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A quasi-static boundary value problem in multi-surface elastoplasticity: Part 1—Analysis

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SUMMARY

The quasi-static evolution of an elastoplastic body with a multi-surface constitutive law of linear kinematic hardening type allows the modelling of curved stress–strain relations. It generalizes classical small-strain elastoplasticity from one to various plastic phases. This paper presents the mathematical models and proves existence and uniqueness of the solution of the corresponding initial-boundary value problem. The analysis involves an explicit estimate for the effective ellipticity constant. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: variational inequalities; elastoplasticity; kinematic hardening; rate independence; multi-surface model; Prandtl–Ishlinskii model

1. INTRODUCTION

Solid bodies undergo deformations when subjected to forces. Usually, the response is elastic for sufficiently small stresses. Moreover, many materials begin to exhibit plastic flow when the stresses reach a regime of critical values, given by the so-called yield surface. The precise relation between stress and strain is specified by the constitutive law. In mechanics, a large variety of constitutive laws has been developed in order to describe the elastoplastic behaviour of solid materials in a phenomenologically correct manner. We refer in particular to the books [1,2] and the surveys [3,4].

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Many of those models feature certain non-linearities in their defining equations, and the solvability of the corresponding initial-boundary value problems are open questions. On the other hand, several models can be transformed into a variational formulation for which a mathematical existence and uniqueness theory is known. This is true in particular for viscoplasticity, because the presence of viscosity regularizes the problem, and for models which attempt only to describe a purely increasing loading (like Hencky's model and its variants), because the situation becomes much simpler if one neglects the memory phenomena induced by cyclic loading. For elastoplastic (that is, rate-independent) models which encompass cyclic loading, in particular models of kinematic hardening, results are sparse. Basic results are due to Johnson [5].

In this two-part article we consider the quasi-static initial-boundary value problem for small strain elastoplasticity with a multi-surface constitutive law of linear kinematic hardening type. The main goal is the construction and error analysis of a discrete solution method which takes care of the multi-surface aspect of the constitutive law. This will be done in the second part. In the first part, we present the precise formulation of the initial-boundary value problem and prove existence and uniqueness of its solution.

Indeed, the existence of such solutions in the quasi-static case has been obtained by Visintin [6], chapter VII, Theorem 2.3, using the theory of variational inequalities. He proves first that the dynamic problem has a unique solution, and then considers the quasi-static case as a singular limit. Thus, he obtains existence with regularity $L^\infty(0,T;L^2(\Omega))$ for the strain $\varepsilon=(Du+Du^{\mathrm{T}})/2$ (the strain rates ε_t being Radon measures), but not uniqueness, the latter stated as an open problem in Reference [6]. Our approach differs from his in that we use the functional framework of Reference [7] which has been already used extensively for numerical approximation and analysis of problems in elastoplasticity [7,8]. For the case of a single yield surface, it is shown there how to obtain unique solvability of the quasi-static problem from a suitable variational inequality formulation, with regularity $H^1(0,T;H^1(\Omega))$ for the displacement u. We extend these results to the multi-surface case. In particular, we also derive an estimate for the ellipticity constant whose size is critical for the performance of numerical methods based on the variational formulation.

2. THE CONSTITUTIVE LAW

The constitutive law furnishes the relationship between the stress tensor σ and the strain tensor ε. The classical law of kinematic hardening goes back to Melan [9] and Prager [10]. It is local in the sense that any given material point x it involves only the time histories $\sigma=\sigma(t)$ and $\varepsilon=\varepsilon(t)$ at that point. It is given by the following system of equations and an evolution variational inequality:

$$\varepsilon = e + p \tag{1}$$

$$\sigma = \sigma^{\mathrm{b}} + \sigma^{\mathrm{p}}$$

$$\sigma = \mathbb{C}e \tag{2}$$

$$\sigma^{\mathrm{b}} = \mathbb{H}p \tag{3}$$

$$\sigma^{\mathrm{p}} \in Z, \quad \dot{p} : (\tau - \sigma^{\mathrm{p}}) \leqslant 0 \quad \text{for all } \tau \in Z \tag{4}$$

Equation (1) represents the additive decomposition of the strain ε into its elastic part e and its plastic part p as well as of the stress σ into the backstress σ^{b} and the plastic stress σ^{p}. Equation (2) denotes a linear elastic law, in the isotropic case one has

$$\mathbb{C}\varepsilon = 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)\mathbb{I} \tag{5}$$

where the (positive) coefficients μ and λ are called Lamé coefficients. Here $\mathbb{I}$ denotes the second order identity tensor (an identity matrix) and $\operatorname{tr}: \mathbb{R}^{d\times d} \to \mathbb{R}$ defines the trace of a matrix, $\operatorname{tr}\varepsilon := \sum_{j=1}^{d} \varepsilon_{jj}$, for $\varepsilon \in \mathbb{R}^{d\times d}$, where d is the problem dimension. Equation (3) couples the backstress σ^{b} and the plastic strain p through a linear mapping with a positive definite hardening matrix $\mathbb{H}$. For this reason, the model (1)–(4) is also called linear kinematic hardening. A typical choice will be $\mathbb{H} = h\mathbb{I}$, where $h > 0$ is a hardening coefficient. Variational inequality (4) formalizes the Prandtl-Reuß normality rule, also called the principle of maximal dissipation. The set $Z \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ describes the admissible (plastic) stresses, its boundary ∂Z is called the yield surface. We will exclusively use the standard von Mises cylinder with yield stress σ^{y}

$$Z = \{\sigma \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \|\operatorname{dev}\sigma\| \leqslant \sigma^{\mathrm{y}}\} \tag{6}$$

