ZAMM · Z. Angew. Math. Mech. **82** (2002) 4, 284–288

Carstensen, C.; Rynne, B. P.

A Posteriori Error Control for Finite Element Approximations of the Integral Equation for Thin Wire Antennas

In this paper we discuss a finite element approximation method for solving the Pocklington integro-differential equation for the current induced on a straight, thin wire by an incident harmonic electromagnetic field. We obtain an a posteriori error estimate for finite element approximations of the equation, and we prove the reliability of this estimate. The theoretical results are then used to motivate an adaptive mesh-refining algorithm which generates very efficient meshes and yields optimal convergence rates in numerical experiments.

Key words and phrases: Electromagnetic scattering, thin wire antennas, finite element approximations, a posteriori error estimate.

MSC (2000): 45J05, 45L05, 49M25

1. Introduction

The problem of electromagnetic scattering from a thin wire antenna by an incident harmonic electromagnetic field is extremely important in electrical engineering. A standard model for this problem is the Pocklington integro-differential equation for the current induced on the wire by the incident electromagnetic field. Once the induced current is known the scattered field can then easily be obtained from this current by means of a standard integral representation. Suppose that the wire, modeled as a thin, perfectly conducting, hollow tube lies along the z-axis in the interval $U = (-1, 1)$. Let the radius of the wire be $a > 0$ and the wave number of the incident field be $k > 0$. Then the current u satisfies the Pocklington equation

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_U u(\zeta)\, G(z-\zeta)\, d\zeta = f(z)\,, \tag{1.1}$$

on U and the boundary conditions $u(\pm 1) = 0$. Here $G\colon \mathbb{R} \to \mathbb{C}$ is defined, for $z \in \mathbb{R}$, by

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp\{-ik(z^2 + 4a^2 \sin^2 \frac{1}{2}\theta)^{1/2}\}}{(z^2 + 4a^2 \sin^2 \frac{1}{2}\theta)^{1/2}}\, d\theta$$

and $f\colon U \to \mathbb{C}$ is a known function, being a constant multiple of the z-component of the incident electric field on the z-axis, see [9].

Many numerical methods of solving Pocklington's equation have been considered before, especially in the electrical engineering literature (see [2] for an extensive discussion and bibliography of such methods). Generally these have been Galerkin type methods (called the 'method-of-moments' in the electrical engineering literature). However, few rigorous convergence results have been proved for these methods (see [13] for some results). In this paper we establish a residual-based a posteriori error estimate for a finite element approximation scheme. The estimate motivates a particular adaptive algorithm for automatic mesh-refinement which performed efficiently in numerical experiments, see [6] (numerical results for many algorithms are described and compared in [6] so, for brevity, we simply refer to [6] for numerical results).

The operator Pu on the left-hand side of (1.1) has similar mapping properties to those of a hypersingular integral operator [14–16]. We refer to the survey [5] for references to the literature regarding adaptive algorithms and a posteriori error control for hypersingular equations which is mainly concerned with closed curves; open antennas are not boundaries of Lipschitz domains and so require mesh-adaptation, a different functional analytical setting, and the efficiency of the a posteriori error estimate remains an open question since the arguments of [3] are not applicable.

The paper is organized as follows. The mapping properties of the operators involved in (1.1) in suitable Sobolev spaces are recalled from recent literature in Section 2. Existence and regularity of weak solutions is explained in Section 3, while we report in Section 4 on the finite element discretization and a priori error estimates for non-uniform meshes. The a posteriori error estimate is proved to be reliable in Section 5 and a resulting mesh-refining algorithm is described.

