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# Some remarks on the history and future of averaging techniques in a posteriori finite element error analysis

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Given a flux or stress approximation $p_h$ from a low-order finite element simulation of an elliptic boundary value problem, averaging or (gradient-)recovery techniques aim the computation of an improved approximation $Ap_h$ by a (simple) post processing of $p_h$. For instance, frequently named after Zienkiewicz and Zhu, $Ap_h$ is the elementwise interpolation of the nodal values $(Ap_h)(z)$ obtained as the integral mean of $p_h$ on a neighbourhood of $z$. This paper gives an overview over old and new arguments in the proof of reliability and efficiency of the error estimator $\eta_{\mathcal{A}} := \|p_h - Ap_h\|$ as an approximation of the error $\|p - p_h\|$ in (an energy norm) $\|\cdot\|$. High-lighted are the general class of meshes, averaging operators, or finite elements (conforming, nonconforming, or mixed). Emphasis is on old and new aspects of superconvergence and arguments to circumvent superconvergence at all within proofs of a posteriori finite element error estimates.



## 1 Introduction

This section presents some main concepts in a posteriori finite element error control and their purpose. Sect. 1.1 gives a vague but brief introduction to reliability and efficiency of (a posteriori) error estimators. Sects. 1.2 and 1.3 describe their usage in termination criteria for error control and in adaptive mesh-refining. Some remarks on references in Sect. 1.4 and the outline of the remaining part of this paper in Sect. 1.5 conclude this introduction.

### 1.1 Reliability and efficiency

Throughout the first section there is no formal need to have a boundary value problem

$$Lu = f \tag{1}$$

at hand for an elliptic (second-order) differential operator $L$ plus boundary conditions. In fact, it suffices to think of $p$ (the flux or stress) as the derivative of an unknown $u$ which is approximated by a (known) finite element solution $u_h$ and so yields a discrete (flux or stress) $p_h$. It is important that $p_h$ is known as well as a right-hand side $f$ in the underlying domain $\Omega$ and prescribed data on the boundary $\partial\Omega$ (the domain $\Omega$ is not visible in (1) but shall arise in applications below). Given a norm $\|\cdot\|$, the aim is to approximate the (unknown) error $\|p - p_h\|$ by a computable quantity $\eta$ called error estimator.

**Definition 1.1.** (Error Estimator). A (computable) quantity $\eta$ which is thought as an approximation to $\|p - p_h\|$ is called *a posteriori error estimator,* or *estimator* for brevity, if it is a function of known quantities $f, \Omega, \partial\Omega$ and $u_h, p_h$ etc. of the data and the discrete solution.

This definition is very vague and not very useful on its own: The estimator $\eta := +\infty$ and $\eta := 0$ is even reliable and efficient, respectively, but certainly not interesting. The real purpose of an estimator is to provide lower *and* upper error bounds, in the following sense.

**Definition 1.2.** (Reliability). An estimator $\eta$ is called *reliable* if

$$\|p - p_h\| \le C_{\mathrm{rel}}\, \eta + \mathrm{h.o.t.}_{\mathrm{rel}}\,. \tag{2}$$

**Definition 1.3.** (Efficiency). An estimator $\eta$ is called *efficient* if

$$\eta \le C_{\mathrm{eff}}\, \|p - p_h\| + \mathrm{h.o.t.}_{\mathrm{eff}}\,. \tag{3}$$

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**Definition 1.4.** (Asymptotic Exactness). An estimator is called *asymptotically exact* if it is reliable and efficient and $C_{\rm rel} = C_{\rm eff}$ in (2)–(3).

Herein, $C_{\rm rel}$ and $C_{\rm eff}$ are multiplicative constants which do not depend on the mesh-size of an underlying finite element mesh $\mathcal{T}$ for the computation of $p_h$ and $\mathrm{h.o.t.}$ denotes higher-order terms. The latter are generically much smaller than $\eta$ or $\|p-p_h\|$, but usually this depends on the (unknown) smoothness of the exact solution or the (known) smoothness of the given data. At this moment we have only one finite element mesh $\mathcal{T}$ in mind, but could certainly think of a family $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of regular triangulations with a (maximal) mesh-size $h$, in a parameter set $\mathcal{H}\subset(0,\infty)$, which can be arbitrarily small. Then, the higher-order terms $\mathrm{h.o.t.}$ can be defined by the property

$$\lim_{\mathcal{H}\ni h\to 0} \mathrm{h.o.t.}_{\rm rel}/\|p-p_h\| = 0 = \lim_{\mathcal{H}\ni h\to 0} \mathrm{h.o.t.}_{\rm eff}/\|p-p_h\|. \tag{4}$$

The constants $C_{\rm rel}$ and $C_{\rm eff}$ may (mildly) depend on $\mathcal{T}_h$ but are supposed to be bounded,

$$\limsup_{\mathcal{H}\ni h\to 0} C_{\rm rel} < \infty \quad \text{and} \quad \limsup_{\mathcal{H}\ni h\to 0} C_{\rm eff} < \infty. \tag{5}$$

It is the mathematical task of a posteriori error analysis to provide sufficient and necessary conditions for the reliability and efficiency estimates (2)–(3) to hold and characterise or estimate the constants $C_{\rm rel}$, $C_{\rm eff}$ and the higher-order terms $\mathrm{h.o.t.}_{\rm rel}$, $\mathrm{h.o.t.}_{\rm eff}$. The properties (4)–(5) may well serve as a guideline for the design of estimators, but they are usually not sufficient for practical purposes in error control.

### 1.2 A posteriori error control

There are at least two aspects where an error estimator is useful in practice: error estimation and adaptive mesh-refinement. For the first main usage, one is given a tolerance $\mathrm{Tol} > 0$ and interested in a termination (of successively adapted mesh-refinements) based on the criterion

$$\|p-p_h\| \le \mathrm{Tol}\,.$$

Since the error $\|p-p_h\|$ is unknown, this is replaced by its upper bound (2) and then reads

$$C_{\rm rel}\eta + \mathrm{h.o.t.}_{\rm rel} \le \mathrm{Tol}\,. \tag{6}$$

For a verification of (6), it is evident that we require not only $\eta$ but also $C_{\rm rel}$ and $\mathrm{h.o.t.}_{\rm rel}$. A qualitative knowledge such as in (4)–(5) does not suffice. Notice that one may call the computable upper bound

$$\tilde\eta := C_{\rm rel}\eta + \mathrm{h.o.t.}_{\rm rel}$$

an error estimator (which then satisfies (2) with a reliability constant 1 and vanishing higher-order terms). If $\tilde\eta$ is not computable, the error control is incomplete and so useless. Fortunately, in the examples below, $\tilde\eta$ is computable and this enables guaranteed error control.

Notice that the error control is exclusively with respect to the norm $\|\cdot\|$. This paper focuses on energy norms and so ignores goal-oriented error control. The latter is important and the reader is referred (i) to [1, 6] for arguments that reduce the more general problem to energy-norm arguments and (ii) to [13] for a survey of the work of Rannacher et al. on a very successful computational approach.

### 1.3 Adaptive mesh-refining

The second main usage of an estimator $\eta$ employs its local character, e.g. in the form

$$\eta^2 = \sum_{T\in\mathcal{T}} \eta_T^2 \tag{7}$$

with (computable) elementwise contributions $\eta_T$. The interpretation of $\eta_T$ is often that of a local *indicator*. The idea then is to refine the element $T$ if this indicator $\eta_T$ is relatively large. The adaptive meshes in our work are generated by a refinement criterion such as

$$\tfrac12 \max\{\eta_K : K\in\mathcal{T}\} \le \eta_T. \tag{8}$$

Notice that further strategies are required to avoid hanging nodes and degenerated elements. The use of $\eta_T$ as a refinement criterion is essentially based on heuristics; one should therefore speak of a *refinement indicator* $\eta_T$ (and not of an error indicator).

Notice that $C_{\rm rel}, C_{\rm eff}$ do not enter in (8) while $\mathrm{h.o.t.}_{\rm rel}, \mathrm{h.o.t.}_{\rm eff}$ are simply ignored therein.

The rigorous justification of adaptive mesh-refining algorithms by a proof of an error-reduction property started with [36, 37, 43]. Therein, one deduces convergence and even a *convergence rate*, but the number of elements generated by the adaptive algorithm is not under control. Hence optimal convergence rates are not guaranteed although expected and observed in many numerical experiments. Coarsening steps may be involved to prove optimal performance of adaptive mesh-refinement [14]. However, the coarsening seems rather a theoretical concept and less likely necessary in practice.

### 1.4 References

Amongst the most influential early publications on a posteriori error control are [4, 5, 38] followed by many others. It is not the aim of this paper to give credit to all the authors who enlighten the field but to guide the unexperienced reader into the area. We therefore refer to the survey articles [13, 39] and the books [1, 6, 39, 46]. However, further work in the context of averaging techniques and contributions from the author's group will be quoted throughout the remainder of this paper.

### 1.5 Outline of the paper

The abstract level of the introduction will be continued in Sect. 2 where an abstract overview concerns different classes of explicit and implicit estimators. Sect. 3 illustrates the estimators in a numerical example for a Laplace model equation. The focus then is on averaging techniques and their analysis. Sect. 4 discusses their asymptotic exactness based on superconvergence for structured grids. Reliability and efficiency of averaging techniques is shown in Sects. 5 and 6, respectively. Sect. 7 gives an overview over other applications. Finally, more global averaging techniques are addressed in Sect. 8. The paper finishes with some conclusion in Sect. 9.

## 2 Overview

This section is devoted to four classes of the most important error estimators presented under minimal notational assumptions. We adopt the (abstract) notation of Sect. 1 specified for model examples and particular applications in the sequel. More explanations will follow in Sect. 3 for the Poisson problem.