Here,

$$\|a\|^2 = a : a, \quad a : b = \sum_{i,j=1}^{d} a_{ij} b_{ij} \tag{7}$$

defines the (Frobenius) norm and the corresponding scalar product, and the deviator of σ is defined as $\operatorname{dev}\sigma := \sigma - (1/d)(\operatorname{tr}\sigma)\mathbb{I}$. The decomposition

$$\mathbb{R}^{d\times d}_{\mathrm{sym}} = X_{\mathrm{D}} \times X_I, \quad X_{\mathrm{D}} = \{\sigma : \operatorname{tr}\sigma = 0\}, \quad X_I = \{t\mathbb{I} : t \in \mathbb{R}\} \tag{8}$$

is orthogonal with respect to the scalar product (7) and, according to (8), $\operatorname{dev}: \mathbb{R}^{d\times d}_{\mathrm{sym}} \to X_{\mathrm{D}}$ represents the orthogonal projection. The following lemma reformulates the variational inequality (4) as a variational inequality with a dissipation function $\mathscr{D}$ (see Reference [7], p. 90).

Lemma 1

Let $(\dot{p}, \sigma^{\mathrm{p}}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Then

$$\sigma^{\mathrm{p}} \in Z, \quad \dot{p} : (\tau - \sigma^{\mathrm{p}}) \leqslant 0 \quad \text{for all } \tau \in Z \tag{9}$$

together with $\operatorname{tr}\dot{p} = 0$ hold if and only if

$$\sigma^{\mathrm{p}} : (q - \dot{p}) \leqslant \mathscr{D}(q) - \mathscr{D}(\dot{p}) \quad \forall q \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \tag{10}$$

where $\mathscr{D}: \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R} \cup \{\infty\}$,

$$\mathscr{D}(q) = \begin{cases} \sigma^{\mathrm{y}}\|q\| & \text{if } \operatorname{tr} q = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{11}$$

Proof

($\Rightarrow$) We rewrite (9) as

$$\sigma^{\mathrm{p}} : (q - \dot{p}) \leqslant \sigma^{\mathrm{p}} : q - \tau : \dot{p} \quad \forall q \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \forall \tau \in Z$$

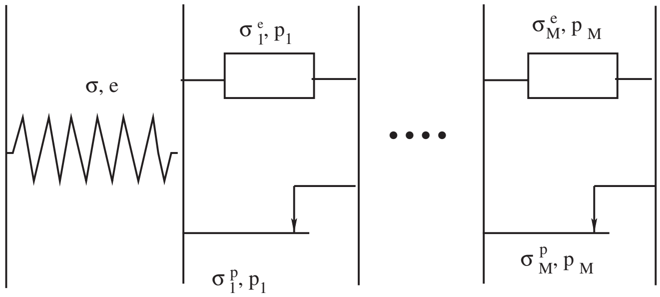

Figure 1. Prandtl-Ishlinskii model of play type.

Setting $\tau = \sigma^{\mathrm{y}}(\dot{p}/\|\dot{p}\|)$ if $\dot{p} \neq 0$, we obtain

$$\sigma^{\mathrm{p}} : (q - \dot{p}) \leqslant \sigma^{\mathrm{p}} : q - \mathscr{D}(\dot{p}) \quad \forall q \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \tag{12}$$

which obviously holds also for $\dot{p} = 0$. Furthermore, if $\operatorname{tr} q = 0$ then

$$\sigma^{\mathrm{p}} : q = \operatorname{dev}\sigma^{\mathrm{p}} : q \leqslant \|\operatorname{dev}\sigma^{\mathrm{p}}\|\|q\| \leqslant \sigma^{\mathrm{y}}\|q\| = \mathscr{D}(q)$$

which together with (12) proves (10).

