11}.$$

The inclusion disks $D_1^{(1)}, \ldots, D_4^{(1)}$ defined in Procedure (Ib) have the same centers as the disks $K_1^{(2)}, \ldots, K_4^{(2)}$ and the radii given by the already calculated values $\frac{1}{4}|W(z_i^{(1)})|, \ldots, \frac{1}{4}|W(z_4^{(1)})|$. Thus, this procedure possesses a low computational cost but the produced error bounds are very rough; for example, one obtains

$$r_1^{(1)} = 1.61 \times 10^{-6},\ r_2^{(1)} = 3.92 \times 10^{-6},\ r_3^{(1)} = 3.92 \times 10^{-6},\ r_4^{(1)} = 1.61 \times 10^{-6},$$

where $r_i^{(1)} := \mathrm{rad}\ D_i^{(1)}$.

Procedure (Ic) is realized using the interval formula (3.2) with the complex approximations $z_1^{(2)}, \ldots, z_4^{(2)}$ and the disks $D_i^{(1)} = \{z_i^{(2)}; r_i^{(1)}\}$ (given above). The following inclusion disks are obtained:

$$Z_1^{(2,1)} = \{7.996505070219710254 + \mathrm{i}\ 11.99932088106339498; 1.83 \times 10^{-17}\},$$
$$Z_2^{(2,1)} = \{6.010455791182352056 + \mathrm{i}\ 9.002056973291392465; 1.93 \times 10^{-16}\},$$
$$Z_3^{(2,1)} = \{3.989544208817647944 + \mathrm{i}\ 5.997943026708607535; 1.93 \times 10^{-16}\},$$
$$Z_4^{(2,1)} = \{2.003494929780289745 + \mathrm{i}\ 3.000679118936605022; 1.83 \times 10^{-17}\}.$$

From the above lists we observe that Procedure (Ic) gives the best estimates for the zeros. To illustrate a very good performance of this procedure we note that three "point" iterations (by (1.3)) and one interval iteration (by (3.2)) produce the inclusion disks with the radii in the interval $[7.83 \times 10^{-33}, 9.96 \times 10^{-32}]$.

In the case of the hybrid method (II) we first apply one "point" iteration by (1.3) and obtain the inclusion disks

$$D_1^{(0)} = \{7.99651 + \mathrm{i}\ 11.999317; 0.00356\},$$
$$D_2^{(0)} = \{6.010469 + \mathrm{i}\ 9.002048; 0.01067\},$$
$$D_3^{(0)} = \{3.989531 + \mathrm{i}\ 5.997951; 0.01067\},$$
$$D_4^{(0)} = \{2.003489 + \mathrm{i}\ 3.000683; 0.00356\}.$$

Then we take these disks as the initial inclusion disks for the zeros of the characteristic polynomial (3.4) (that is, the eigenvalues of the matrix H), $Z_i^{(0)} = D_i^{(0)}$ ($i = 1, 2, 3, 4$).

With these disks one iterates according to (1.4) getting the following inclusion intervals:

$$Z_1^{(2)} = \{7.996505070219710254 + \mathrm{i}\ 11.99932088106339497; 4.62 \times 10^{-18}\},$$
$$Z_1^{(2)} = \{6.010455791182352056 + \mathrm{i}\ 9.002056973291392465; 1.06 \times 10^{-17}\},$$
$$Z_1^{(2)} = \{3.989544208817647944 + \mathrm{i}\ 5.997943026708607535; 1.07 \times 10^{-17}\},$$
$$Z_1^{(2)} = \{2.003494929780289745 + \mathrm{i}\ 3.000679118936605023; 5.19 \times 10^{-18}\}.$$

These results are comparable with those produced by the hybrid method (I) – Procedure (Ic), but the hybrid method (II) requires somewhat more numerical operations since it needs more interval iterations.