### 2.1 Explicit error estimators

In many cases, the underlying partial differential equation (PDE) in (1) gives rise to a residual

$$\mathrm{Res} := f - Lu_h$$

which, often, is a linear and bounded functional $\mathrm{Res} : X \to \mathbb{R}$ in a (real) Hilbert or Banach space $X$. Although $f, L$, and $u_h$ are given, the dual norm

$$\|\mathrm{Res}\|_{X^*} = \sup_{v \in X \setminus \{0\}} \frac{\mathrm{Res}(v)}{\|v\|_X} \tag{9}$$

is usually difficult or laborious to compute. If the boundary value problem (1) is well posed and $\|\cdot\| = \|\cdot\|_X$ is a norm in $X$, there holds

$$c_1 \|\mathrm{Res}\|_{X^*} \le \|p - p_h\| \le c_2 \|\mathrm{Res}\|_{X^*}$$

with global positive constants $c_1$ and $c_2$ which solely depend on the bounds of $L$ and $L^{-1}$. For the Laplace equation and the energy norm below there holds $c_1 = 1 = c_2$ and so the error equals the norm of the residual. Moreover, the computation of the real number (9) appears to be as expensive as the computation of the error $p - p_h$ itself and so (as $p_h$ is known) of the exact solution $p$. Thus, the exact computation of (9) is not the goal of a posteriori error control, but the less expensive computation of lower and upper bounds is.

Lower bounds of (9) are easily deduced by any particular choice of $v$ in $X$ as edge or element bubble functions [46]. Upper bounds usually require weak interpolation operators in Sobolev spaces [18, 35]. The most popular residual-based explicit error estimator reads

$$\eta_E = \left( \sum_{T \in \mathcal{T}} h_T^q \, \|f - Lu_h\|_{L^q(T)}^q \right)^{1/q} + \left( \sum_{E \in \mathcal{E}} h_E \, \|[p_h]\|_{L^q(E)}^q \right)^{1/q} . \tag{10}$$

Here, $\mathcal{T}$ is a triangulation into elements $T$ of diameter $h_T$ and $\mathcal{E}$ is the set of edges therein; $h_E$ denotes the diameter of the edge $E \in \mathcal{E}$. If $\|\cdot\|$ is the Lebesgue norm in $L^p(\Omega)$ then, typically, $q$ is the coefficient of the dual $L^q(\Omega)$ defined by $1/p + 1/q = 1$. It is assumed that $L$ can be evaluated for $u_h$ on each element and that $[p_h]$ denotes the jump of a $\mathcal{T}$-piecewise uniformly continuous $p_h$ over an interior edge $E$ (with modifications along the boundary $\partial\Omega$).

### 2.2 Implicit error estimators

Localisation is the main step towards more accurate residual-based error estimators. At least two approaches are established, namely localisation by a partition of unity and localisation by domain decomposition with interfaces corrections. In fact, the localisation techniques are related to overlapping and non-overlapping domain decomposition techniques.

Given a mesh $\mathcal{T}$, the hat functions $\varphi_z$ for each node $z$ form the nodal basis $(\varphi_z : z \in \mathcal{N})$ for (conforming) first-order finite element schemes (without boundary restrictions). The partition of unity property

$$\sum_{z\in\mathcal{N}} \varphi_z = 1 \quad \text{in} \quad \Omega$$

yields

$$\operatorname{Res}(v) = \sum_{z\in\mathcal{N}} \operatorname{Res}(\varphi_z Dv) \quad \text{and} \quad \|Dv\|_p^p = \sum_{z\in\mathcal{N}} \|\varphi_z^{1/p} Dv\|_p^p,$$

for a Sobolev semi-norm in (a power of) $X = W^{m,p}(\Omega)$ and the functional matrix $Dv$ of all $m$th-order partial derivatives. Since

$$\operatorname{Res}(\varphi_z v) \le \sup_{\substack{w_z\in X\\ \left\|\varphi_z^{1/p} Dw_z\right\|_p=1}} \operatorname{Res}(\varphi_z w_z) \left\|\varphi_z^{1/p} Dv\right\|_p =: \eta_z \left\|\varphi_z^{1/p} Dv\right\|_p$$

we infer

$$\|\operatorname{Res}\| \le \sup_{v\in X\setminus\{0\}} \sum_{z\in\mathcal{N}} \eta_z \left\|\varphi_z^{1/p} Dv\right\|_p \le \left(\sum_{z\in\mathcal{N}} \eta_z^q\right)^{1/q} =: \eta_L. \tag{11}$$

The local estimator $\eta_L$ assumes that the patchwise interface problems behind the definition of $\eta_z$ can be solved exactly. In practice, $\eta_z$ has to be approximated by a finite element method with proper accuracy. We refer to [26, 42] for details and mention that the idea goes back to [5].

The non-overlapping domain decomposition schemes employ artifical unknowns $\mu_{\partial T}$ at the interfaces which allow a representation of the form

$$\operatorname{Res}(v) = \sum_{T\in\mathcal{T}} \left(\int_T R_T v\,dx + \int_{\partial T} (p_h\cdot\nu + \mu_{\partial T})\,v\,ds\right).$$

Ladeveze suggested a certain choice of the interface corrections $\mu_{\partial T}$ such that the local problem on $T \in \mathcal{T}$ satisfies

$$\eta_T := \sup_{\substack{v\in X\\ \|Dv\|_{L^p(T)}=1}} \left(\int_T R_T v\,dx + \int_{\partial T} (p_h\cdot\nu + \mu_{\partial T}) v\,ds\right) < \infty. \tag{12}$$

Depending on the differential operator $L$, several considerations are necessary to ensure $\eta_T < \infty$; for the Laplace and Lamé equation the constants and the rigid body motions have to vanish for the equilibrated local residual, respectively. The resulting equilibration error estimator

$$\eta_{EQ} = \left(\sum_{T\in\mathcal{T}} \eta_T^q\right)^{1/q} \tag{13}$$

seems to be very popular. Details on the implementation can be found in [1], a related correction technique in [6]. As for $\eta_L$, it is assumed that the local problem (12) is solved with sufficient accuracy.

### 2.3 Multilevel estimators

While the preceding estimators evaluate or estimate the residual of one finite element solution $p_h$, multilevel estimators concern at least two meshes $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$ and two discrete (solutions or) fluxes $p_h$ and $p_{h/2}$. The interpretation is that $p_{h/2}$ is computed on a finer mesh, say, of halved mesh-size and $\mathcal{T}_{h/2}$ is obtained by one refinement of $\mathcal{T}_h$ or that $p_{h/2}$ is computed with higher polynomial order. The point is that one believes that the error $\|p-p_{h/2}\|$ of the solution on the finer mesh $\mathcal{T}_{h/2}$ is systematically smaller than the error $\|p - p_h\|$ on the coarser mesh $\mathcal{T}_h$ in the sense that

$$\|p - p_{h/2}\| \le \varrho\, \|p - p_h\| \tag{14}$$

for some constant $\varrho < 1$. If this saturation assumption (14) holds, one obtains (2)–(3) for

$$\eta_{ML} = \|p_{h/2} - p_h\| \tag{15}$$

and $\mathrm{h.o.t.}_{\mathrm{rel}} = 0 = \mathrm{h.o.t.}_{\mathrm{eff}}$ with

$$C_{\mathrm{rel}} = (1-\varrho)^{-1} \quad \text{and} \quad C_{\mathrm{eff}} = 1+\varrho \tag{16}$$

by simple arguments with a triangle inequality [7]. Efficiency is robust in $\varrho \to 1$, but reliability is not: The reliability constant $(16)_{\mathrm{a}}$ tends to infinity as $\varrho$ approaches 1. The limit $\varrho = 1$ in (14) reflects that $\|p - p_{h/2}\|$ is indeed smaller than or equal to $\|p - p_h\|$ which is natural for Galerkin-schemes and energy norms. However, the saturation assumption (14) assumes that $\varrho$ is smaller than 1 and bounded away from 1 uniformly for the reliability of the multilevel estimator $\eta_{ML}$. This property is observed for fine meshes in practice and can be monitored during the calculation. The precise conditions under which (14) holds for coarse meshes lead to mild restrictions on the mesh and smooth, namely non-oscillatory, right-hand sides [37, 42]. Details are under current investigation; it should be stressed that $p_{h/2}$ is not really computed (but approximated) in practice [7].

### 2.4 Averaging estimators

The preceding estimator used two meshes and two discrete solutions for error control. Averaging techniques focus on one mesh and one known flux approximation $p_h$ and do not need any underlying residual or boundary value problem at all. The procedure is simply to take a piecewise smooth $p_h$ and approximate it by some globally continuous piecewise polynomials of higher degree $Ap_h$. A simple example, frequently named after Zienkiewicz and Zhu and sometimes even called ZZ estimator, reads as follows: For each node $z \in \mathcal{N}$ and its patch $\omega_z$ let

$$(Ap_h)(z) = \int_{\omega_z} p_h \, dx \Big/ \int_{\omega_z} 1 \, dx \quad \in \mathbb{R}^d \tag{17}$$

be the integral mean of $p_h$ over $\omega_z$. Then, define $Ap_h$ by interpolation with (conforming, i.e. globally continuous) hat functions $\varphi_z$,

$$Ap_h = \sum_{z \in \mathcal{N}} (Ap_h)(z)\, \varphi_z .$$

Given $Ap_h \in \mathcal{S}^1(\mathcal{T})^d$, where $\mathcal{S}^1(\mathcal{T}) = \mathrm{span}\{\varphi_z : z \in \mathcal{N}\}$ denotes the (conforming) first-order finite element space, let the averaging estimator be defined by

$$\eta_A := \|p_h - Ap_h\|. \tag{18}$$

Notice that there is a minimal version

$$\eta_M := \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|p_h - q_h\| \le \eta_A . \tag{19}$$

The efficiency of $\eta_M$ follows from a triangle inequality, namely

$$\eta_M \le \|p - p_h\| + \|p - q_h\| \quad \text{for all} \quad q_h \in \mathcal{S}^1(\mathcal{T}),$$

and the fact that $\|p - p_h\| = \mathcal{O}(h)$ while (in all the examples of this paper)

$$\min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|p - q_h\| = \mathrm{h.o.t.}(p) =: \mathrm{h.o.t.}_{\mathrm{eff}} \, .$$

The latter is of higher order in the sense of $(4)_b$ for a smooth solution $p$ and (3) follows for $\eta = \eta_M$ and $C_{\mathrm{eff}} = 1$.