2. Preliminaries

We introduce some function spaces. For $s \geq 0$, let $H^s_{2\pi}$ denote the fractional order (Hilbert) Sobolev space of functions on $\mathbb{R}$ which are 2π-periodic, with norm $\|\cdot\|_{H^s_{2\pi}}$, see Definition 8.1 of [10] for the specific definition that we use, and

 0044-2267/02/0404-0284 $ 17.50+.50/0

Chapter 8 of [10] for a general discussion of these spaces (these spaces are also discussed in Chapter I of [17], see in particular, Exercise 4.4). We use the spaces $H^s_{2\pi}$ in this situation since they are particularly simple to define, in terms of Fourier series rather than Fourier transforms, and are sufficient for our present 1-dimensional setting – the relationship between the 2π-periodic Sobolev spaces and the usual Sobolev spaces is discussed in [17]. We also let $H^{-s}_{2\pi}$ denote the dual space, $(H^s_{2\pi})'$, of $H^s_{2\pi}$, see Definition 8.8 in [10].

We will also require spaces of functions, or distributions, defined on U. For any 2π-periodic function, or distribution, T we let $T|_U$ denote its restriction to the set U (see Section 1.4 of [17] for the definition of the restriction of a distribution); we let $\operatorname{supp} T$ denote the support of T in the interval $[-\pi,\pi]$ (see Definition 1.3 of [17] for the definition of the support of a distribution). Now, for $s\in\mathbb{R}$, we define the following spaces and norms:

$$\begin{aligned} H^s(u) &= \{u : u = u^*|_U \text{ with } u^*\in H^s_{2\pi}\}\,,\\ \|u\|_{H^s(U)} &= \inf\{\|u^*\|_{H^s_{2\pi}} : u^*\in H^s_{2\pi} \text{ and } u^*|_U = u\}\,,\\ \tilde{H}^s(U) &= \{u : u = \tilde{u}|_U \text{ with } \tilde{u}\in H^s_{2\pi} \text{ and } \operatorname{supp}\tilde{u}\subset \bar{U}\}\,,\\ \|u\|_{\tilde{H}^s(U)} &= \|\tilde{u}\|_{H^s_{2\pi}}\,; \end{aligned}$$

for any $u\in\tilde{H}^s(U)$, we let $\tilde{u}\in H^s_{2\pi}$ denote the extension, by zero, of the function u to $[-\pi,\pi]$, and then to $\mathbb{R}$ by 2π-periodicity.

It can be shown that, for $s>0$, $H^{-s}(U)=\tilde{H}^s(U)'$ (in fact, $H^{-s}(U)$ can be defined to be $\tilde{H}^s(U)'$). Let $\langle\cdot,\cdot\rangle$ denote the usual $L^2(U)$ inner product, or its extension to the dual pair of Sobolev spaces $H^{-1/2}(U)\times\tilde{H}^{1/2}(U)$.

The space $\tilde{H}^s(U)$, with $s>0$, can also be defined by interpolation, see (1.9) in [15], but the above definition is equivalent to the interpolation definition, see Lemma 1.1 of [15]. We also note that $\tilde{H}^s(U)=H^s(U)$, when $0<s<\frac{1}{2}$, $\tilde{H}^{1/2}(U)=H^{1/2}_{00}(U)\subset H^{1/2}_0(U)$, and $\tilde{H}(U)=H^s_0(U)$, when $\frac{1}{2}<s<1$ (where these spaces are defined in [11] and these results are proved there, see Theorems 11.4 and 11.7).

When $\frac{1}{2}<s<1$ the elements of $\tilde{H}^s(U)$ are continuous, so they satisfy the boundary condition $u(\pm1)=0$ in the classical sense, whereas the elements of $\tilde{H}^{1/2}(U)$ satisfy $u(\pm1)=0$ in a generalized sense.