4. The case of polynomials with interval coefficients

In applications the coefficients of polynomials are often not given as real or complex numbers since they must be computed (cf. Weidner's transformations [28]) or are based on perturbed measurements. As it was mentioned in the book [16], the presence of *uncertainty* in initial data appears in case studies of the behaviour of mathematical models; for instance, "parameter studies, sensitivity analysis, design analysis, effects of inaccurate measurements or observational errors... ". Since algebraic polynomial are often involved in mathematical models in various scientific and engineering disciplines, the problem of finding the zeros (or, more precisely, ranges of zeros) of polynomials with interval coefficients is of evident interest. In these cases we are lead to consider an interval polynomial, i.e. a polynomial

$$\mathbb{P}(z) = A_0 + A_1 z + \ldots + A_{n-1} z^{n-1} + z^n \tag{4.1}$$

with interval coefficients $A_0, \ldots, A_{n-1}$, $A_i = \{a_i; \epsilon_i\}$, $i = 0, \ldots, n-1$. For a given interval polynomial (4.1) we will write $P \in \mathbb{P}$ if

$$P(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + z^n \quad \text{with} \quad b_i \in A_i \quad (i = 0, \ldots, n-1). \tag{4.2}$$

Since we are concerned with a set of polynomials we have to deal with the set of zeros

$$\Lambda := \{z \in \mathbb{C} | \exists P \in \mathbb{P} \quad P(z) = 0\}$$

whose structure, in general, is involved. Assuming that P, given in (4.2) by $b_i := \mathrm{mid}(A_i) = a_i$, has simple zeros and the coefficients of $\mathbb{P}$ are sufficiently small, Λ can be partitioned in n disjoint subsets. We say that $\Lambda_1, \ldots, \Lambda_n$ are *zero sets of* $\mathbb{P}$ if $\Lambda_1, \ldots, \Lambda_n$ are disjoint and cover Λ,

$$\Lambda = \Lambda_1 \cup, \ldots, \cup \Lambda_n,$$

and any $P \in \mathbb{P}$ has zeros $\xi_1, \ldots, \xi_n$ with $\xi_i \in \Lambda_i$ $(i = 1, \ldots, n)$. In this case we may ask for disjoint inclusions of the zero sets of $\mathbb{P}$, i.e. we are lead to the problem of first to compute some pairwise disjoint intervals $Z_1, \ldots, Z_n$ with the property that any $P \in \mathbb{P}$ has exactly one zero in Z_j for $j = 1, \ldots, n$. In this way we provide that $Z_1, \ldots, Z_n$ *include all the zero-sets of* $\mathbb{P}$. Secondly, we are going to make these inclusion disks as small as possible.

Let us note that, if the interval coefficients are too large or the zeros of P are reasonable close, then the inclusion disks will be intersecting. We will not consider this case here since Weierstrass' interval method is defined only for disjoint disks. Thus, in this section, we always assume that the interval coefficients of $\mathbb{P}$ are sufficiently small. Then, in a preliminary step we compute good approximations of one complex polynomial $P \in \mathbb{P}$, e.g. P given in (4.2) by $b_i = \operatorname{mid}(A_i) = a_i$. Let $z_1, ..., z_n$ be approximations of the simple zeros of this polynomial P. Then $\mathbb{P}(z_j)$ is a disk computed by Horner's scheme using circular complex interval arithmetic,

$$\mathbb{P}(z_i) := (\cdots(((z_i + A_{n-1}) \cdot z_i + A_{n-2}) \cdot z_i + ... + A_2) \cdot z_i + A_1) \cdot z_i + A_0.$$

In that case Weierstrass' correction in the form of a disk is given by

$$\mathbf{W}_i := \frac{\mathbb{P}(z_i)}{\prod\limits_{k=1, k\neq i}^{n} (z_i - z_k)} \qquad (i = 1, ..., n) \tag{4.4}$$

Throughout this section the absolute value (modulus) of a disk $\{c; r\}$ is defined by $|\{c; r\}| := |c| + r$. If the interval coefficients of $\mathbb{P}$ are small enough and the approximations are sufficiently good then the following result leads to disk including all the zero-sets of $\mathbb{P}$.

Theorem 4.1. *Let $z_1, ..., z_n \in \mathbb{C} \setminus \{z_1, ..., z_n\}$ be pairwise distinct and set δ_i and σ_i as in Theorem B, where $|\mathbf{W}_j|$ is now the modulus of disk (4.4).*

If $\sqrt{1+\delta_i} > \sqrt{\delta_i} + \sqrt{\sigma_i}$ for any $i = 1, ..., n$ then the disks with center $z_i - \operatorname{mid}(\mathbf{W}_i)$ and radius

$$|W_i| \cdot \left(1 - \frac{2(1-2\sigma_i-\delta_i)}{1-\sigma_i-2\delta_i+\sqrt{(1-\sigma_i-2\delta_i)^2+4\delta_i(1-2\sigma_i-\delta_i)^2)}}\right) + \operatorname{rad}(\mathbf{W}_i). \tag{4.5}$$

includes all the zero-sets Λ_i of $\mathbb{P}$.