It turns out that $\eta_A$ and $\eta_M$ are very close and accurate estimators in many numerical examples. This and the fact that the calculation of $\eta_A$ is an easy post processing made $\eta_A$ extremely popular. This paper is devoted to old and new arguments for the reliability of $\eta_M$ and the efficiency of $\eta_A$ in Sects. 4, 5, 6, and 7.

## 3 Estimator competition

This section is devoted to an elliptic model problem

$$1 + \Delta u = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{20}$$

for the $L$-shaped domain $\Omega = (-1,+1)^2 \backslash ([0,1] \times [-1,0])$ and its boundary $\partial\Omega$. The aim is the illustration of the practical performance of error estimators in a model situation.

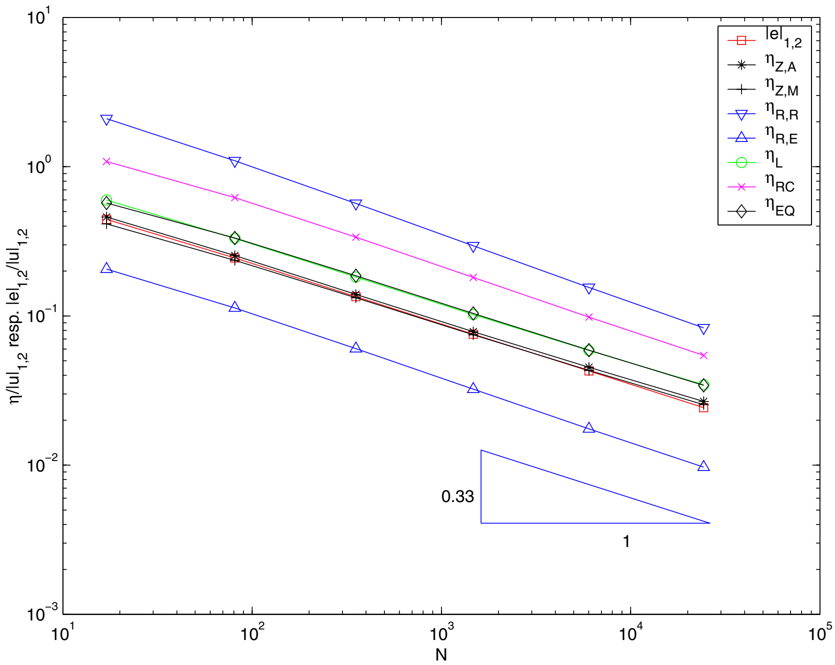


**Fig. 1** Experimental results for Problem (20) with meshes $\mathcal{T}_1, \ldots, \mathcal{T}_6$ and FE spaces (21). The relative error $|e|_{1,2}/|u|_{1,2}$ and various estimators $\eta/|u|_{1,2}$ are plotted as functions of the number of degrees of freedom $N$.

### 3.1 Energy error

The first mesh $\mathcal{T}_1$ consists of 17 free nodes and 48 elements and is obtained by, first, a decomposition of $\Omega$ in 12 congruent squares of size 1/2 and, second, a decomposition of each of the boxes along its two diagonals into 4 congruent triangles. The subsequent meshes are successively red-refined (i.e., each triangle is partitioned into four congruent sub-triangles). Given any mesh $\mathcal{T}_k$ the (conforming) $P_1$ finite element space reads

$$\mathcal{S}_k := \{v_h \in C(\overline{\Omega}) : \forall T \in \mathcal{T}_k,\, v_h|_T \quad \text{affine and} \quad v_h = 0 \quad \text{on} \quad \partial\Omega\} \tag{21}$$

and is of dimension $N = \dim(\mathcal{S}_k)$. The finite element solution $u_h$ is defined by $u_h \in \mathcal{S}_k$ and

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega v_h \, dx \quad \text{for all} \quad v_h \in \mathcal{S}_k. \tag{22}$$

Given the unique (weak) solution $u$ of (20), the error

$$e := u - u_h$$

is estimated in the energy norm (the Sobolev semi-norm in $H^1(\Omega)$)

$$|e|_{1,2} := \left( \int_\Omega |\nabla e|^2 \, dx \right)^{1/2}.$$

There is no analytic formula for an exact solution $u$ and so $|e|_{1,2}$ was computed according to

$$|e|_{1,2}^2 = \int_\Omega \nabla e \cdot \nabla (u + u_h) \, dx = |u|_{1,2}^2 - |u_h|_{1,2}^2 \tag{23}$$

with an extrapolated value $|u|_{1,2}^2 \approx 0.21407315683398$. Fig. 1 displays the values of $|e|_{1,2}$ for different meshes $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_6$ as a function of the number of degrees of freedom $N = 17,\ 81,\ 353,\ 1473,\ 6017,\ 24321,\ 97793$. It is this curve that will be estimated by computable upper and lower bounds. Notice that the two axis in Fig. 1 scale logarithmically such that any algebraic curve of growth $\alpha$ is mapped into a straight line of slope $-\alpha$. The experimental convergence rate is 2/3 in agreement with the (generic) singularity of the domain and resulting theoretical predictions.

### 3.2 Explicit error estimators

The reliable estimator (10) reduces to

$$\eta_{R,R} := \left( \sum_{T\in\mathcal{T}} h_T^2 \|1\|_{L^2(T)}^2 \right)^{1/2} + \left( \sum_{E\in\mathcal{E}_\Omega} h_E \int_E [\partial u_h/\partial \nu_E]^2\, ds \right)^{1/2}, \tag{24}$$

where only the normal component $[p_h]\cdot\nu_E = [\partial u_h/\partial\nu_E]$ of $p_h = \nabla u_h$ jumps across the interior edge $E$; $\mathcal{E}_\Omega$ denotes the subset of $\mathcal{E}$ of all interior edges. Written in the form (24), the estimator $\eta_{R,R}$ is reliable with constant $C_{\mathrm{rel}} = 1$ and $\mathrm{h.o.t.}_{\mathrm{rel}} = 0$ for the class of meshes under consideration [26]. Fig. 1 displays $\eta_{R,R}$ as a function of the number of unknowns $N$. One clearly observes that $\eta_{R,R}$ is an upper bound of the right convergence rate (parallel to the line for $|e|_{1,2}$) and it is indeed well known that $\eta_{R,R}$ is efficient [46]. The efficiency constant $C_{\mathrm{eff}}$ is less established and the higher order terms $\mathrm{h.o.t.}_{\mathrm{eff}}$ usually involve $\|f - f_T\|_{L^2(T)}$ for the elementwise integral mean $f_T$ of the right-hand side. Here, $f\equiv 1$ and so $\mathrm{h.o.t.}_{\mathrm{eff}} = 0$. Following an attempt in [25], Fig. 1 also displays an efficient variant $\eta_{R,E}$ which is a guaranteed lower error bound. The constants in $\eta_{R,E}$ involve geometric data and are derived in [25] to where we refer for details. It is important to observe that

$$\eta_{R,E} \le |e|_{1,2} \le \eta_{R,R}$$

is correct but the range between the lower and the upper bound is very large. One reason is that various geometries (the shape of the patches) lead to different constants and $C_{\mathrm{rel}} = 1$ reflects the worst possible situation in $\mathcal{T}_k$. A local estimation yields an estimator $\eta_{R,C}$ [26] displayed in Fig. 1 as well. This reliable estimator requires the computation of local (patchwise) analytical eigenvalues and hence is very expensive. However, the explicit estimators $\eta_{R,C}$ and $\eta_{R,E}$ still overestimate and underestimate the true error by a huge factor (up to 10 and even more) in the simplest situation possible. This experimental evidence supports the design of more elaborate estimators: Used as a termination criterion (6), the reliable estimators based on (24) may be very cheap and easy. But the decision (6) may be too coarse and be satisfied for a mesh much finer than really necessary and so may yield an unnecessary overkill computation. Said differently, a cheap criterion can be too expensive. It pays to invest into a sharper error estimator.

For the model problem at hand, the volume contribution $\|h_{\mathcal{T}} f\|_2$ can be replaced by the edge contributions plus higher order terms. Different proofs are given in [33, 44]; this fact has been noticed before in [12, 43]. We will come back to this detail in Sect. 5.

### 3.3 Implicit estimators

For comparison, the two implicit estimators $\eta_L$ and $\eta_{EQ}$ are displayed in Fig. 1 as functions of $N$. It is stressed that both estimators are efficient and reliable with

$$C_{\mathrm{rel}} = 1 \quad \text{and} \quad \mathrm{h.o.t.}_{\mathrm{rel}} = 0 = \mathrm{h.o.t.}_{\mathrm{eff}}\,.$$

For the class of meshes $\mathcal{T}_1, \mathcal{T}_2, \ldots$, [26] showed $C_{\mathrm{eff}} = 2.37$ and so

$$|e|_{1,2} \le \eta_L \le 2.37\,|e|_{1,2}.$$

The practical performance of $\eta_L$ and $\eta_{EQ}$ in Fig. 1 is comparable and in fact much sharper than that of $\eta_{R,E}$ and $\eta_{R,R}$ (cf. [25] for more details).

### 3.4 Averaging estimator

The averaging estimators $\eta_A$ and $\eta_M$ of (17)–(19) are as well displayed in Fig. 1 as a function of $N$. Here, $\eta_M$ is efficient up to higher order terms (since the exact solution $u\in H^{5/3-\varepsilon}(\Omega)$ is singular this is not really guaranteed) while its reliability is open, i.e., the corresponding constants have not been computed. Nevertheless, the picture for $\eta_A$ and $\eta_M$ is exclusively seen here from an experimental point of view. The striking numerical result is an amazing high accuracy of $\eta_M \approx \eta_A$ as empirical guesses of $|e|_{1,2}$. If we took $C_{\mathrm{rel}}$ into account, this effect would be destroyed. We observe $\eta_M \le |e|_{1,2} \le \eta_A$ except, perhaps, for the last mesh $\mathcal{T}_6$ (a possible numerical artefact from the extrapolation procedure (23)).