We now define various linear operators on $H^s(U)$ and $\tilde{H}^s(U)$. Firstly, for any $s\in\mathbb{R}$ and $u\in H^s_{2\pi}$, let Du denote the distributional derivative of u; if $u\in H^s(U)$, with $u=u^*|_U$, $u^*\in H^s_{2\pi}$, then we defined $Du=(Du^*)|_U$ (this definition is independent of the extension u^* of u). Next, it can be shown that the kernel G may be written in the form

$$G(z) = -\frac{1}{a\pi}\log|z| + R(z)\,,\qquad z\in\mathbb{R}\,, \tag{2.1}$$

where the function $R\in C^\infty(\mathbb{R}\setminus\{0\})$, and, near $z=0$,

$$|R(z)| + \left|\frac{\partial R}{\partial z}\,(z)\right| \le c\,,\qquad \left|\frac{\partial^2 R}{\partial z^2}\,(z)\right| \le |\log|z||\,, \tag{2.2}$$

for some constant $c>0$, see Lemma 4.1 in [12]. Now, for any $u\in C^\infty_0(U)$ and $z\in U$, let

$$Vu(z) = -\frac{1}{\pi}\int_U u(\zeta)\log|z-\zeta|\,d\zeta \quad\text{and}\quad Lu(z) = \int_U u(\zeta)\,R(z-\zeta)\,d\zeta\,.$$

It is well-known that for each $s\in\mathbb{R}$ the operator V can be extended to the space $\tilde{H}^s(U)$ to yield a bounded operator $V:\tilde{H}^s(U)\to H^{s+1}(U)$, while it follows from the smoothness properties (2.2) of the function R that L can be extended to a bounded operator $L:\tilde{H}^s(U)\to H^{s+3}(U)$ (see, for example, the discussion on pp. 368–371 of [16]). In view of these results we can define a bounded operator $P:\tilde{H}^s(U)\to H^{s-1}(U)$ by

$$Pu = (D^2+k^2)\,(a^{-1}V+L)\,u\,,\qquad u\in\tilde{H}^s(U)\,.$$

The operator P represents the left hand side of eq. (1.1), together with the boundary conditions $u(\pm1)=0$, in the setting of the space $\tilde{H}^s(U)$. Then, given $f\in H^{s-1}(U)$, the problem can be rewritten as the equation

$$Pu = f\,. \tag{2.3}$$

3. Existence and regularity results

It has recently been shown that P is non-singular and satisfies a Gårding inequality.

Theorem 3.1 (Theorem 2.1 in [13])**:** *For $0<s<1$ the operator $P:\tilde{H}^s(U)\to H^{s-1}(U)$ is non-singular. Furthermore, when $s=\frac{1}{2}$ there is a compact operator $C:\tilde{H}^{1/2}(U)\to H^{-1/2}(U)$ and a constant $\gamma>0$ such that, for all $u\in\tilde{H}^{1/2}(U)$,*

$$-\operatorname{Re}\langle(P+C)\,u,u\rangle \ge \gamma\|u\|^2_{\tilde{H}^{1/2}(U)}\,. \tag{3.1}$$

Solutions of (2.3) typically have a square root type singular behavior at the end points ± 1 which precludes Theorem 3.1 from holding for $s \geq 1$. Despite this, higher regularity results can be obtrained by augmenting the usual Sobolev spaces with functions which describe this singular behavior.

We define the following functions: let $\varrho_\pm(z) = |z \mp 1|$, for $z \in U$, and let $\chi_\pm$ be C^∞ functions on $\mathbb{R}$ such that $0 \leq \chi_\pm \leq 1$, $\chi_\pm \equiv 1$ near ± 1 and $\chi_\pm \equiv 0$ near ∓ 1. For each $s \in [1, 2)$ we define the space $\mathscr{Z}^s(U)$, consisting of functions $w : \bar{U} \to \mathbb{C}$ of the form

$$w = \alpha_- \varrho_-^{1/2} \chi_- + \alpha_+ \varrho_+^{1/2} \chi_+ + v\,, \tag{3.2}$$

where $\alpha_\pm \in \mathbb{R}$ and $v \in \tilde{H}^s(U)$ (Lemma A.2 in [15] shows that $\varrho_\pm^{1/2}\chi_\pm \in \tilde{H}^s(U)$ for $s < 1$). We can identify $\mathscr{Z}^s(U)$ with $\mathbb{R}^2 \times H^s(U)$, and we write elements of $\mathscr{Z}^s(U)$ in the form $\{\alpha_-, \alpha_+, v\}$. The norm on $\mathscr{Z}^s(U)$ is defined by

$$\|\{\alpha_-, \alpha_+, v\}\|_{\mathscr{Z}^s(U)} = |\alpha_-| + |\alpha_+| + \|v\|_{\tilde{H}^s(U)}\,.$$

The operator P acts on elements of $\mathscr{Z}^s(U)$ (see Theorem 1.8 in [16]), and solutions of (2.3) can be represented in the form (3.2).