Proof. Let $P \in \mathbb{P}$ be given by (4.2). Let $W_1^P, ..., W_n^P$ denote the Weierstrass corrections,

$$W_i^P := \frac{P(z_i)}{\prod\limits_{\substack{k=1\\k\neq i}}^{n} (z_i - z_k)} \in \mathbb{C}.$$

Because of the inclusion property (using circular complex arithmetic for the computation of (4.4)), we have $P(z_i) \in \mathbb{P}(z_i)$, whence

$$W_i^P \in \mathbf{W}_i, \quad |W_i^P| < |\mathbf{W}_i| \qquad (i = 1, ..., n). \tag{4.6}$$

We apply Theorem B for the polynomial P with $\xi = z_i - W_i^P$. Let δ_i^P, σ_i^P denote numbers defined in Theorem B while δ_i, σ_i are defined in Theorem 4.1. Then, by (4.6),

$$\delta_i^P < \delta_i, \quad \sigma_i^P < \sigma_i \qquad (i = 1, ..., n).$$

Thus, Theorem B gives that there lies exactly one zero of P in the disk with center $z_i - W_i^P$ and radius

$$|W_i^P| \cdot (1 - \frac{2(1 - 2\sigma_i^P - \delta_i^P)}{1 - \sigma_i^P - 2\delta_i^P + \sqrt{(1 - \sigma_i^P - 2\delta_i^P)^2 + 4\delta_i^P(1 - 2\sigma_i^P - \delta_i^P)^2)}})$$

By using the triangle inequality and the upper bounds for δ_i^P and σ_i^P we conclude the proof. □

Remark 6. If $\epsilon := \max\{\epsilon_1, ..., \epsilon_n\}$ tends towards zero, then the right hand side of (4.5) tends towards the right hand side of (2.4). Hence, if we know good approximations of the zeros of one P from $\mathbb{P}$, then $|\mathbf{W}_1|, ..., |\mathbf{W}_n|$ are small so that Theorem 4.1 gives disks including all the zero-sets of $\mathbb{P}$ provided ϵ is sufficiently small.

Remark 7. Theorems A, B and 2.1 can also be applied giving inclusion disks for all the polynomials of $\mathbb{P}$. The proofs are analogous to the proof of Theorem 4.1 and they are also based on the inclusion property so that we omit details.

In view of Theorem 4.1 we assume in the following that we know some pairwise disjoint disks $Z_1, ..., Z_n$ including all the zeros of $\mathbb{P}$. Then, we are interested in new disks $\hat{Z}_1, ..., \hat{Z}_n$ having also this property but smaller radii.

We consider the following method, a natural generalization of the Weierstrass inclusion method:

$$\hat{Z}_i := z_i - \frac{\mathbb{P}(z_i)}{\prod_{\substack{k=1\\k\neq i}}^{n}(z_i - Z_k)} \quad (z_i = \mathrm{mid}\, Z_i;\ i = 1, ..., n). \tag{4.7}$$

Let $r_j := \mathrm{rad}\, Z_j,\ \hat{r_j} := \mathrm{rad}\, \hat{Z}_j,\ r := \max\{r_1, ..., r_n\}$, and $\hat{r} := \max\{\hat{r_1}, ..., \hat{r_n}\}$.

Theorem 4.2. *Assume that disks $Z_1, ..., Z_n$ contain all the zero-sets of $\mathbb{P}$. Then for the method (4.7) there holds the following:*
(i) $\hat{Z}_1, ..., \hat{Z}_n$ *include all the zero-sets of* $\mathbb{P}$;
(ii) $\hat{r} = O(\epsilon + r^2)$.