### 3.5 Further results and remarks

**a. Adaptive meshes.** Fig. 1 displays numerical results from [25] for a sequence of uniformly refined meshes. The local refinement indicators $\eta_T$ from (7) yield the basis for refinement rules (8) and resulting automatic mesh-refinement. In the example at hand, the corresponding convergence rates are improved from 2/3 to the [optimal] value 1. This is true for all refinement indicators motivated by the estimators of Sects. 3.1–3.4. For the purpose of adaptive mesh-refining, the cheapest, namely the residual-based explicit estimator, yields the best results. The refinement indicators based on averaging are slightly less effective, at least for the examples reported in [25].

Apart from the optimal convergence rate 1, the pictures in [25] that correspond to Fig. 1 for adapted meshes are similar; the factors of over- and underestimation are quite the same. It is interesting to observe that the three lines for the error $|e|_{1,2}$ and its estimators $\eta_M$ and $\eta_A$ are narrow in Fig. 1 but even closer (almost identical) for the adaptive meshes. One might speculate that the adapted meshes compensate the singularity at the origin and so lead to even higher accuracy, possibly to asymptotic exactness. (Here, the latter property is expected to be based on higher regularity.) Although not published, collaborators observed similar effects for $3D$ problems.

**b. Nonconforming finite element methods.** The error analysis of nonconforming finite element methods is more delicate than for conforming schemes ([15,16,34]). The Crouzeix-Raviart $P_1^{\rm nc}$ finite element method has been analysed in [11, 18] and empirically studied in [24]. Given the discrete space

$$\mathcal{S}_k^{\rm nc} := \big\{v_h \in L^\infty(\overline{\Omega}) : \forall T \in \mathcal{T}_k,\ v_h|_T \text{ affine}, \quad v_h \text{ continuous at midpoints of interior edges} \\ \text{and zero on midpoints of boundary edges}\big\} \tag{25}$$

and the $\mathcal{T}_k$-piecewise gradient $\nabla_{\mathcal{T}_k}$ based on the regular triangulation $\mathcal{T}_k$, the discrete solution $u_h$ is defined by $u_h \in \mathcal{S}_k^{\rm nc}$ and

$$\int_\Omega \nabla_{\mathcal{T}_k} u_h \cdot \nabla_{\mathcal{T}_k} v_h\, dx = \int_\Omega f\, v_h\, dx \quad \text{for all} \quad v_h \in \mathcal{S}_k^{\rm nc}.$$

The discrete flux (i.e. the elementwise gradient) reads

$$p_h := \nabla_{\mathcal{T}_k} u_h$$

and leads to estimators $\eta_E$ in (10), $\eta_L$ in (11), $\eta_{EQ}$ in (13), $\eta_{ML}$ in (15), $\eta_A$ in (18), and $\eta_M$ in (19).

Numerical experiments in [24] support a high accuracy of the averaging estimators $\eta_A \approx \|p - p_h\| \approx \eta_M$ for adaptively refined meshes.

**c. Mixed finite element methods.** The error analysis of mixed finite element methods is even more delicate than for conforming and nonconforming finite element schemes ([15,16,34]). The Raviart Thomas $\mathrm{RT}_0$ finite element method involves the flux $p_h$ as the main variable in

$$\mathrm{RT}_0(\mathcal{T}_k) := \{q_h \in H(\mathrm{div}, \Omega) : \forall T \in \mathcal{T}\ \exists a_T \in \mathbb{R}^d\ \exists b_T \in \mathbb{R},\ q_h(x) = a_T + b_T x \quad \text{for} \quad x \in T\}$$

while the discrete displacements $u_h$ are $\mathcal{T}_k$-piecewise constant, denoted $u_h \in \mathcal{L}^0(\mathcal{T}_k)$. The discrete solution $(p_h, u_h)$ is a pair in $\mathrm{RT}_0(\mathcal{T}_k) \times \mathcal{L}^0(\mathcal{T}_k)$ with

$$\begin{aligned} &\int_\Omega p_h \cdot q_h\, dx + \int_\Omega u_h \operatorname{div} q_h\, dx = 0 && \text{for all } q_h \in \mathrm{RT}_0(\mathcal{T}_k), \\ &\int_\Omega v_h \operatorname{div} p_h\, dx = -\int_\Omega f\, v_h\, dx && \text{for all } v_h \in \mathcal{L}^0(\mathcal{T}_k). \end{aligned} \tag{26}$$

The Sobolev space $H(\mathrm{div}, \Omega)$ is defined as the vector space of all $L^2$ functions whose distributional divergence is in $L^2$. Hence the terms $\operatorname{div} q_h$ and $\operatorname{div} p_h$ make sense in (26) while no derivative of $u_h$ is included. The relation $p = \nabla u$ has only a very weak counterpart in $(26)_{\rm a}$ by an integration by parts [irreversible in the discrete situation]. This gives rise to further residuals analysed in [18, 24]. The reliability of the averaging estimators $\eta_A$ and $\eta_M$ of (18) and (19), respectively, can be found in [24]. The numerical results of that paper show similar effects and support the high accuracy of $\eta_A \approx \|p - p_h\| \approx \eta_M$ for adaptively refined meshes.

**d. Higher order finite element methods.** A local version of averaging schemes is suggested in [10]. For each interior edge $E$ with patch $\omega_E$, $\overline{\omega}_E = T_+ \cup T_-$ for $T_\pm \in \mathcal{T}_k$ with $E = T_+ \cap T_-$, one considers

$$\min_{q_E} \|p_h - q_E\|_{L^2(\omega_E)}\,.$$

Here, $q_E$ is a global polynomial (on $\omega_E$) of degree higher than that of $p_h|_{T_+}$ or $p_h|_{T_-}$. This leads to $\eta_N$, with

$$\eta_N^2 := \tfrac{1}{3} \sum_{E \in \mathcal{E}_\Omega} \min_{q_E} \|p_h - q_E\|^2_{L^2(\omega_E)}. \tag{27}$$

This reliable and efficient estimator has been tested empirically for the $p$-version and polynomial degrees $1, 2, \ldots, 6$. The numerical results underline that (27) is, particularly for higher degrees, an accurate estimator. Although the reliability and efficiency constants depend on the polynomial degrees, this dependency appears to be very mild (in the examples considered) and even an $hp$-version seems to work [10].

## 4 Superconvergence and asymptotic exactness

This section is devoted to review the first justifications of averaging error estimators and to discuss their strong assumptions. Given a regular triangulation $\mathcal{T}_k$ and the discrete space $\mathcal{S}_k$ of (21) the nodal interpolation operator is defined by $(I_h f)(z) = f(z)$, i.e.

$$I_h : C(\overline{\Omega}) \cap H_0^1(\Omega) \to \mathcal{S}_k,\ f \mapsto \sum_{z\in\mathcal{N}} f(z)\varphi_z .$$

Superconvergence means that $u_h - I_h u$ is much smaller than $u_h - u = -e$; for instance, if

$$\|D(u_h - I_h u)\|_{L^2(\Omega)} \le c(u)\, h^{1+\tau}$$

for the maximal mesh-size $h$ in $\mathcal{T}_h$ and some $\tau > 0$. One can read in [1, p. 96] that *the precise assumptions used to obtain such estimator differ according to the type of finite element approximation being used* and *it should be stressed that superconvergence only occurs in very special circumstances*. One old positive result due to M. Wheeler and J. Whiteman illustrates the difficulties.

**Theorem 1.** ([48]) *Let $\Omega_0 \subset \Omega_1 \subset \Omega$ be compactly included for $d = 2$ let $\mathcal{T}_k$ be parallel in $\Omega_1$ (i.e., each edge-patch $\omega_E$ is a parallelogram) and let $u|_{\Omega_1} \in H^3(\Omega_1)$. Then there holds*

$$\|\nabla(u_h - I_h u)\|_{L^2(\Omega_0)} \le C\left(h^2\,\|u\|_{H^3(\Omega_1)} + \|e\|_{L^2(\Omega_1)}\right). \tag{28}$$

The assumptions are extremely strong: No boundary conditions can be involved, very high regularity is required, and the grid is the affine image of a uniform standard triangulation. All this contradicts adaptive mesh-refinement as this a) yields unstructured grids, b) is performed to compensate for singularities of the exact solutions, and, of course, c) aims global error control and aims to reflect the effects of boundary approximation errors. The state of the art of superconvergence results cannot completely overcome those problems [6, 47]. There is, however, the vision that local symmetry in patches plus some regularity yields superconvergence phenomena and the bad influence of singularities, in the solution etc. is observed, but partly compensated by good meshes and so not dominant globally.

The following result provides one example of how superconvergence and (local) symmetry leads to asymptotically exact averaging estimators. To be realistic, we adapt the notation of Theorem 1 and define

$$\eta := \|\nabla u_h - A(\nabla u_h)\|_{L^2(\Omega_0)}$$

as the local version of $\eta_M$.