($\Leftarrow$) From (10) it immediately follows that $\operatorname{tr}\dot{p} = 0$. Setting $q = 2\dot{p}$ in (10) it follows that $\operatorname{dev}\sigma^{\mathrm{p}} : \dot{p} = \sigma^{\mathrm{p}} : \dot{p} \leqslant \mathscr{D}(\dot{p})$, so for all q with $\operatorname{tr}(q) = 0$ we have $\operatorname{dev}\sigma^{\mathrm{p}} : q = \sigma^{\mathrm{p}} : q \leqslant \mathscr{D}(q)$, thus $\|\operatorname{dev}\sigma^{\mathrm{p}}\| \leqslant \sigma^{\mathrm{y}}$, i.e. $\sigma^{\mathrm{p}} \in Z$. On the other hand, $q = 0$ yields $-\sigma^{\mathrm{p}} : \dot{p} \leqslant -\mathscr{D}(\dot{p})$, so for any $\tau \in Z$ we get

$$\dot{p} : (\tau - \sigma^{\mathrm{p}}) \leqslant \tau : \dot{p} - \mathscr{D}(\dot{p}) \leqslant \operatorname{dev}\tau : \dot{p} - \mathscr{D}(\dot{p}) \leqslant (\|\operatorname{dev}\tau\| - \sigma^{\mathrm{y}})\|\dot{p}\| \leqslant 0 \qquad \square$$

The standard model of linear kinematic hardening as described above introduces essentially one additional internal state variable of tensor type, the plastic strain p, whose evolution is governed by (4). In particular, $\dot{p}(t) \neq 0$ only if $\sigma^{\mathrm{p}} \in \partial Z$. More complicated models for the constitutive law involve additional surfaces and internal state variables. We treat here a specific model which goes back in the 1D case to Prandtl [11] and Ishlinskii [12] and in the multi-dimensional case to Besseling [13] and Iwan [14]. The model discussed here is the one called Prandtl-Ishlinskii model of play type [6,15] with finitely many surfaces, whose rheological structure is depicted in Figure 1. The plastic strain p is decomposed as

$$p = \sum_{r\in I} p_r, \quad I = \{1, \ldots, M\} \tag{13}$$

we have backstresses σ^{b}_r,

$$\sigma^{\mathrm{b}}_r = \mathbb{H}_r p_r, \quad r \in I$$

and plastic stresses σ^{p}_r

$$\sigma = \sigma^{\mathrm{b}}_r + \sigma^{\mathrm{p}}_r, \quad r \in I$$

and a family of a variational inequalities

$$\sigma^{\mathrm{p}}_r \in Z_r, \quad \dot{p}_r : (\tau_r - \sigma^{\mathrm{p}}_r) \leqslant 0 \quad \forall \tau_r \in Z_r, \; r \in I \tag{14}$$

with convex restrictions Z_r, $r \in I$. If one wants to have infinitely many surfaces, a natural way to do this is to replace (13) by

$$p = \int_I p_r \,\mathrm{d}\mu(r) \tag{15}$$

where μ is a (finite) measure on some set I. In that case, (14) represents an infinite system of variational inequalities.

3. THE BOUNDARY VALUE PROBLEM

The elastoplastic continuum is assumed to occupy a bounded domain $\Omega \subset \mathbb{R}^d$, with a Lipschitz boundary $\Gamma = \partial\Omega$. The boundary Γ is split into a Dirichlet boundary Γ_D, a closed subset of Γ with a positive surface measure, and the remaining (relatively open and possibly empty) Neumann part $\Gamma_\mathrm{N} := \Gamma \backslash \Gamma_\mathrm{D}$. We pose essential and static boundary conditions, namely

$$u = 0 \quad \text{on } \Gamma_\mathrm{D} \quad \text{and} \quad \sigma \cdot n = g \quad \text{on } \Gamma_\mathrm{N} \tag{16}$$

where g is a given applied surface force and n denotes the outer normal to the boundary Γ_N. Our analysis will be restricted to the study of a boundary value problem defined in these functional spaces:

$$H^1_\mathrm{D}(\Omega) = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_\mathrm{D}\}$$

$$Q = \{q : q \in \operatorname{dev} \mathbb{R}^{d\times d}_\mathrm{sym},\ q_{ij} \in L^2(\Omega)\}$$

where $H^1(\Omega)$ and $L^2(\Omega)$ are the usual Sobolev and Lebesgue spaces. The equilibrium between external and internal forces in the quasi-static case is given by

$$\operatorname{div} \sigma(x,t) + f(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T) \tag{17}$$

where σ satisfies the boundary condition (16). With the relation

$$\varepsilon(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{18}$$

the variational formulation of (17) becomes

$$\int_\Omega \sigma : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v\,\mathrm{d}S(x) \tag{19}$$

valid for all $t \in [0,T]$ and all $v \in H^1_\mathrm{D}(\Omega)$. According to Lemma 1, we express the constitutive law by the form given in (10)

$$\sigma^\mathrm{p}_r : (q_r - \dot{p}_r) \leqslant \mathscr{D}_r(q_r) - \mathscr{D}_r(\dot{p}_r) \quad \forall q_r \in Q, r \in I \tag{20}$$

where (note that we only consider arguments with zero trace here)

$$\mathscr{D}_r(q_r) = \sigma^\mathrm{y}_r \|q_r\| \tag{21}$$

The integral form of (20) over Ω is given by

$$\int_\Omega \sigma_r^{\mathrm{p}} : (q_r - \dot{p}_r)\,\mathrm{d}x \leqslant \int_\Omega \mathscr{D}_r(q_r)\,\mathrm{d}x - \int_\Omega \mathscr{D}_r(\dot{p}_r)\,\mathrm{d}x \tag{22}$$