Proof. In order to prove (i) let $P \in \mathbb{P}$ have the zeros $\zeta_1, ..., \zeta_n$. By assumption we have $\zeta_i \in Z_i$ for all $i = 1, ..., n$. Due to the inclusion property and $P(z_i) \in \mathbb{P}(z_i)$, from (4.7) we have

$$\zeta_i \in z_i - \frac{P(z_i)}{\prod_{\substack{k=1\\k\neq i}}^{n}(z_i - Z_k)} \subseteq \hat{Z}_i.$$

Note that $\zeta_i \in Z_i$ and $P(\zeta_j) = 0$ imply $|P(z_i)| = O(r_i)$. Since $\mathrm{rad}\,\mathbb{P}(z_i) = O(\epsilon)$ and $\mathrm{rad}\,(\prod_{k=1,k\neq i}^{n}(z_i - Z_k)) = O(r)$, using circular arithmetic operations we obtain

$$\mathrm{rad}\, \hat{Z}_i = O(\epsilon + r_i \cdot \max_{k=1,\ldots,n,k\neq i} r_k), \tag{4.8}$$

which concludes the proof. □

Remark 8. Theorem 4.2 states that, for sufficiently small ϵ and r, the interval method (4.7) behaves like quadratic convergent method (i.e. the radii of the inclusion disks decrease quadratically in any iteration step) at least in the first iteration steps where we have $r^2 > \epsilon$. Since ϵ is fixed and r decreases we arrive within a finite number of iteration steps at $r^2 < \epsilon$ when we conclude from Theorem 4.2 that the radii will not be decreased quadratically furthermore.

Remark 9. The iteration should be terminated if $\hat{r}$ is not smaller than, for instance, $r/2$, i.e. if the radii are not improved considerably.

After termination one may use Theorem 4.1 to estimate the inclusions more closely.

Remark 10. Note that, by $\epsilon > 0$, one cannot expect that the inclusion disks can become arbitrarily small, their radii are at least $O(\epsilon)$. This shows that the Weierstrass' interval method (4.7) is optimal (up to constant factors) in the sense that the radii of the disks are decreased to the smallest possible bound $O(\epsilon)$.

Remark 11. From (4.8) we see that a greater R-order of convergence can be obtained by the single step version of method (4.7),

$$\hat{Z}_i := z_i - \frac{\mathbb{P}(z_i)}{\prod_{k=1}^{i-1}(z_i - \hat{Z}_k)\prod_{k=i+1}^{n}(z_i - Z_k)},$$

at least in the first steps when $r^2 >> \epsilon$. If we apply only one iteration step (or a few) we observe only the improvement of a few disks compared with the total step method (4.7).

According to Remark 4 and the previous consideration it follows that, under the condition (3.1), exactly one zero of P lies in the disk $\{z_i - W_i^P; \frac{1}{4}|W_i^P|\}$, and hence, in the disk $\{z_i; \frac{5}{4}|W_i^P|\}$. In practice, instead of $|W_i^P|$ we can take the modulus of the disk $\mathbf{W}_i$ given by (4.4) (see (4.6)). Applying the mentioned facts we are able to construct the following practical algorithms for the inclusion of zeros of interval polynomials:

1. For some reasonably good initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ apply M iterations (usually two or three) of Weierstrass' method (1.3) in ordinary complex arithmetic to the polynomial (4.2) in order to obtain complex approximations $z_1^{(M)}, \ldots, z_n^{(M)}$ to the desired accuracy.

2. Construct the inclusion disks

$$Z_i := \{z_i^{(M)}; r_i^{(M)}\} \quad \text{with} \quad r_i^{(M)} := \frac{5}{4}|\mathbf{W}(z_j^{(M)})| \quad (i = 1, \ldots, n). \tag{4.9}$$

3. Apply only one iterative step of the interval method

$$Z_i^{(M,1)} = z_i^{(M)} - \frac{\mathbb{P}(z_i^{(M)})}{\prod_{k\neq i}(z_i^{(M)} - Z_k)} \quad (i = 1, \ldots, n) \tag{4.10}$$

(formula (4.7)) to obtain the inclusion disks for the zeros-sets of a given interval polynomial $\mathbb{P}$.

Example 2. Let us consider the polynomial

$$\begin{aligned} P(z) = {} & z^5 + \{-4 - 5\mathrm{i}; \delta\}z^4 + \{6 + 20\mathrm{i}; \delta\}z^3 \\ & + \{-4 - 30\mathrm{i}; \delta\}z^2 + \{-15 + 20\mathrm{i}; \delta\}z + \{75\mathrm{i}; \delta\} \end{aligned}$$

which coefficients are disks with the radius δ. For demonstration, we have chosen $\delta = 10^{-k}$ with $k = 3, 6, 9, 12, 15$. First, we have applied three iterations by (1.3) with the initial complex approximations

$$z_1^{(0)} = 1.2 + 2.2\mathrm{i},\ z_2^{(0)} = 0.8 - 2.2\mathrm{i},\ z_3^{(0)} = -1.2 - 0.1\mathrm{i},$$
$$z_4^{(0)} = 2.8 + 0.1\mathrm{i},\ z_5^{(0)} = 0.2 + 4.9\mathrm{i}$$

to the polynomial whose coefficients are equal to the centers of disk-coefficients. Clearly, for all five values of δ we have obtained the same complex approximations given below.