**Theorem 2.** ([44]) *Under the assumptions of Theorem 1 and if* $\|e\|_{L^2(\Omega_1)} = \text{h.o.t.}$ *there holds*

$$\eta = \|\nabla e\|_{L^2(\Omega_0)} + \text{h.o.t.}$$

P r o o f. It is understood that $h$ is sufficiently small such that $A(\nabla u_h)|_{\Omega_0}$, computed on $\cup\{\omega_z : z \in \mathcal{N} \cap \overline{\Omega}_0\} \subset \Omega$, is based on the structured grid in $\Omega_1$. Given any element $T \in \mathcal{T}_k$ in $\overline{\Omega}_0$ let $\omega_T$ denote the patch of $T$, i.e.

$$\overline{\omega}_T = \cup\{K \in \mathcal{T}_k : K \cap T \ne \emptyset\} \subset \overline{\Omega}_1 .$$

If $u$ is a quadratic polynomial on $\omega_T$, $\nabla u$ is affine and so is $A(\nabla I u)$. The value of $\nabla u - A(\nabla I_h u)$ at any vertex $z$ of $T$ is zero and so $\nabla u - A(\nabla I_h u) \equiv 0$ on $T$. For a proof let $\overline{\omega}_z^{\rm ref} = \operatorname{conv}\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$ be a standard patch of the node $z = (0,0)$. Owing to symmetry one checks for $u(x) = x_1^2$, $u(x) = x_1x_2$, and $u(x) = x_2^2$ by direct calculations that the gradients of $I_h u$ are asymmetric with respect to $z = 0$. Hence $A(\nabla I_h u) = 0$. This means that $\nabla u - A(\nabla I_h u)$ vanishes at $z = 0$ for all polynomials $u$ of degree 2. Since $I_h$ is exact for affine functions and $A$ is exact for constant functions, $A(\nabla u - A(\nabla I_h u))(z) = 0$ for all polynomials $u$ of degree $\le 2$. Finally, an affine transformation of $\omega_z^{\rm ref}$ onto $\omega_z$ yields the assertion on the mesh at hand. Since $z$ is any vertex, $\nabla u - A(\nabla I_h u) \equiv 0$ on $T \in \mathcal{T}_k$.

Since $\nabla v - A(\nabla I_h v)$ vanishes on $T$ for a quadratic polynomial $v$ and since $A$ and $I_h$ are linear operators, we deduce for $u \in H^3(\omega_T)$ that

$$\|\nabla u - A(\nabla I_h u)\|_{L^2(T)} \le \|\nabla(u-v) - A(\nabla I_h(u-v))\|_{L^2(T)} \le \|u-v\|_{H^1(T)} + \|A(\nabla I_h(u-v))\|_{L^2(T)} .$$

Let $v$ be the nodal piecewise quadratic interpolation of $u$. Then $u - v$ vanishes at each node (and at the midpoints of edges) and so $I_h(u-v) = 0$. Hence

$$\|\nabla u - A(\nabla I u)\|_{L^2(T)} \le c_3 h^2\,\|u\|_{H^3(T)}$$

from standard approximation estimates. Since $T$ has been arbitrary in $\overline{\Omega}_0$, we infer

$$\|\nabla u - A(\nabla I_h u)\|_{L^2(\Omega_0)} = \text{h.o.t.} \tag{29}$$

From the $L^2$ continuity of $A$ and from (28) there holds

$$\|A(\nabla u_h - \nabla I_h u)\|_{L^2(\Omega_0)} = \text{h.o.t.} \tag{30}$$

The assertion follows from (29)–(30) and a triangle inequality (twice), namely

$$\begin{aligned} |\eta - \|\nabla e\|_{L^2(\Omega)}| &\le \|\nabla u - A(\nabla u_h)\|_{L^2(\Omega_0)} \\ &\le \|\nabla u - A(\nabla I_h u)\|_{L^2(\Omega_0)} + \|A(\nabla u_h - \nabla I_h u)\|_{L^2(\Omega_0)} \\ &= \text{h.o.t.} + \text{h.o.t.} \end{aligned}$$ □

Historically, the result goes back to Duran, Muschietti, and Rodriguez (1992), Duran and Rodriguez (1992), and Rodriguez (1994) and is reported in [46].

## 5 Reliability on unstructured grids

An old argument for $P_1$ finite elements on regular (unstructured) triangulations and the Laplace equation $L = -\Delta$ compares the edge contributions $h_E \,\|[p_h]\|^q_{L^q(E)}$ and the volume contributions $h_T^q \,\|f\|^q_{L^q(T)}$ of the explicit error estimator (10) (with $p = 2 = q$). In the sequel, $h_\mathcal{T}$ and $h_\mathcal{E}$ denote the piecewise constant mesh-sizes on $\Omega$ and $\cup\mathcal{E}$, respectively; $\cup\mathcal{E}$ is the skeleton of all edges and $\nu_\mathcal{E}$ is the unit normal along $\cup\mathcal{E}$. Let us assume that each element has at least one vertex inside the domain.

**Theorem 3.** *There is an $h$-independent constant $c_4 > 0$ such that there holds*

$$\|h_\mathcal{T} f\|_{L^q(\Omega)} \le c_4 \left\| h_\mathcal{E}^{1/q} [p_h] \right\|_{L^q(\cup\mathcal{E}_\Omega)} + c_4 \left( \sum_{z \in \mathcal{N}\cap\Omega} \min_{f_z \in \mathbb{R}} h_z^q \,\|f - f_z\|^q_{L^q(\omega_z)} \right)^{1/q} .$$

**Remark 1.** The point is that, for $P_1$ finite element schemes, the volume contribution is dominated by the edge contribution and higher order terms.

**Remark 2.** The theorem holds for conforming and nonconforming finite element schemes. In the first case, only the jump of the normal components enter the upper bound.

**Remark 3.** The theorem is found in [45] but the arguments have been employed [independently], e.g., in [12, 43].

Proof of Theorem 3. Since each element has at least one vertex inside the domain, each boundary node $a$ has at least one neighbouring node $b$ (i.e., $a$ and $b$ belong to the same element) inside the domain. Fix one selection $\mathcal{I}$ of such pairs $(a, b)$ and add $(z, z)$ for all interior nodes. Then let

$$\psi_z := \sum_{\substack{x \in \mathcal{N} \\ (x,z) \in \mathcal{I}}} \varphi_x \quad \text{for} \quad z \in \mathcal{K} := \mathcal{N} \cap \Omega. \tag{31}$$

For interior nodes $z$ without any link to a neighbouring node on the boundary, $(z, z)$ is the only chosen pair in (31) and $\psi_z = \varphi_z$. The purpose of (31) is simply not to enlarge the patches ($\operatorname{supp} \psi_z = \overline{\omega}_z$) too much but to have a partition of unity (with parameter se $\mathcal{K}$ of free nodes)

$$\sum_{z \in \mathcal{K}} \psi_z \equiv 1 \quad \text{in} \quad \Omega.$$

Because of this and since $h_\mathcal{T} \le h_z$ on $\omega_z$, there holds

$$\|h_\mathcal{T} f\|^q_{L^q(\Omega)} = \sum_{z \in \mathcal{K}} \left\| \psi_z^{1/q} h_\mathcal{T} f \right\|^q_{L^q(\omega_z)} \le \sum_{z \in \mathcal{K}} h_z \left\| \psi_z^{1/q} f \right\|^q_{L^q(\omega_z)} . \tag{32}$$

For some fixed $f_z \in \mathbb{R}$ there holds

$$\left\| \psi_z^{1/q} f \right\|_{L^q(\omega_z)} \le \left\| \psi_z^{1/q} f_z \right\|_{L^q(\omega_z)} + \left\| \psi_z^{1/q} (f - f_z) \right\|_{L^2(\omega_z)} . \tag{33}$$

With $c_5 := \max\{\|\psi_z\|_{L^q(\omega_z)} / \|\varphi_z\|_{L^q(\omega_z)} : z \in \mathcal{K}\} \le (d+1)!$ it remains to estimate

$$\left\|\psi_z^{1/q} f_z\right\|_{L^q(\omega_z)} \le c_5 \left\|\varphi_z^{1/q} f_z\right\|_{L^q(\omega_z)} = c_5 |f_z|^{1/p} \, \|\varphi_z f_z\|_{L^1(\omega_z)}^{1/q} . \tag{34}$$

The main argument for the estimation of

$$\|\varphi_z f_z\|_{L^1(\omega_z)} = \left| \int_\Omega f \, \varphi_z \, dx + \int_\Omega \varphi_z \, (f_z - f) \, dx \right| \tag{35}$$

is the discrete equation with $\varphi_z$ as a test function followed by an integration by parts in the model problem, i.e.

$$\int_\Omega f \, \varphi_z \, dx = \int_\Omega p_h \cdot \nabla \varphi_z \, dx = \int_{\cup\mathcal{E}} \varphi_z \, [p_h] \cdot \nu_\mathcal{E} \, ds \tag{36}$$

for the $\mathcal{T}_h$-piecewise constant discrete flux $p_h = \nabla_{\mathcal{T}_h} u_h$. The combination of (35)–(36) with Hölder's inequality yields

$$\begin{aligned}
\|\varphi_z f_z\|_{L^1(\omega_z)} &\le \|\varphi_z \, [p_h] \cdot \nu_\mathcal{E}\|_{L^1(\cup\mathcal{E})} + \|\varphi_z (f - f_z)\|_{L^1(\omega_z)} \\
&\le \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})} \left\|h_\mathcal{E}^{-1/q} \varphi_z^{1/p}\right\|_{L^p(\cup\mathcal{E})} + \left\|\varphi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} \left\|\varphi_z^{1/p}\right\|_{L^p(\omega_z)} \qquad (37) \\
&\le \left\|\varphi_z^{1/p}\right\|_{L^p(\omega_z)} \left( \left\|\varphi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} + c_6 h_z^{-1} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\omega_z)} \right) .
\end{aligned}$$

Here we considered the mesh-size independent constant

$$c_6 := \max_{z \in \mathcal{K}} h_z \left\|h_\mathcal{E}^{-1/q} \varphi_z^{1/p}\right\|_{L^p(\cup\mathcal{E}_z)} \Big/ \left\|\varphi_z^{1/q}\right\|_{L^p(\omega_z)} .$$

The combination of (34) and (37) plus Young's inequality show

$$\begin{aligned}
&\left\|\psi_z^{1/q} f_z\right\|_{L^q(\omega_z)}^q \\
&\le c_5^q |f_z|^{q/p} \, \|\varphi_z f_z\|_{L^1(\omega_z)} \\
&\le \left\|\varphi_z^{1/p}\right\|_{L^p(\omega_z)} |f_z|^{q/p} \left( c_5^q \left\|\varphi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} + c_5^q c_6 h_z^{-1} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})} \right) \\
&\le \frac{1}{p} \, |f_z|^q \, \|\varphi_z\|_{L^1(\omega_z)} + \frac{1}{q} \left( c_5^q \left\|\varphi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} + c_5^q c_6 h_z^{-1} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})} \right)^q .
\end{aligned}$$