We equivalently replace v by $v - \dot{u}$ in the force equilibrium (19), sum the inequalities (22) over r and subtract (19) to obtain

$$\begin{aligned}
&\int_\Omega \sigma : \left(\varepsilon(v) - \sum_{r\in I} q_r\right)\mathrm{d}x - \int_\Omega \sigma : \left(\varepsilon(\dot{u}) - \sum_{r\in I} \dot{p}_r\right)\mathrm{d}x \\
&\quad + \sum_{r\in I}\int_\Omega \sigma_r^{\mathrm{b}} : (q_r - \dot{p}_r)\,\mathrm{d}x + \sum_{r\in I}\int_\Omega \mathscr{D}_r(q_r)\,\mathrm{d}x - \sum_{r\in I}\int_\Omega \mathscr{D}_r(\dot{p}_r)\,\mathrm{d}x \\
&\quad - \int_\Omega f\cdot(v-\dot{u})\,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} g\cdot(v-\dot{u})\,\mathrm{d}S(x) \geqslant 0
\end{aligned} \tag{23}$$

In the case of a single yield surface, i.e. $I=\{1\}$, this corresponds to the primal variational formulation discussed in Section 7.1 of Reference [7]. Next, we eliminate $\sigma = \mathbb{C}e = \mathbb{C}(\varepsilon(u) - p)$, $\sigma_r^{\mathrm{b}} = \mathbb{H}_r p_r$ and collect the remaining unknowns as a vector of functions

$$w = (u, (p_r)_{r\in I})$$

We consider w as an element of the Hilbert space (the scalar product will be defined below)

$$\mathscr{H} = H_{\mathrm{D}}^1(\Omega) \times \prod_{r\in I} Q \tag{24}$$

Writing $z = (v, (q_r)_{r\in I})$, we define a bilinear form $a(\cdot,\cdot)$, a linear functional $\ell(\cdot)$ and a non-linear functional $\psi(\cdot)$ by

$$\begin{aligned}
a : \mathscr{H}\times\mathscr{H} \to \mathbb{R}, \quad & a(w,z) = \int_\Omega \mathbb{C}\left(\varepsilon(u) - \sum_{r\in I} p_r\right) : \left(\varepsilon(v) - \sum_{r\in I} q_r\right)\mathrm{d}x \\
& \qquad + \sum_{r\in I}\int_\Omega \mathbb{H}_r p_r : q_r\,\mathrm{d}x \\
\ell(t) : \mathscr{H} \to \mathbb{R}, \quad & \langle \ell(t), z\rangle = \int_\Omega f(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g(t)\cdot v\,\mathrm{d}S(x) \\
\psi : \mathscr{H} \to \mathbb{R}, \quad & \psi(z) = \sum_{r\in I}\int_\Omega \mathscr{D}_r(q_r)\,\mathrm{d}x
\end{aligned} \tag{25}$$

From (23) we thus obtain the time-dependent variational inequality

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \geqslant \langle \ell(t), z - \dot{w}(t)\rangle, \quad \text{for all } z\in\mathscr{H} \tag{26}$$

We assume zero initial conditions

$$w(0) = 0 \tag{27}$$

We thus have arrived at the following formulation of the boundary value problem of quasi-static elastoplasticity.

Problem 1 (*BVP of quasi-static multi-surface elastoplasticity*)
For given $\ell \in H^1(0,T;\mathscr{H}^*)$ with $\ell(0)=0$, find $w \in H^1(0,T;\mathscr{H})$ with $w(0)=0$, such that (26) holds for almost all $t \in (0,T)$.

The case of infinitely many surfaces (15) again leads to Problem 1, see Reference [16]. We set

$$\mathscr{H} = H^1_{\mathrm{D}}(\Omega) \times L^2_\mu(I;Q) \tag{28}$$

where

$$L^2_\mu(I;Q) := \left\{ f \,|\, f : I \to Q, \int_{r\in I} \|f_r\|^2_{L^2}\,\mathrm{d}\mu(r) < \infty \right\}$$

The linear functional $\ell(\cdot)$ is defined as in (25). The bilinear form $a(\cdot,\cdot)$ and the non-linear functional $\psi(\cdot)$ are given by

$$\begin{aligned} a : \mathscr{H} \times \mathscr{H} \to \mathbb{R},\ \ a(w,z) &= \int_\Omega \mathbb{C}\left(\varepsilon(u) - \int_I p_r\,\mathrm{d}\mu(r)\right) : \left(\varepsilon(v) - \int_I q_r\,\mathrm{d}\mu(r)\right)\mathrm{d}x \\ &\quad + \int_\Omega \int_I \mathbb{H}_r\, p_r : q_r\,\mathrm{d}\mu(r)\,\mathrm{d}x \\ \psi : \mathscr{H} \to \mathbb{R},\quad \psi(z) &= \int_\Omega \int_I \mathscr{D}_r(q_r)\,\mathrm{d}\mu(r)\,\mathrm{d}x \end{aligned} \tag{29}$$