$$z_1^{(3)} = 1.000000006292 + \mathrm{i}\ 2.000000011752,$$
$$z_2^{(3)} = 0.999999990316 - \mathrm{i}\ 2.000000013267,$$
$$z_3^{(3)} = -1.000000004366 + \mathrm{i}1.24 \times 10^{-8},$$
$$z_4^{(3)} = 3.000000005817 - \mathrm{i}\ 1.25 \times 10^{-8},$$
$$z_5^{(3)} = 1.94 \times 10^{-9} + \mathrm{i}\ 5.00000000158.$$

According to (4.9) we have constructed the inclusion disks $Z_i(\delta) = \{z_i^{(3)}; r_i(\delta)\}$ with the radii $r_i(\delta)$ given in Table 1.

Table 1.

δ	$r_1(\delta)$	$r_2(\delta)$	$r_3(\delta)$	$r_4(\delta)$	$r_5(\delta)$
10^{-15}	1.67×10^{-8}	2.05×10^{-8}	1.65×10^{-8}	1.72×10^{-8}	3.13×10^{-9}
10^{-12}	1.67×10^{-8}	2.05×10^{-8}	1.65×10^{-8}	1.72×10^{-8}	3.13×10^{-9}
10^{-8}	2.91×10^{-8}	2.61×10^{-8}	1.69×10^{-8}	4.16×10^{-8}	7.66×10^{-8}
10^{-6}	1.26×10^{-6}	5.75×10^{-7}	6.24×10^{-8}	2.46×10^{-6}	7.35×10^{-6}
10^{-3}	1.24×10^{-3}	5.54×10^{-4}	4.59×10^{-5}	2.44×10^{-3}	7.34×10^{-3}

Finally, applying only one step of the interval formula (4.10) we have obtained the disks $Z_i^{(3,1)}(\delta) = \{z_i^{(3,1)}; R_i(\delta)\}$ which contain the zeros of the polynomial P with the interval coefficients of the radius δ. The upper index "(3,1)" indicates that these disks are obtained after 3 point iterations and 1 interval iteration. For a fixed i the centers $z_i^{(3,1)}$ of all disks were the same for various δ and given by

$$z_1^{(3,1)} = 1.0000000000000000 + \mathrm{i}\,2.0000000000000000,$$
$$z_2^{(3,1)} = 0.9999999999999998 - \mathrm{i}\,2.0000000000000000,$$
$$z_3^{(3,1)} = -1.000000000000000 + \mathrm{i}\,1.1 \times 10^{-17},$$
$$z_4^{(3,1)} = 3.0000000000000000 - \mathrm{i}\,1.3 \times 10^{-17},$$
$$z_5^{(3,1)} = 5.2 \times 10^{-18} + \mathrm{i}\,5.0000000000000000.$$

The radius $R_i(\delta)$ of the disks $Z_i^{(3,1)}(\delta)$ are given in Table 2.

Table 2.

δ	$R_1(\delta)$	$R_2(\delta)$	$R_3(\delta)$	$R_4(\delta)$	$R_4(\delta)$
10^{-15}	1.23×10^{-15}	7.15×10^{-16}	2.75×10^{-16}	2.20×10^{-15}	5.91×10^{-15}
10^{-12}	9.92×10^{-13}	4.44×10^{-13}	3.70×10^{-14}	1.95×10^{-12}	5.87×10^{-12}
10^{-8}	9.91×10^{-9}	4.43×10^{-9}	3.68×10^{-10}	1.95×10^{-8}	5.87×10^{-8}
10^{-6}	9.91×10^{-7}	4.43×10^{-7}	3.68×10^{-8}	1.95×10^{-6}	5.87×10^{-6}
10^{-3}	9.94×10^{-4}	4.44×10^{-4}	3.69×10^{-5}	1.95×10^{-3}	5.88×10^{-3}

5. The zeros of analytic functions

Let D be a given closed region in the complex plane with the simple smooth contour Γ and the interior $\operatorname{int}\Gamma$. Let $z \mapsto f(z)$ be an analytic function which has exactly n simple zeros $\zeta_1, \ldots, \zeta_n$ inside D. Then following Smirnov [26] f can be represented as

$$f(z) = \exp\left(y(z)\right) \prod_{j=1}^{n} (z - \zeta_j) \quad (z \in \operatorname{int}\Gamma), \tag{5.1}$$

where $z \mapsto y(z)$ is an analytic function in $\operatorname{int}\Gamma$ such that $x(z) := \exp\left(y(z)\right) \neq 0$ for all $z \in \operatorname{int}\Gamma$.