Since $1 - \frac{1}{p} = \frac{1}{q}$ and $\left\|\varphi_z^{1/q} f_z\right\|_{L^q(\omega_z)} \le \left\|\psi_z^{1/q} f_z\right\|_{L^q(\omega_z)}$ this (after taking the $q$-th root) leads to

$$\left\|\psi_z^{1/q} f_z\right\|_{L^q(\omega_z)} \le c_5^q \left\|\varphi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} + c_5^q c_6 h_z^{-1} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})} . \tag{38}$$

The combination of (33) and (38) plus another Hölder inequality (in $\mathbb{R}^2$) shows

$$\begin{aligned}
\left\|\psi_z^{1/q} f\right\|_{L^q(\omega_z)} &\le (1 + c_4^q) \left\|\psi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)} + c_5^q c_6 h_z^{-1} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})} \\
&\le c_4 \left( \left\|\psi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)}^q + h_z^{-q} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})}^q \right)^{1/q}
\end{aligned}$$

for $c_4 := \left((1 + c_5^q)^p + (c_5^q c_6)^p\right)^{1/p}$. This and (32) yield

$$\|h_\mathcal{T} f\|_{L^q(\Omega)}^q \le c_4^q \sum_{z \in \mathcal{K}} h_z^q \left\|\psi_z^{1/q} (f - f_z)\right\|_{L^q(\omega_z)}^q + c_4^q \sum_{z \in \mathcal{K}} \left\|h_\mathcal{E}^{1/q} \varphi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})}^q .$$

With $\sum_{z \in \mathcal{K}} \left\|h_\mathcal{E}^{1/q} \psi_z^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})}^q = \left\|h_\mathcal{E}^{1/q} [p_h] \cdot \nu_\mathcal{E}\right\|_{L^q(\cup\mathcal{E})}^q$, this concludes the proof of Theorem 3. □

**Theorem 4.** *There exists a mesh-size independent constant $c_7 > 0$ such that there holds*

$$\eta_E := \left\|h_\mathcal{E}^{1/q} [p_h]\right\|_{L^2(\cup\mathcal{E}_\Omega)} \le c_7 \min_{q_h \in \mathcal{S}^1(\mathcal{T})^d} \|p_h - q_h\|_{L^q(\Omega)} \quad \textit{for all} \quad p_h \in \mathcal{L}^0(\mathcal{T})^d .$$

**Remark 4.** The combination of Theorem 3 and 4 shows reliability of $\eta_M$ and $\eta_A$, in terms of $\eta_R$, namely

$$\eta_R \le c_4 c_7 \min_{q_h\in\mathcal{S}^1(\mathcal{T})^d} \|p_h - q_h\|_{L^q(\Omega)} + \text{h.o.t.}$$

**Remark 5.** Theorem 4 is mentioned in [24] and possibly expected by some experts since [44]. A proof of the corollary

$$\eta_E \le c_8\,\eta_A$$

can be found in [46].

P r o o f of Theorem 4. The main argument is an equivalence of (semi-)norms on finite-dimensional spaces which leads to inverse estimates, here,

$$h_E \int_E |[p_h]|^q\,ds \le c_8 \min_{q_h\in\mathcal{S}^1(\mathcal{T}_z)^d} \|p_h - q_h\|^q_{L^q(\omega_z)}\,. \tag{39}$$

This estimate involves an arbitrary edge or face $E \in \mathcal{E}_\Omega$ with one fixed vertex $z \in E\cap\mathcal{N}$ and its patch $\omega_z$. The estimate holds for all $p_h \in \mathcal{L}^0(\mathcal{T}_z)^d$ where $\mathcal{T}_z := \{T \in \mathcal{T} : z \in T\cap\mathcal{N}\}$ is the sub-grid of $\mathcal{T}$ on $\omega_z$. The $q$-th root of the left- and right-hand side of (39), respectively, defines a seminorm on $\mathcal{L}^0(\mathcal{T}_z)^d$ (where $p_h$ belongs to). If the right-hand side vanishes for some $p_h \in \mathcal{L}^0(\mathcal{T}_z)^d$ then $p_h$ is $\mathcal{T}_z$-piecewise constant and (equal to some $q_h \in \mathcal{S}^1(\mathcal{T}_z)^d$ and so) continuous on $\omega_z$. Thus $p_h$ is constant on $\omega_z$ and so $[p_h]$ vanishes on $E$ as well. Hence the left-hand side of (39) is associated with a seminorm on $\mathcal{L}^0(\mathcal{T}_z)^d$ stronger than the right-hand side. Since $\dim\mathcal{L}^0(\mathcal{T}_z)^d$ is bounded by $d\,\mathrm{card}(\mathcal{T}_z)$ and hence uniformly in $h$, there follows an estimate (39) with some constant $c_8$ that may depend on the geometry $\mathcal{T}_z$. A scaling argument (make $\omega_z \ni 0$ bigger by a constant factor and compute the left- and right-hand side of (39) for a proof) shows that $c_8$ does *not* depend on the diameter of $\omega_z$. This proves (39).

The sum over all edges in (39) with some choice of related vertices $z = z_E$ leads to

$$\sum_{E\in\mathcal{E}_\Omega} h_E \int_E |[p_h]|^q\,ds \le c_8 \sum_{E\in\mathcal{E}_\Omega} \min_{q_h\in\mathcal{S}^1(\mathcal{T}_{z_E})^d} \|p_h - q_h\|^q_{L^q(\omega_{z_E})} \le c_8 \min_{q_h\in\mathcal{S}^1(\mathcal{T})^d} \sum_{E\in\mathcal{E}_\Omega} \|p_h - q_h\|^q_{L^q(\omega_{z_E})}. \tag{40}$$

(Notice that $q_h$ plays a different role in the last two terms and $q_h|_{\omega_{z_E}}$ gives less flexibility in the latter term and so leads to the inequality.) The patches $(\omega_{z_E} : E \in \mathcal{E}_\Omega)$ have a finite overlap bounded by $\max_{T\in\mathcal{T}} \mathrm{card}\{E \in \mathcal{E} : T\cap E \ne \emptyset\} \le c_9$. Hence the upper bound in (40) is smaller than or equal to the $q$-th power of the right-hand side of Theorem 4 with $c_7 = (c_8 c_9)^{1/q}$. □

An alternative proof of reliability of $\eta_M$ follows [24, 25] with an argument of [33].

## 6 Efficiency on unstructured grids

The converse to Theorem 4 holds on unstructured grids for the residual estimators $\eta_E$. The next theorem therefore implies efficiency for $\eta_A$ as well.

**Theorem 5.** *There exists a mesh-size-independent constant $c_{10}$ such that for any $p_h \in \mathcal{L}^0(\mathcal{T})$ and $Ap_h$ from* (17) *there holds*

$$\min_{q_h\in\mathcal{S}^1(\mathcal{T})^d} \|p_h - q_h\|_{L^q(\Omega)} \le \|p_h - Ap_h\|_{L^q(\Omega)} \le c_{10} \left\| h_{\mathcal{E}}^{1/q}[p_h] \right\|_{L^q(\cup\mathcal{E}_\Omega)}.$$

**Remark 6.** Theorem 5 is shown in the context of mixed boundary conditions in [20] and, furthermore, provides a quantitative bound on $c_{10}$.

**Remark 7.** Similar estimates have been observed before [10] and have certainly been known to the experts.

P r o o f of Theorem 5. The first inequality of the assertion is obvious (since $Ap_h \in \mathcal{S}^1(\mathcal{T})^d$) and the proof concentrates on the second. An inverse estimate

$$\|p_h - \fint_{\omega_z} p_h\,dx\|^q_{L^q(\omega_z)} \le c_{11} \sum_{E\in\mathcal{E}_z} h_E \int_E |[p_h]|^q\,ds \tag{41}$$

is the main argument similar to the first step in the proof of Theorem 4. In (41), $z$ is an arbitrary node with patch $\omega_z$ and $\fint_{\omega_z} p_h\,dx$ denotes the average of any $p_h \in \mathcal{L}^0(\mathcal{T}_z)^d$ over $\omega_z$. Finally, $\mathcal{E}_z = \{E \in \mathcal{E}_\Omega : z \in E\cap\mathcal{N}\}$ denotes the set of all edges with vertex $z$. The main point is to see in (41) that any $p_h$ with right-hand side zero has no jumps on $\omega_z$ and is therefore constant. Thus, $p_h$ annulates the left-hand side of (41) as well. The remaining arguments are the same as in the proof of Theorem 4 and hence neglected.

The assertion of Theorem 5 follows with Hölder inequalities from (41): Let $p_z := (Ap_h)(z)$ and calculate

$$\begin{aligned}
\|p_h - Ap_h\|^q_{L^q(\Omega)} &= \int_\Omega \left| \sum_{z\in\mathcal{N}} (p_h - p_z)\varphi_z \right|^q dx \\
&\le \int_\Omega \left( \sum_{z\in\mathcal{N}} \varphi_z \right)^{q/p} \left( \sum_{z\in\mathcal{N}} |p_h - p_z|^q \varphi_z \right) dx \\
&= \sum_{z\in\mathcal{N}} \int_{\omega_z} \varphi_z \, |p_h - p_z|^q \, dx \\
&\le \sum_{z\in\mathcal{N}} \left\| p_h - \fint_{\omega_z} p_h \, dx \right\|^q_{L^q(\omega_z)} \\
&\le c_{11} \sum_{z\in\mathcal{N}} \sum_{E\in\mathcal{E}_z} h_E \int_E |[p_h]|^q \, ds \\
&= c_{11} \sum_{E\in\mathcal{E}_\Omega} \sum_{z\in E\cap\mathcal{N}} h_E \int_E |[p_h]|^q \, ds \\
&\le c_{11} \max_{E\in\mathcal{E}} \operatorname{card}(E\cap\mathcal{N}) \sum_{E\in\mathcal{E}} h_E \int_E |[p_h]|^q \, ds.
\end{aligned}$$

□

## 7 Applications

This section gives an overview of some applications to Stokes and Lamé equations or to elastoplastic, obstacle, and degenerated problems in which averaging techniques are known to work. The arguments for the reliability and efficiency proof are partly those from the previous Sects. 5 and 6 plus particular techniques.