Theorem 3.2 (Theorem 2.2 in [13])**:** *For $1 < s < 2$ the operator $P : \mathscr{Z}^s(U) \to H^{s-1}(U)$ is non-singular.*

4. Discretization and a priori error estimates

The results of Section 3 can now be applied to Galerkin methods for the numerical solution of eq. (2.3). The abstract formulation of general Galerkin methods, and the derivation of convergence results from Gårding inequalities, is well-known so the discussion will be brief and proofs can be omitted; we refer to [7], [8], or [14], for example, for more details.

To apply Galerkin methods to the problem we first reformulate eq. (2.3) in an equivalent form: given $f \in H^{-1/2}(U)$, find $u \in \tilde{H}^{1/2}(u)$ such that, for all $\xi \in \tilde{H}^{1/2}(U)$,

$$\langle Pu, \xi \rangle = \langle f, \xi \rangle\,. \tag{4.1}$$

Next, we define a mesh $\pi := \{U_1, \ldots, U_N\}$ to be a partition of the interval U into the subintervals $U_j = [x_{j-1}, x_j]$, $j = 1, \ldots, N$, where $-1 = x_0 < x_1 < \ldots < x_N = 1$ is an arbitrary collection of points in U. For each $j = 1, \ldots, N$, let h_j denote the length of U_j and let $h := \max\{h_1, \ldots, h_N\}$. Given a mesh π, let $S^1_\pi(U)$ be the space of continuous functions, with support in U, which are polynomials of degree one on each element of the mesh π.

Now, a Galerkin solution of (4.1) is a function $u_\pi \in S^1_\pi(U)$ such that, for all $\xi_\pi \in S^1_\pi(U)$,

$$\langle Pu_\pi, \xi_\pi \rangle = \langle f, \xi_\pi \rangle\,. \tag{4.2}$$

As a consequence of the non-singularity of the operator P and the Gårding inequality (3.1), Galerkin solutions u_π of (4.2) exist and convergence quasi-optimally to the exact solution u of (4.1) as $h \to 0$.

Theorem 4.1 (Theorem 3.1 in [13])**:** *If $f \in H^{-1/2}(U)$, then there exist constants $h_0 \in (0, 1)$, $C > 0$, such that for $h \in (0, h_0)$ the problem* (4.2) *has a unique solution $u_\pi \in S^1_\pi(U)$ and*

$$\|u - u_\pi\|_{\tilde{H}^{1/2}(U)} \leq C \inf_{\phi_\pi \in S^1_\pi(U)} \|u - \phi_\pi\|_{\tilde{H}^{1/2}(U)}\,, \tag{4.3}$$

where $u \in \tilde{H}^{1/2}(U)$ is the unique solution of (4.1).

It follows from Theorems 11.5–7 in [11] that $S^1_\pi(U) \subset \tilde{H}^s(U)$ for $s < 3/2$. The approximation properties of the spaces $S^1_\pi(U)$ are well-known and can be summarized as follows, see Ch. 4 of [1] or p. 677 of [7].

Theorem 4.2: *If $0 \leq t \leq s \leq 2$, $t < 3/2$, then there exists a constant $C = C(s, t) > 0$ such that, for $h > 0$ and any $v \in \tilde{H}^s(U)$,*

$$\inf_{\phi_\pi \in S^1_\pi(U)} \|v - \phi\|_{\tilde{H}^t(U)} \leq C h^{s-t} \|v\|_{\tilde{H}^s(U)}\,. \tag{4.4}$$

Convergence properties of the scheme can now be derived from Theorems 4.1 and 4.2 (note that by Theorem 3.1, if $f \in L^2(U)$ then the solution $u \in \tilde{H}^{1-\epsilon}(U)$ for all $\epsilon > 0$, and $\|u\|_{\tilde{H}^{1-\epsilon}(U)} \leq \gamma_\epsilon \|f\|_{L^2(U)}$, for some constant $\gamma_\epsilon > 0$).