4. EXISTENCE AND UNIQUENESS

In this section, we will prove the unique solvability of Problem 1. We pose the natural assumption that the elastic and hardening tensors are symmetric and positive definite,

$$\begin{aligned} \xi : \mathbb{C}\lambda = \mathbb{C}\xi : \lambda \quad &\text{for all } \xi,\lambda \in \mathbb{R}^{d\times d} \\ \xi : \mathbb{H}_r\lambda = \mathbb{H}_r\xi : \lambda \quad &\text{for all } \xi,\lambda \in \mathbb{R}^{d\times d}, r = 1,\ldots,M \end{aligned} \tag{30}$$

and there exist constants $c, h_r > 0$ such that

$$\begin{aligned} \mathbb{C}\xi : \xi \geqslant c\|\xi\|^2 \quad &\text{for all } \xi \in \mathbb{R}^{d\times d} \\ \mathbb{H}_r\xi : \xi \geqslant h_r\|\xi\|^2 \quad &\text{for all } \xi \in \mathbb{R}^{d\times d}, r = 1,\ldots,M \end{aligned} \tag{31}$$

We now state the main theorem of this paper.

Theorem 1
Assume that (30) and (31) hold, let $\ell \in H^1(0,T;\mathscr{H}^*)$ with $\ell(0)=0$. Then there exists a unique solution $w \in H^1(0,T;\mathscr{H})$ of Problem 1.

We will prove that Theorem 1 is implied by the following theorem, which in turn constitutes a special case of Theorem 7.3 in Reference [7].

Theorem 2 (*Han and Reddy* [7])
Let $\mathscr{H}$ be a Hilbert space, $a : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ be a bilinear form that is symmetric, bounded, and $\mathscr{H}$-elliptic; $\ell \in H^1(0,T;\mathscr{H}^*)$ with $\ell(0)=0$; and $\psi : \mathscr{H} \to \mathbb{R}$ non-negative, convex, positively homogeneous, and Lipschitz continuous. Then there exists a unique $w \in H^1(0,T;\mathscr{H})$ with $w(0)=0$ which satisfies the variational inequality (26) for almost all $t \in (0,T)$.

In order to prove Theorem 1, we have to prove that the assumptions of Theorem 2 are satisfied. As mentioned above, for a finite index set $I=\{1,\ldots,M\}$ we set

$$\mathscr{H} = H^1_{\mathrm{D}}(\Omega) \times \prod_{r=1}^{M} Q \tag{32}$$

The scalar product and the induced norm are given by

$$(w,z)_{\mathscr{H}} := (u,v)_{H^1} + \sum_{r=1}^{M} (p_r, q_r)_{L^2}, \quad \|w\|^2_{\mathscr{H}} := (u,u)^2_{H^1} + \sum_{r=1}^{M} (p_r, p_r)^2_{L^2}$$

where

$$(p_r, q_r)_{L^2} = \int_\Omega p_r : q_r \,\mathrm{d}x, \quad \|p_r\|^2_{L^2} = (p_r, p_r)_{L^2}$$

Proposition 1 (*Boundedness of the bilinear form* $a(\cdot,\cdot)$)
The bilinear form $a(\cdot,\cdot)$ is bounded in the space $\mathscr{H}$,

$$|a(w,z)| \leqslant \left((M+1)\|\mathbb{C}\| + \max_{r=1,\ldots,M} \|\mathbb{H}_r\| \right) \|w\|_{\mathscr{H}} \|z\|_{\mathscr{H}} \tag{33}$$

Proof
We have

$$\begin{aligned}
&\left| \int_\Omega \left(\mathbb{C}(\varepsilon(u) - \sum_{r=1}^{M} p_r) \right) : \left(\varepsilon(v) - \sum_{r=1}^{M} q_r \right) \mathrm{d}x \right| \\
&\quad \leqslant \|\mathbb{C}\| \cdot \left\| \varepsilon(u) - \sum_{r=1}^{M} p_r \right\|_{L^2} \cdot \left\| \varepsilon(v) - \sum_{r=1}^{M} q_r \right\|_{L^2}
\end{aligned} \tag{34}$$