### 7.1 Stokes' equations

The stationary viscous flow inside a bounded volume $\Omega \subset \mathbb{R}^2$ of viscosity $\mu > 0$ is described by the velocity field $u : \Omega \to \mathbb{R}^2$ and the pressure variable $p : \Omega \to \mathbb{R}$. Since it appears to be the more complicated situation, mixed boundary conditions are assumed, i.e. $\Gamma := \partial\Omega = \Gamma_N \cup \Gamma_D$ for the closed Dirichlet boundary $\Gamma_D$ and the relatively open Neumann (or traction) boundary $\Gamma_N$. Suppose that both, $\Gamma_D$ and $\Gamma_N$, are of positive surface measure. Let $\nu$ denote the exterior unit normal vector along the Lipschitz boundary $\Gamma$. Then the *Stokes problem* reads: Given $f \in L^2(\Omega)^2$, $u_D \in H^1(\Omega)^2$, and $g \in L^2(\Gamma_N)^2$, find $(u, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ with

$$\begin{aligned}
\operatorname{div}\sigma + f &= 0 \quad \text{in} \quad \Omega, \\
\sigma &:= 2\mu\,\varepsilon(u) - pI, \\
\varepsilon(u) &:= (\nabla u + \nabla u^T)/2, \\
\operatorname{div} u &= 0 \quad \text{in} \quad \Omega, \\
u &= u_D \quad \text{on} \quad \Gamma_D, \\
\sigma\nu &= g \quad \text{on} \quad \Gamma_N.
\end{aligned}$$

(Here, $I$ denotes the $2 \times 2$-identity matrix and $u_D$, $f$, and $g$ are smoother for the results below.)

**Remark 8.** In case $\Gamma_N = \emptyset$ and $\Gamma_D = \Gamma$ one requires an additional constraint such as $\int_\Omega p(x)\,dx = 0$ to fix a global additive constant in the pressure variable.

**Remark 9.** The above formulation is based on the symmetric strain rate $\varepsilon(u)$ and leads to the correct model from the physical point of view. This is important since traction boundary conditions $\sigma\nu = g$ are present. In the literature, one often finds the non-symmetric form with $\sigma = 2\mu\nabla u - pI$. It is known that the two models are equivalent if $\Gamma_N = \emptyset$, $\Gamma = \Gamma_D$; but it is emphasised that this is untrue in the present situation.

The weak form of the (above symmetric) Stokes problem is straightforward and leads to a mixed FEM with unknown $u_h$ and $p_h$. The proper choice of finite spaces for the discrete velocities $u_h$ and the discrete pressures $p_h$ is less trivial; particularly for piecewise polynomials of low-order. For instance, Kouhia and Stenberg considered $u_h$ in $\mathcal{S}_k \times \mathcal{S}_k^{\mathrm{nc}}$ (cf. (21) and (25) for $\mathcal{T}_k$) and piecewise constant $p_h$.

It is known that, for sufficiently fine meshes, there exist unique exact and discrete solutions $(u,p)$ and $(u_h,p_h)$, respectively, with quasioptimal a priori error bounds [41]. A posteriori error estimates are studied in [9,27]. In particular, for a discrete stress field

$$\sigma_h := 2\mu\varepsilon_{\mathcal{T}}(u_h) - p_h I,$$

there holds reliability and efficiency of

$$\eta_A := \|\sigma_h - A\sigma_h\|_{L^2(\Omega)} \leq \eta_M := \min \|\sigma_h - \tau_h\|_{L^2(\Omega)} \quad \text{for} \quad \tau_h \in \mathcal{S}^1_N(\mathcal{T}).$$

Here $\mathcal{S}^1_N(\mathcal{T})$ is the subspace of symmetric matrices in $\mathcal{S}_1(\mathcal{T})^{d\times d}$ which satisfy the traction boundary conditions at each node on $\overline{\Gamma}_N$ and $A\sigma_h \in \mathcal{S}^1_N(\mathcal{T})$. For proofs and details see [27]. It is surprising, but is a result of the mathematical analysis, that the averaging concerns the stress field $\sigma_h$ only and not the variables $\varepsilon_{\mathcal{T}}(u_h)$ and $p_h$ separately. The numerical examples in [27] underline the high accuracy of $\eta_A$ in this application.

### 7.2 Linear elasticity

The small deformations $u : \Omega \to \mathbb{R}^2$ of a 2D elastic body are modelled by the *Navier-Lamé equations*: Given $f \in L^2(\Omega)^2$, $u_D \in H^1(\Omega)^2$, $g \in L^2(\Gamma_N)^2$, find $u \in H^1(\Omega)^2$ with

$$\begin{aligned}
\operatorname{div}\sigma + f &= 0 \quad \text{in} \quad \Omega,\\
\sigma &:= \mathbb{C}\,\varepsilon(u) := \lambda \operatorname{tr}(\varepsilon(u))I + 2\mu\,\varepsilon(u),\\
\varepsilon(u) &:= (\nabla u + \nabla u^T)/2,\\
u &= u_D \quad \text{on} \quad \Gamma_D,\\
\sigma\nu &= g \quad \text{on} \quad \Gamma_N.
\end{aligned}$$

The setting $\Gamma = \Gamma_D \cup \Gamma_N$ is the same as in the preceding application. The new aspect is compressibility with the material constant $\lambda$ which tends to infinity as the Poisson ratio tends to 1/2 for rubber-like materials. It is known in the incompressible limit for $\lambda \to \infty$, that the elastic problem turns into the Stokes problem as $\lambda \operatorname{tr}(\varepsilon(u)) \to p$ and $\operatorname{tr}\varepsilon(u) = \operatorname{div}(u) \to 0$. In principle, the linear elastic problem can be discretized by $u_h$ in $\mathcal{S}_1 \times \mathcal{S}_1$. This leads to quasioptimal convergence in the energy norm

$$\left\|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\right\| \leq C(\lambda) h_{\max} \left\|D^2 u\right\|_{L^2(\Omega)}$$

for a smooth exact solution $u$. Therein, the multiplicative constant $C(\lambda)$ may deteriorate if $\lambda \to \infty$. Fig. 2 displays a numerical example from [28] for three different materials with Poisson ratios $nu = 0.3$, $0.49$, and $0.499$. For a uniform sequence of meshes on an $L$-shaped domain $\Omega$ with known singular exact solution $u$, we computed discrete solutions $u_h \in \mathcal{S}_1 \times \mathcal{S}_1$ within a standard $P_1$ FE. The error of the corresponding discrete stress $\sigma_h := \mathbb{C}\,\varepsilon(u_h)$, with the fourth-order material tensor $\mathbb{C}$ in $\sigma = \mathbb{C}\,\varepsilon(u)$, reads, in its energy norm

$$\left\|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\right\|^2 := \int_\Omega (\sigma - \sigma_h) : \varepsilon(u - u_h)\, dx.$$

The quantity $\|\,\mathbb{C}^{-1/2}(\sigma - \sigma_h)\|$ is plotted as a function of the number of degrees of freedom in Fig. 2. For uniform meshes we observe a suboptimal convergence rate caused by the singularity of $u$. The experimental convergence rate appears to be independent of the Poisson ratio $nu$ (i.e. of $\lambda$) in contrast to the multiplicative constant $C(\lambda)$. This phenomenon is called locking [15]: In Fig. 2, the numerical result for a uniform mesh with $N = 10000$ degrees of freedom and $nu = 0.499$ is worse than that for the coarsest mesh with $N = 16$ degrees of freedom for $nu = 0.3$. The situation is even more dramatic for larger and larger $\lambda \to \infty$.

Fig. 2 displays three sequences of adaptively refined meshes for $nu = 0.3$, $0.49$, and $0.499$ as well. The coarse meshes coincide with the results for the uniformly refined ones but then adaptivity improves with a convergence rate larger than 1 until the error is much smaller. Said differently, the effect of the multiplicative constant $C(\lambda)$ is seen in the beginning for $N \leq 100$ and decreases for larger $N$. Does this indicate the conjecture that adaptivity overcomes locking?

The error control by averaging schemes via

$$\eta_M := \min_{\tau_h \in \mathcal{S}^1_N(\mathcal{T})} \left\|\,\mathbb{C}^{-1/2}(\sigma_h - \tau_h)\right\|_{L^2(\Omega)} \leq \eta_A := \left\|\,\mathbb{C}^{-1/2}(\sigma_h - A\sigma_h)\right\|_{L^2(\Omega)}$$

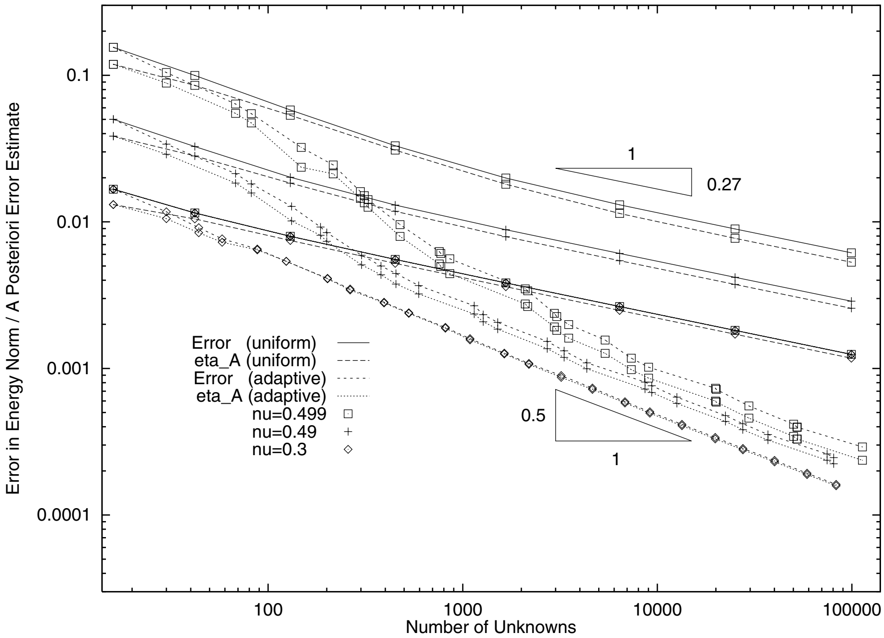


**Fig. 2** Locking in compressible linear elasticity: The energy norm $\left\|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\right\|$ and estimators $\eta_A$ are plotted versus the number of unknowns $N$ for uniform and adapted meshes in the $P_1 \times P_1$ conforming FEM.

is displayed in Fig. 2 as well. It is proved in [28] that $\eta_M \leq \eta_A$ is reliable up to $\lambda$-depending constants and, clearly, $\eta_M$ is efficient with respect to $\lambda$-independent constants. As observed in Fig. 2, even a very poor finite element solution is estimated very accurately.