In this section we will consider Weierstrass' method in ordinary complex arithmetic for the determination of all zeros inside a given region D of an analytic function of the form (5.1). We note that the number of zeros n of f that belong to $\operatorname{int}\Gamma$ can be determined by the *argument principle*. Furthermore, the analytic function y involved in (5.1) is given by

$$y(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\log[(w-c)^{-n} f(w)]}{w - z} \, dw, \tag{5.2}$$

where c is an arbitrary point inside Γ such that $f(c) \neq 0$ (see [2]). Methods for finding zeros of analytic functions belonging to this class have been considered in the papers [13], [21], [22] and [23]. As it was advised in [13], the contour integral (5.2) should be computed with satisfactory effect using trapezoidal quadrature rule. Computational aspect of the calculation of the value $y(z_i)$ at $z = z_i$ and the determination of the number of zeros n were studied in details in the papers [13], [22], [23] so that we will not discuss these points here.

Starting from (5.1) we find

$$\zeta_i = z - \frac{f(z)}{x(z) \prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z - \zeta_j)} \quad (i = 1, \ldots, n). \tag{5.3}$$

Substituting the exact zeros $\zeta_1, \ldots, \zeta_n$ on the right-hand side of (5.3) by their approximations $z_1, \ldots, z_n$ and taking $z = z_i$, we will obtain an approximation of ζ_i, say $\hat{z}_i$,

$$\hat{z}_i = z_i - \frac{f(z_i)}{x(z_i) \prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)} \quad (i = 1, \ldots, n), \tag{5.4}$$

with

$$x(z_i) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\log[(w-c)^{-n} f(w)]}{w - z_i} \, dw \right). \tag{5.5}$$

Formula (5.4) evidently resembles Weierstrass' formula (1.2). In practice, the contour integral involved in (5.5) should be computed by numerical integration, for example, by the trapezoidal quadrature rule. Assuming that we have found initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ to the zeros $\zeta_1, \dots, \zeta_n$ of f (for the zero searching procedure see, e.g. [7] [15]), from (5.4) we can establish

Algorithm (W): *For each* $m = 0, 1, \dots$ *let for* $i = 1, \dots, n$

$$w_i^{(m)} = \frac{f(z_i^{(m)})}{x(z_i^{(m)}) \prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i^{(m)} - z_j^{(m)})} \quad (i = 1, \dots, n), \tag{5.6}$$

$$z_i^{(m+1)} = z_i^{(m)} - w_i^{(m)}, \tag{5.7}$$

supposing that all approximations $z_1^{(m)}, \dots, z_n^{(m)}$ *belong to* $\operatorname{int} \Gamma$.

Theorem 5.1. *If the initial approximations* $z_1^{(0)}, \dots, z_n^{(0)}$ *are reasonably close to the zeros* $\zeta_1, \dots, \zeta_n$ *of* f*, then the iterative method (W) has a quadratic convergence.*

The proof of Theorem 5.1 will be given in the following as a consequence of the analysis of numerical stability of the iterative method (**W**) concerning the error of numerical integration in calculation of $y(z_i)$.