Because $\left(\sum_{r=0}^{M} a_r\right)^2 \leqslant (M+1)\sum_{r=0}^{M} a_r^2$ in $\mathbb{R}$, and because $\|\varepsilon(u)\|_{L^2} \leqslant \|u\|_{H^1}$, we have

$$\begin{aligned}
\left\| \varepsilon(u) - \sum_{r=1}^{M} p_r \right\|^2_{L^2} &\leqslant \left(\|\varepsilon(u)\|_{L^2} + \sum_{r=1}^{M} \|p_r\|_{L^2} \right)^2 \\
&\leqslant (M+1) \left(\|\varepsilon(u)\|^2_{L^2} + \sum_{r=1}^{M} \|p_r\|^2_{L^2} \right) \\
&\leqslant (M+1)\|w\|^2_{\mathscr{H}}
\end{aligned} \tag{35}$$

likewise for the rightmost term in (34). Moreover, we have

$$\left|\sum_{r=1}^{M}\int_{\Omega}\mathbb{H}_r p_r : q_r \,\mathrm{d}x\right| \leqslant \left(\max_{r=1,\dots,M}\|\mathbb{H}_r\|\right)\sum_{r=1}^{M}\|p_r\|_{L^2}\|q_r\|_{L^2} \tag{36}$$

and

$$\sum_{r=1}^{M}\|p_r\|_{L^2}\|q_r\|_{L^2} \leqslant \left(\sum_{r=1}^{M}\|p_r\|_{L^2}^2\right)^{1/2}\left(\sum_{r=1}^{M}\|q_r\|_{L^2}^2\right)^{1/2} \leqslant \|w\|_{\mathscr{H}}\|z\|_{\mathscr{H}} \tag{37}$$

Putting together (34)–(37), we obtain the assertion. □

We now turn to the problem to find an ellipticity constant $c_{\mathrm{e}}>0$ satisfying

$$a(w,w)\geqslant c_{\mathrm{e}}\|w\|_{\mathscr{H}}^2 \quad \text{for all } w\in\mathscr{H}$$

We first determine the largest constant $k(M)$, $M\in\mathbb{N}$, such that

$$\left(x_0-\sum_{r=1}^{M}x_r\right)^2+\sum_{r=1}^{M}x_r^2\geqslant k(M)\sum_{r=0}^{M}x_r^2 \tag{38}$$

holds for all $x_0,x_1,\dots,x_M\in\mathbb{R}$. Indeed, we have

$$\left(x_0-\sum_{r=1}^{M}x_r\right)^2+\sum_{r=1}^{M}x_r^2 = x^{\mathrm{T}}Ax \tag{39}$$

where

$$A=D+a\otimes a,\quad D=\mathrm{diag}(0,1,\dots,1),\quad a=(1,-1,\dots,-1) \tag{40}$$

Thus, the optimal constant $k(M)$ in (38) is equal to the smallest eigenvalue of A, which we will compute with the aid of the following Lemma.

Lemma 2
Let $D\in\mathbb{R}^{N\times N}$ be a diagonal matrix, $D=\mathrm{diag}(d_1,\dots,d_{\mathrm{N}})$, $d_j\neq 0$ for $j=1,\dots,N$, let $a\in\mathbb{R}^N$. Then there holds

$$\det(D+a\otimes a)=\left(\prod_{j=1}^{N}d_j\right)\left(1+\sum_{j=1}^{N}a_j^2/d_j\right) \tag{41}$$

Proof
The assertion follows from the identity

$$\begin{aligned}\det(D+a\otimes a) &= \det\begin{pmatrix}D+a\otimes a & -a\\ 0 & 1\end{pmatrix} = \det\begin{pmatrix}D & -a\\ a^{\mathrm{T}} & 1\end{pmatrix}\\ &= \det\begin{pmatrix}D & -a\\ 0 & 1+\sum_{j=1}^{N}a_j^2/d_j\end{pmatrix}\end{aligned} \tag{42}$$

Table I. Values of k for different values of M.

M	k
1	0.3819
2	0.2679
3	0.2087
4	0.1715
5	0.1458
10	0.0839
100	0.0098
1000	9.98 10e−4

To see that the second equality holds, for $j=1,\dots,N$ we multiply the last column

$$\begin{pmatrix} -a \\ 1 \end{pmatrix} \quad \text{of} \quad B := \begin{pmatrix} D & -a \\ a^{\mathrm{T}} & 1 \end{pmatrix}$$

by $-a_j$ and add it to the jth column of B. Similarly, we obtain the third inequality in (42), if for $j=1,\dots,N$ we multiply the jth row of B by $-a_j/d_j$ and add it to the last row of B. □

We now determine the smallest eigenvalue $\lambda_{\min}$ of A in (40). By (39), we obviously have $\lambda_{\min}>0$. By Lemma 2 we have, if $\lambda \neq 0,1$,

$$\det(A-\lambda I) = -\lambda(1-\lambda)^M \left(1+\frac{1}{-\lambda}+\frac{M}{1-\lambda}\right) \tag{43}$$

Besides 0 and 1, the zeroes of (43) are given by $\lambda_{1,2}=1+M/2\pm 1/2\sqrt{4M+M^2}$. Thus,

$$k(M)=\lambda_{\min}=1+\frac{M}{2}-\frac{1}{2}\sqrt{4M+M^2} \tag{44}$$

Table I displays some values of k. Now we prove the ellipticity of the bilinear form $a(\cdot,\cdot)$. By Korn's inequality,

$$\int_\Omega \|\varepsilon(u)\|^2 \,\mathrm{d}x \geqslant K\|u\|_{H^1}^2 \quad \text{for all } u\in H_{\mathrm{D}}^1(\Omega) \tag{45}$$

holds for some constant $K=K(\Omega,d)$.