Theorem 4.3 (Theorem 3.3 in [13])**:** *If $f \in L^2(U)$, then for each $\epsilon > 0$ there exist constants $h_1 \in (0, h_0)$, $C_\epsilon > 0$ such that, for $h \in (0, h_1)$,*

$$\|u - u_\pi\|_{\tilde{H}^{1/2}(U)} \leq C_\epsilon h^{1/2-\epsilon} \|f\|_{L^2(U)} \tag{4.5}$$

(u_π exists by Theorem 4.1*).*

The rate of convergence of the Galerkin scheme can be improved by augmenting the approximation spaces $S^1_\pi(U)$ with the singular functions $\varrho_\pm^{1/2}\chi_\pm$ of Section 3, that is, by defining the space

$$\mathscr{Z}^1_\pi(u) = \{\varrho_-^{1/2}\chi_-, \varrho_+^{1/2}\chi_+\} \oplus S^1_\pi(U)\,.$$

It follows from Theorems 11.5–7 in [11] that $\mathscr{Z}^1_\pi(U) \subset \mathscr{Z}^s(U)$, for $s < 3/2$. Theorem 4.1 also holds for the spaces $\mathscr{Z}^1_\pi(U)$ and we have the following analogue of Theorem 4.2, see Lemma 4.1 in [7].

Theorem 4.4: *If $1 < s < 2$, then there exists a constant $C = C(s) > 0$ such that, for each $h > 0$ and any $w \in \mathscr{Z}^s(U)$,*

$$\inf_{\phi_\pi \in \mathscr{Z}^1_\pi(U)} \|w - \phi_\pi\|_{\tilde{H}^{1/2}(U)} \le Ch^{s-1/2}\|w\|_{\mathscr{Z}^s(U)}\,. \tag{4.6}$$

The convergence properties of the augmented Galerkin scheme now follow from Theorems 4.1 and 4.4, see Theorems 4.4, 4.5 in [7] for details of the proof.

Theorem 4.5 (Theorem 3.5 in [13])**:** *If $1 < s < 2$, $1 \le t \le 3/2$, $t \le s$, and $f \in H^{s-1}(U)$, then there exist constants $h_2 \in (0, h_0)$, $C > 0$ such that, for $h \in (0, h_2)$,*

$$\|u - u_\pi\|_{\tilde{H}^{1/2}(U)} \le Ch^{s-1/2}\|f\|_{H^{s-1}(U)} \quad \textit{and} \quad \|u - u_\pi\|_{\mathscr{Z}^t(U)} \le Ch^{s-t}\|f\|_{H^{s-1}(U)}\,. \tag{4.7}$$

5. A posteriori error estimate

In this section a residual-based a posteriori error estimate is derived for arbitrary meshes. Given a mesh π, define $\kappa := \max\{h_j/h_k : |j-k| = 1\}$.

Theorem 5.1 (Theorem 2 in [4]): *For $0 \le \alpha \le 1$, there exists a constant C_α such that for any $\phi \in L^2(U)$ which is $L^2(U)$-orthogonal to $S^1_\pi(U)$ we have*

$$\|\phi\|_{H^{-\alpha}(U)} \le c(\alpha,\pi)\left(\sum_{j=1}^N h_j^{2\alpha}\|\phi\|^2_{L^2(U_j)}\right)^{1/2}, \tag{5.1}$$

where $c(\alpha,\pi)$ is given by

$$c(\alpha,\pi) := \begin{cases} C_\alpha & \textit{if} \quad \alpha \ne \frac12\,, \\ C_{1/2}(\log(1+\kappa))^{1/2} & \textit{if} \quad \alpha = \frac12\,. \end{cases} \tag{5.2}$$