For simplicity we omit the iteration index and write z_i and $\hat{z}_i$ instead of $z_i^{(m)}$ and $z^{(m+1)}$. Let $\alpha = O_M(\beta)$ mean that $|\alpha| = O(|\beta|)$ (the same order of moduli), α and β being real or complex numbers, where O is Landau's symbol. Furthermore, let

$$\hat{\epsilon}_i := \hat{z}_i - \zeta_\mathrm{i}, \quad \epsilon_i = z_i - \zeta_\mathrm{i}, \quad |\epsilon| = \max_\mathrm{i} |\epsilon_i|, \quad \rho = \max_\mathrm{i} \rho_\mathrm{i},$$

where ρ_i is the upper error bound obtained in calculation of $y(z)$ given by (5.2) at the point $z = z_i$. As recommended by Henrici [12], to control the error of calculation (in our case, the error of numerical integration) it is desirable to deal with a small disk $Z = \{c; r\}$ instead of a (approximate, uncertain) complex value c. Using the centered form of the exponential complex interval function (see [18]) we introduce the disk

$$X_i = \mathrm{e}^{\{y_i;\rho_i\}} := \left\{\mathrm{e}^{y_i}; |\mathrm{e}^{y_i}|(\mathrm{e}^{\rho_i} - 1)\right\}.$$

where $y_i = y(z_i)$ is exactly the value for which $f(z_i) = \exp(y_i) \prod_{j=1}^{n} (z_i - \zeta_j)$. This is possibly to achieve by the fitting, increasing slightly the radius ρ_i. Assuming that the error of numerical integration is reasonably small, we have

$$\mathrm{e}^{\rho_i} - 1 = (1 + \rho_i + \frac{\rho_i^2}{2} + \cdots) - 1 \cong \rho_i,$$

so that

$$X_i \cong \{\mathrm{e}^{y_i}; |\mathrm{e}^{y_i}|\rho_i\} = \mathrm{e}^{y_i}\{1; \rho_i\}.$$

Following the technique for the analysis of numerical stability by circular arithmetic, presented in [17], we start from (5.7) and obtain

$$\hat{\epsilon}_i = \hat{z}_i - \zeta_i = z_i - \zeta_i - \frac{f(z_i)}{x(z_i)\prod_{j\neq i}(z_i - z_j)} \in E_i := \epsilon_i - \frac{f(z_i)}{X_i \prod_{j\neq i}(z_i - z_j)}$$

$$= \epsilon_i - \frac{\epsilon_i \prod_{j\neq i}(z_i - \zeta_j)}{\{1;\rho_i\}\prod_{j\neq i}(z_i - z_j)} = \epsilon_i\Big(1 - \frac{\{1;\rho_i\}}{1-\rho_i^2}\big(1 + \sum_{j\neq i}\frac{\epsilon_j}{z_i - z_j} + O_M(\epsilon^2)\big)\Big).$$

Hence, using the properties of circular arithmetic,

$$\operatorname{rad} E_i = |\epsilon_i| O(\rho_i), \quad \operatorname{mid} E_i = \frac{\epsilon_i}{1-\rho_i^2}\Big(-\rho_i^2 - \sum_{j\neq i}\frac{\epsilon_i}{z_i - z_j} - O_M(\epsilon^2)\Big).$$

It is possible to find some positive constants $\alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i}$ and $\beta_{1,i}$ so that

$$\operatorname{rad} E_i \leq \alpha_{1,i}|\epsilon_i|\rho_i, \quad |\operatorname{mid} E_i| \leq |\epsilon_i|(\alpha_{2,i}\rho_i^2 + \alpha_{3,i}|\epsilon|),$$

whence

$$|\hat{\epsilon}_i| \leq |\operatorname{mid} E_i| + \operatorname{rad} E_i \leq |\epsilon_i|(\alpha_{2,i}\rho_i^2 + \alpha_{3,i}\epsilon) + \alpha_{1,i}|\epsilon_i|\rho_i = |\epsilon_i|(\alpha_{1,i}\rho_i + \beta_{1,i}|\epsilon|).$$

Let $\epsilon_i^{(m)} = z_i^{(m)} - \zeta_i$ and $|\epsilon^{(m)}| = \max_i |\epsilon_i^{(m)}|$. From the last relation we can conclude the following:

(i) Theoretically, if $\rho_i^{(m)} = 0$, then we have

$$|\epsilon_i^{(m+1)}| \leq |\epsilon^{(m+1)}| = O(|\epsilon^{(m)}|^2),$$

which means that Algorithm (**W**) has a *quadratic convergence*. In this way we have proved Theorem 5.1.

(ii) If in each iteration step $m = 0, 1, \ldots$ the errors of numerical integration $\rho_i^{(m)}$ are at least of the same order as $|\epsilon^{(m)}|$, then Weierstrass-like method (5.7) *preserves* a quadratic convergence. This fact points to good numerical stability of this method in the presence of the error of numerical integration involved in the iterative formula (5.7). Numerical results of Example 3 shown in Table 3 confirm this assertion.