Proposition 2 (*Ellipticity of the bilinear form* $a(\cdot,\cdot)$)
The bilinear form $a(\cdot,\cdot)$ is $\mathscr{H}$-elliptic,

$$a(w,w)\geqslant (k(M)\,\min\{c,h_1,\dots,h_M\}\,\min\{1,K(\Omega,d)\})\,\|w\|_{\mathscr{H}}^2 \tag{46}$$

where $k(M)$ is given in (44) and c,h_r are given in (31).

Proof
We can bound the integrand in the scalar product $a(w,w)$ from below as

$$\begin{aligned}
&\mathbb{C}\left(\varepsilon(u)-\sum_{r=1}^{M} p_r\right):\left(\varepsilon(u)-\sum_{r=1}^{M} p_r\right)+\sum_{r=1}^{M}\mathbb{H}_r\, p_r : p_r \\
&\geqslant c\left\|\varepsilon(u)-\sum_{r=1}^{M} p_r\right\|^2+\sum_{r=1}^{M} h_r\|p_r\|^2 \\
&\geqslant \min\{c,h_1,\dots,h_M\}\left(\left\|\varepsilon(u)-\sum_{r=1}^{M} p_r\right\|^2+\sum_{r=1}^{M}\|p_r\|^2\right)
\end{aligned} \tag{47}$$

The assertion now follows from (38) and Korn's inequality. Note that, if (38) is valid for all scalars $x_r \in \mathbb{R}$, it is also valid for all tensors $x_r \in \mathbb{R}^{d\times d}$. □

The functional

$$\psi(z)=\sum_{r=1}^{M}\int_{\Omega}\mathscr{D}_r(q_r)\,\mathrm{d}x,\quad \mathscr{D}_r(q_r)=\sigma_r^{\mathrm{y}}\|q_r\| \tag{48}$$

is a convex, non-negative and positively homogeneous functional, because $\mathscr{D}_r$ has those properties.

Proposition 3 (*Lipschitz continuity of the functional* $\psi(\cdot)$)
The functional $\psi(\cdot)$ is a Lipschitz continuous functional in the space $\mathscr{H}$ with the Lipschitz constant

$$L=\left(\max_{r=1,\dots,M}\sigma_r^{\mathrm{y}}\right)\mathrm{meas}(\Omega)^{1/2}M^{1/2} \tag{49}$$

Proof
Let us define $z^1=(v^1,q_1^1,\dots,q_M^1), z^2=(v^2,q_1^2,\dots,q_M^2)$. Then

$$\begin{aligned}
|\psi(z^1)-\psi(z^2)| &= \sum_{r=1}^{M}\left|\int_{\Omega}\sigma_r^{\mathrm{y}}(\|q_r^1\|-\|q_r^2\|)\,\mathrm{d}x\right| \\
&\leqslant \left(\max_{r=1,\dots,r}\sigma_r^{\mathrm{y}}\right)\sum_{r=1}^{M}\int_{\Omega}\|q_r^1-q_r^2\|\,\mathrm{d}x
\end{aligned} \tag{50}$$

Moreover,

$$\begin{aligned}
\sum_{r=1}^{M}\int_{\Omega}\|q_r^1-q_r^2\|\,\mathrm{d}x &\leqslant \mathrm{meas}(\Omega)^{1/2}\sum_{r=1}^{M}\|q_r^1-q_r^2\|_{L^2} \\
&\leqslant \mathrm{meas}(\Omega)^{1/2}M^{1/2}\left(\sum_{r=1}^{M}\|q_r^1-q_r^2\|_{L^2}^2\right)^{1/2}
\end{aligned} \tag{51}$$

Putting (50) and (51) together, the assertion follows. □

We now have shown that all assumptions of Theorem 2 are satisfied in Problem 1. Thus, Theorem 1 is proved.

5. THE CASE OF INFINITELY MANY SURFACES

The main existence and uniqueness theorem (Theorem 1) can be extended to the case of infinitely many surfaces given by (28) and (29). We present the results corresponding to Propositions 1, 2 and 3 and sketch the changes in the arguments, more details are given in Reference [16]. Firstly, note that the estimate (35) in the proof of the boundedness of $a(\cdot,\cdot)$ can be modified to

$$\begin{aligned}\|\varepsilon(u) - \int_I p_r \,\mathrm{d}\mu(r)\|_{L^2}^2 &\leqslant 2\left(\|\varepsilon(u)\|_{L^2}^2 + \mu(I)\cdot\int_I \|p_r\|_{L^2}^2 \,\mathrm{d}\mu(r)\right)\\ &\leqslant 2\max\{1,\mu(I)\}(\|\varepsilon(u)\|_{L^2}^2 + \|p\|_{L^2_\mu(I;Q)}^2)\end{aligned} \tag{52}$$

and consequently the constant $(M+1)$ in Proposition 1 is replaced by $2\max\{1,\mu(I)\}$, i.e. the following proposition holds.