Now suppose that $f \in L^2(U)$ and $h \in (0, h_1)$, so that Theorem 4.3 holds, and let R_π be the residual of the Galerkin solution u_π, that is,

$$R_\pi := f - Pu_\pi = P(u - u_\pi)\,, \tag{5.3}$$

where u and u_π are the solutions of (4.1) and (4.2), respectively. We note that $R_\pi \in L^2(U)$ because $f \in L^2(U)$ and $\{0, 0, u_\pi\} \in \mathscr{Z}^t(U)$ for $1 < t < 3/2$, whence $Pu_\pi \in H^{t-1}(U) \subset L^2(U)$ by Theorem 3.2. Thus, for $0 < s < 1$ we may define the error indicators

$$\eta_j := h_j^{1-s}\|R_\pi\|_{L^2(U_j)}\,, \qquad j = 1, \dots, N\,. \tag{5.4}$$

As a consequence of the Galerkin conditions and Theorem 5.1 we obtain the following a posteriori error estimate for these error indicators (note that, by Theorem 3.1, the inverse operator $P^{-1} : H^{s-1}(U) \to \tilde{H}^s(U)$ is bounded, that is, the operator norm $\|P^{-1}\|_{L(H^{s-1}(U);\tilde{H}^s(U))}$ is bounded).

Theorem 5.2: *If $0 < s < 1$ and $f \in L^2(U)$ then*

$$\|u - u_\pi\|_{\tilde{H}^s(U)} \le c(1-s,\pi)\,\|P^{-1}\|_{L(H^{s-1}(U);\tilde{H}^s(U))}\left(\sum_{j=1}^N \eta_j^2\right)^{1/2}. \tag{5.5}$$

Proof: The essential observation is that the residual R_π is $L^2(U)$-orthogonal to $S^1_\pi(U)$ because of (4.1)–(4.2). Indeed, for all $\xi_\pi \in S^1_\pi(U)$,

$$\langle R_\pi, \xi_\pi\rangle = \langle f - Pu_\pi, \xi_\pi\rangle = 0\,. \tag{5.6}$$

Therefore, Theorem 5.1 shows that

$$\|R_\pi\|_{H^{s-1}(U)} \le c(1-s,\pi)\left(\sum_{j=1}^N \eta_j^2\right)^{1/2}. \tag{5.7}$$

Now, $u - u_\pi = P^{-1} R_\pi$ so

$$\|u - u_\pi\|_{\tilde{H}^s(U)} = \|P^{-1} R_\pi\|_{\tilde{H}^s(U)} \leq \|P^{-1}\|_{L(H^{s-1}(U);\, \tilde{H}^s(U))}\, \|R_\pi\|_{H^{s-1}(U)}\,. \tag{5.8}$$

Combining (5.7)−(5.8) concludes the proof. □

The question of the sharpness of the estimate (5.5) is clarified only for quasi-uniform meshes on closed curves, see [3] for details. The question is unresolved for open arcs such as U.

We conclude this paper by describing and adaptive algorithm for automatic mesh-refinement.

Algorithm (A). Given a coarse mesh, refine it successively by halving some of its elements as follows: For any mesh π compute $\eta, \ldots, \eta_N$ as defined in (5.4) and refine U_j if and only if

$$\eta_j \geq \theta \max_{k=1,\ldots,N} \eta_k\,. \tag{5.9}$$

The parameter θ is chosen between 0 (uniform refinement; every element is refined) and 1 (very selective refinement; only very few elements are refined); $\theta = \frac{1}{2}$ is expected to generate reasonable meshes.

If $s = \frac{1}{2}$, the Algorithm (A) ignores the mesh-dependence in $c(\frac{1}{2}, \pi)$ for simplicity. Since the dependence on κ is very weak, this is reasonable if we deal with a moderate number of refinement steps only.