(iii) In the cases when the errors $\rho_i^{(m)}$ are larger in size compared to $|\epsilon^{(m)}|$, that is $\rho_i^{(m)} = O(|\epsilon^{(m)}|^k)$ $(0 \leq k < 1)$, then the convergence of Algorithm (**W**) is *superlinear* or even only *linear*.

From (ii) we see that the requirement for preserving quadratic convergence of Algorithm (**W**) needs the increase of the accuracy of numerical integration as the number of iteration steps grows. Practical examples show that, if the error of numerical integration is not reasonable large, then its influence is small, especially in later iterations. Moreover, due to a simple structure of Algorithm (**W**), this influence is smaller compared to some other algorithms of higher order (see [22], [23]).

In connection with the above comments we present the following example for the sake of demonstration.

Example 3. Let us consider the analytic function

$$f(z) = \exp(z) - 2\cos(3z) - 2$$

inside the disk $D = \{z : |z| \leq 1.5\}$. The number of zeros of f was found by the computable argument principle (see [11]) calculating the variation of the argument stepwise along the polygon with vertices $V_1, ..., V_M, V_{M+1} = V_1$ belonging to the contour $\Gamma = \{z : |z| = 1.5\}$. The real numbers $z_1^{(0)} = -1.5, z_2^{(0)} = -0.5, z_3^{(0)} = 0.8$ were taken as starting approximations. For the simulation, we have calculated $y_i^{(m)} = y(z_i^{(m)})$ ($i = 1, 2, 3$; $m = 0, 1, ...$) to very high accuracy (more than 33 significant digits) and then we have incorporated artificially "parasite" errors of the form 10^{-k} ($k = 2, 8, 16$). In other words, we have taken $y_i^{(m)} \pm 10^{-k}$ instead of (almost) exact values $y_i^{(m)}$. The maximal errors $|\epsilon^{(m)}|$ of approximations to the zeros for $m = 3(1)7$ and for the simulated errors of numerical integration $\rho_1 = 10^{-2}, \rho_2 = 10^{-8}, \rho_3 = 10^{-16}$ and $\rho_4 = 10^{-33}$ are given in Table 3. Actually, ρ_4 is the assumed maximal accuracy of the employed arithmetic used for the model "the absence of error of numerical integration".

Table 3.

	$\lvert\epsilon^{(3)}\rvert$	$\lvert\epsilon^{(4)}\rvert$	$\lvert\epsilon^{(5)}\rvert$	$\lvert\epsilon^{(6)}\rvert$	$\lvert\epsilon^{(7)}\rvert$
$\rho_1 = 10^{-2}$	9.56×10^{-4}	4.25×10^{-6}	9.13×10^{-9}	1.95×10^{-11}	4.17×10^{-14}
$\rho_2 = 10^{-8}$	8.85×10^{-4}	1.90×10^{-6}	8.89×10^{-12}	1.95×10^{-20}	4.12×10^{-29}
$\rho_3 = 10^{-16}$	8.85×10^{-4}	1.90×10^{-6}	8.89×10^{-12}	1.94×10^{-22}	1.00×10^{-33}
$\rho_4 = 10^{-33}$	8.85×10^{-4}	1.90×10^{-6}	8.89×10^{-12}	1.94×10^{-22}	1.00×10^{-33}

It is evident from Table 3 that the iterative method (5.7) produces better results when the error of numerical integration is smaller, especially in latter iterations. We see that a crude error 10^{-2} does not permit quadratic convergence of the iterative method (5.7). If this type of error is smaller then the convergence is of the second order (although it is not the case in the first iterations because of crude initial approximations). In the presence of the errors smaller than 10^{-16} the accuracy of the generated approximations is limited due to the finite precision of the applied floating-point arithmetic. We can also observe that the accuracy of the produced approximations, expressed by the values $|\epsilon^{(m)}|$, for $\rho_2 = 10^{-8}, \rho_3 = 10^{-16}$ and $\rho_4 = 10^{-33}$ is almost the same (except in the seventh iteration) which points that the influence of the error of numerical integration is relatively small. Such conclusion can be also drawn according to the results of the first four iterations; the obtained approximations are almost of the same accuracy for all cases.

The presented example coincides very well with the results of the analysis of numerical stability presented previously. The same results have been also obtained in a real case when we performed the numerical integration to calculate $y_i^{(m)}$.

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