Proposition 4 (*Boundedness of the bilinear form* $a(\cdot,\cdot)$*, case of infinitely many surfaces*)
The bilinear form $a(\cdot,\cdot)$ is bounded in the space $\mathscr{H}$,

$$a(w,z)\leqslant\left(2\max\{1,\mu(I)\}\|\mathbb{C}\| + \sup_{r\in I}\|\mathbb{H}_r\|\right)\|w\|_{\mathscr{H}}\|z\|_{\mathscr{H}} \tag{53}$$

Secondly, in order to prove the ellipticity of the bilinear form $a(\cdot,\cdot)$ we will determine a constant $k(\mu)$ such that

$$\left(x_0 - \int_I x_r \,\mathrm{d}\mu(r)\right)^2 + \int_I x_r^2 \,\mathrm{d}\mu(r) \geqslant k(\mu)\left(x_0^2 + \int_I x_r^2 \,\mathrm{d}\mu(r)\right) \tag{54}$$

holds for all $x_0, x_r\in\mathbb{R}, r\in I, \int_I x_r^2 \,\mathrm{d}\mu(r) < \infty$. Indeed, applying the argument from [7], page 168, the left-hand side of (54) can be bounded from below as follows:

$$\begin{aligned}&\left(x_0 - \int_I x_r \,\mathrm{d}\mu(r)\right)^2 + \int_I x_r^2 \,\mathrm{d}\mu(r)\\ &\quad = x_0^2 + \left(\int_I x_r \,\mathrm{d}\mu(r)\right)^2 - 2x_0\left(\int_I x_r \,\mathrm{d}\mu(r)\right) + \int_I x_r^2 \,\mathrm{d}\mu(r)\\ &\quad \geqslant x_0^2 + \left(\int_I x_r \,\mathrm{d}\mu(r)\right)^2 - dx_0^2 - \frac{1}{d}\left(\int_I x_r \,\mathrm{d}\mu(r)\right)^2 + \int_I x_r^2 \,\mathrm{d}\mu(r)\\ &\quad \geqslant (1-d)(x_0)^2 + \left[\left(1-\frac{1}{d}\right)\mu(I) + 1\right]\int_I x_r^2 \,\mathrm{d}\mu(r)\end{aligned} \tag{55}$$

Here $d \in (0,1)$ is arbitrary, and we have used the inequality $2ab \leqslant da^2 + (1/d)b^2$ for all $a, b \in \mathbb{R}$ and the Cauchy-Schwarz inequality

$$\left(\int_I x_r \, \mathrm{d}\mu(r)\right)^2 \leqslant \int_I 1 \, \mathrm{d}\mu(r) \cdot \int_I x_r^2 \, \mathrm{d}\mu(r) = \mu(I) \int_I x_r^2 \, \mathrm{d}\mu(r)$$

Now, for all $d \in (\mu(I)/1 + \mu(I), 1)$ we have $\min\{1-d, 1-\mu(I)\ (1-d)/d\} > 0$. Consequently, (54) holds if we set

$$\begin{aligned} k(\mu) &= \max_{d \in \left(\frac{\mu(I)}{1+\mu(I)}, 1\right)} \min\left\{1-d, 1-\mu(I)\,\frac{1-d}{d}\right\} \\ &= \frac{1}{2}\left(\sqrt{(\mu(I))^2 + 4\mu(I)} - \mu(I)\right) \end{aligned} \tag{56}$$

The following proposition holds.

Proposition 5 (*Ellipticity of the bilinear form $a(\cdot,\cdot)$, case of infinitely many surfaces*)
The bilinear form $a(\cdot,\cdot)$ is $\mathscr{H}$-elliptic,

$$a(w,w) \geqslant \left(k(\mu) \min\left\{c, \inf_{r\in I}\{h_r\}\right\} \min\{1, K(\Omega, d)\}\right) \|w\|_{\mathscr{H}}^2 \tag{57}$$

where $k(\mu)$ is given in (56) and c, h_r are given in (31).

The extension of the proof of Proposition 3 is straightforward.

Proposition 6 (*Lipschitz continuity of the functional $\psi(\cdot)$, case of infinitely many surfaces*)
The functional $\psi(\cdot)$ is Lipschitz continuous on $\mathscr{H}$ with the Lipschitz constant

$$L = \sup_{r\in I}\{\sigma_r^{\mathrm{y}}\} \mathrm{meas}(\Omega)^{1/2} \mu(I)^{1/2} \tag{58}$$

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