The Algorithm (A) is implemented in [6, Section 2.6] and compared with various other schemes. The numerical evaluation of the error indicators η_j is somewhat complicated (it involves elliptic integrals) so is not described here; see [6, Section 3.5]. Fig. 8 in [6] shows that the experimentally observed convergence rate of the error, with respect to the L^2-norm rather than the $H^{1/2}$-norm, is 1 for uniform meshes and 2 for the adaptive meshes generated by Algorithm (A). These observations are in agreement with (4.5) (which, with the usual Aubin-Nitsche argument, yields a convergence rate of 1 for uniform meshes, with respect to the L^2-norm) and the convergence rate of 3/2 in the $H^{1/2}$-norm which is usually observed for adaptive algorithms. From this numerical evidence we conclude that the adaptive algorithm significantly improves the performance of the discretization.

Acknowledgement

We are grateful to the British Council and the German Academic Exchange Service (DAAD) for supporting this work through the ARC program.

References

1 Aziz, A. K. (ed.): The mathematical foundations of the finite element method with applications to partial differential equations. Academic Press 1972.
2 Burke, G. J.: Numerical Electromagnetics Code (NEC-4) − Method of moments, Part II − Theory, UCRL-MA-109338, Pt. II. Lawrence Livermore National Laboratory, CA 1992.
3 Carstensen, C.: Efficiency of a posteriosi BEM error estimates for first kind integral equations on quasiuniform meshes. Math. Comp. **65** (1996), 69−84.
4 Carstensen, C.: An a posteriori error estimate for a first kind integral equation. Math. Comp. **66** (1997), 139−155.
5 Carstensen, C.; Faermann, B.: Mathematical foundation of adaptive mesh-refining algorithms for boundary integral equations of the first kind. Eng. Analysis Boundary Elem. **25** (2001), 497−509.
6 Davies, P. J.; Duncan, D. B.; Funken, S. A.: Accurate and efficient algorithms for frequency domain scattering from a thin wire. J. Comp. Phys. **168** (2001), 155−183.
7 Feistauer, M.; Hsiao, G. C.; Kleinman, R. E.: Asymptotic and a posteriori error estimates for boundary element solutions of hypersingular integral equations. SIAM J. Numer. Anal. **33** (1996), 666−685.
8 Gohberg, I. C.; Feldman, J. A.: Convolution equations and projection methods for their solution. American Mathematical Society 1974.
9 Jones, D. S.: Note on the integral equation for a straight wire antenna. IEE Proc. **128** (1981), 114−116.
10 Kress, R.: Linear integral equations. Springer 1989.
11 Lions, J. L.; Magenes, E.: Non-homogeneous boundary value problems and applications. Vol. I. Springer 1972.
12 Rynne, B. P.: The well-posedness of the integral equations for thin wire antennas. IMA J. Appl. Math. **49** (1992), 35−44.
13 Rynne, B. P.: Convergence of Galerkin method solutions of the integral equation for thin wire antennas. Adv. Comp. Math. **12** (2000), 251−259.
14 Stephan, E.; Wendland, W. L.: Remarks to Galerkin and least squares methods with finite elements for general elliptic problems. Manuscripta Geod. **1** (1976), 93−123.
15 Wendland, W. L.; Stephan, E.; Hsiao, G. C.: On the integral equations method for the plane mixed boundary value problem of the Laplacian. Math. Meth. Appl. Sci. **1** (1979), 265−321.
16 Wendland, W. L.; Stephan, E.: A hypersingular boundary integral method for two dimensional screen and crack problems. Arch. Rat. Mech. Anal. **122** (1990), 363−390.
17 Wloka, J.: Partial differential equations. Cambridge University Press, Cambridge 1992.

Received February 11, 2000, revised March 2, 2001, accepted April 4, 2001

Addresses: Prof. Dr. Carsten Carstensen, Technische Universität Wien, Institut für Angewandte und Numerische Mathematik, Wiedner Hauptstr. 8−10/115, A-1040 Wien, Österreich; Dr. Bryan P. Rynne, Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland, e-mail: bryan@ma.hw.ac.uk