The preceding discussion focused on conforming FEM and large errors caused by locking in the incompressible limit $\lambda \to \infty$. More appropriate FEMs can overcome this locking. The first hint is to use ansatz and test functions which lead to stable FEM for the Stokes problem regarded as the limit problem for incompressibility. The choice of $\mathcal{S}_k \times \mathcal{S}_k^{\mathrm{nc}}$ due to Kouhia and Stenberg [41] is appropriate for that and led in [28] to robust and accurate estimates. Therein, the finite element schemes as well as their error estimators are highly accurate and $\lambda$-independent.

### 7.3 Elastoplasticity

This section briefly describes the perspectives and limitations of averaging techniques in elastoplastic evolution problems. Therein, a time-discretisation is performed followed by a spatial discretisation in each time-step. Averaging error estimators $\eta_M \leq \eta_A$ for the exact and discrete stress field $\sigma$ and $\sigma_h$, respectively, have the same definition as in Sect. 7.2 for each time step. It can be proved for an implicit time-discretisation that

$$\left\|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\right\|^2 \leq \int_\Omega (\sigma - \sigma_h) : \varepsilon(u - u_h)\, dx = \int_\Omega (\sigma - \sigma_h) : \varepsilon(u - u_h - v_h)\, dx \quad \text{for all} \quad v_h \in \mathcal{S}_h \times \mathcal{S}_h .$$

This term is an upper bound for error terms such as the stress error in the energy norm. Moreover, if hardening is present, the stress error controls the displacement error $\|u - u_h\|_{H^1(\Omega)}$ up to hardening-depending multiplicative constants. For details and proofs we refer to [2, 3, 17]. Therefore, one can proceed as in linear elasticity to derive reliability and efficiency of $\eta_M \leq \eta_A$. The constant $C_{\mathrm{rel}}$, however, depends crucially on the hardening; the estimates are *not* valid (in this form) for perfect plasticity.

It should be emphasised that reliability holds solely for the spatial discretisation; the accumulated error in time is *not* controlled by $\eta_M \leq \eta_A$; the result holds for Hencky materials only. The control of the time-discretisation error appears to be an important open question. However, the numerical results in [3] provide numerical evidence that $\eta_A$ is indeed a very accurate (spatial) error estimator.

### 7.4 Obstacle problems

This section briefly reports on the surprising result that, for a nonlinear variational inequality (in a simple situation) the same averaging estimator

$$\eta_M := \min_{q_h \in \mathcal{S}_h(\mathcal{T})^d} \|p_h - q_h\| \leq \eta_A := \|p_h - A p_h\|$$

as for the variational equation is reliable and efficient: An affine obstacle has no further influence! To be more precise, let $K$ denote the set of admissible deformations,

$$K := \{v \in H^1_0(\Omega) : 0 \le v \quad \text{almost everywhere in} \quad \Omega\} \quad \text{with} \quad H^1_0(\Omega) := \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \partial\Omega\}.$$

Then the weak form of the obstacle problem reads: Given $f \in L^2(\Omega)$ find $u \in K$ with

$$\int_\Omega \nabla u \cdot \nabla(u - v)\, dx \le \int_\Omega f(u - v)\, dx \quad \text{for all} \quad v \in K.$$

The FE discretisation replaces $K$ by the discrete version

$$K_h := K \cap \mathcal{S}_1$$

and hence determines $u_h \in K_h$ with

$$\int_\Omega \nabla u_h \cdot \nabla(u_h - v_h)\, dx \le \int_\Omega f(u_h - v_h)\, dx \quad \text{for all} \quad v_h \in K_h.$$

The main difference of the variational inequality and the model example of Sect. 3 can be expressed in the residuals $\varrho \in H^{-1}(\Omega)$ and $\varrho_h \in \mathcal{S}_1$,

$$\varrho(v) := \int_\Omega v\, f\, dx - \int_\Omega \nabla u \cdot \nabla v\, dx \quad \text{for all} \quad v \in H^1(\Omega),$$
$$\varrho_h := \sum_{z \in \mathcal{K}} \left( \int_\Omega f\, \varphi_z\, dx - \int_\Omega \nabla u_h \cdot \nabla \varphi_z\, dx \right) \psi_z \Big/ \int_\Omega \varphi_z\, dx \quad \in \mathcal{S}_1.$$

It is elementary to verify that the error $e := u - u_h$ in the energy norm reads

$$|e|^2_{1,2} = \int_\Omega f(e - e_h)\, dx - \int_\Omega \nabla u_h \cdot \nabla(e - e_h)\, dx + \int_\Omega \varrho_h\, e\, dx - \varrho(e)$$

for some $e_h := \sum_{z \in \mathcal{K}} \left( \int_\Omega e \psi_z\, dx \right) \varphi_z \big/ \int_\Omega \varphi_z\, dx \in \mathcal{S}_1$. The approximation error $e - e_h$ can be analysed as in Sect. 5. The additional terms with $\varrho_h$ and $\varrho$ reflect the variational inequality. Indeed, one can show that $0 \le \varrho(e)$ and

$$\varrho_h(z) \le 0 = \varrho_h(z) u_h(z) \le u_h(z) \quad \text{for all} \quad z \in \mathcal{K}.$$

Hence the arguments of Sect. 5 lead to the a posteriori error estimate [23]

$$|e|^2_{1,2} \le C\eta_M^2 + \text{h.o.t.}(f) - \int_\Omega \varrho_h\, u_h\, dx.$$

The last term can be analysed further. Indeed, $\varrho_h u_h$ vanishes on an element $T \in \mathcal{T}$ or, at least, $\varrho_h(a) < 0 = u_h(a) = \varrho_h(b) < u_h(b)$ for two nodes $a$ and $b$ of $T$. Inverse inequalities based on $\varrho_h \le 0 \le u_h$ yield a bound of $\|\varrho_h u_h\|_{L^1(T)}$ in terms of the mesh-size, $|\varrho_h|_{1,2}$, and $\|\nabla u_h - A(\nabla u_h)\|_2$. The details and proofs can be found in [23]. The final result reads

$$|e|_{1,2} \le C_{\text{rel}} \eta_M + \text{h.o.t.}(f).$$

For non-affine obstacles and nonconforming discretisations (i.e. $K_h \not\subset K$) some consistency terms arise and may dominate the upper bound, cf. [23]. Numerical results in [23] provide empirical evidence for a surprisingly high accuracy of $\eta_A$.

### 7.5 Degenerate problems

The preceding examples concerned uniformly convex minimisation problems on affine or convex subsets. The $p$-Laplacian is a first nonlinear equation with less strong convexity which requires a particular analysis. This is based on an appropriate quasi-norm, a metric that depends on the exact or discrete solution. The techniques of Sects. 5 and 6, however, can be adopted to this setting and then yield reliable and efficient error estimators [31].

The situation is even more difficult and essentially open for convexified problems where the energy minimisation functional is not strictly convex. Very much as a surprise came numerical evidence in a 2-well benchmark example allowing for microstructures that $\eta_A$ yields an accurate stress error estimation [30].

## 8 Averaging on large patches

This section aims to sketch future prospectives of averaging techniques based on an old argument that, the author learned from L.B. Wahlbin, goes back to an Oberwolfach conference in the eighties and is known to the experts since then. The technique is employed in [40] for a posteriori $L^\infty$ estimates. In this section we sketch only one (much) simpler result for the energy norm. Suppose that we are given two meshes $\mathcal{T}_H$ and $\mathcal{T}_h$ such that $\mathcal{T}_h$ is a uniform refinement of $\mathcal{T}_H$ such that typical mesh-sizes $H$ and $h$ of $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively, satisfy

$$H^2 \ll h \ll H.$$

Suppose the notation of Sect. 3 for $\mathcal{T} = \mathcal{T}_h$ with exact solution $u$ and discrete (fine) solution $u_h$. Let $\mathcal{S}_2(\mathcal{T}_H)$ denote the $\mathcal{T}_H$-piecewise quadratic polynomial subspace of $C(\Omega)$. Let $u_H$ and $u_{hH}$ denote the finite element approximations in $\mathcal{S}_2(\mathcal{T}_H)$ of $u$ and $u_h$, respectively.

**Theorem 6.** (Essentially known to the experts). *There exists a constant $c_{12}$ such that for smooth $u$*

$$\eta := \|\nabla u_h - \nabla u_{hH}\| \quad \textit{is computable}$$

*and reliable for small $h/H$ in the sense that*

$$\|\nabla u - \nabla u_h\| \le \frac{\eta + \mathrm{h.o.t.}(u)}{1 - c_{12} h/H}$$

*and always efficient*

$$\eta \le \|u - u_h\| + \mathrm{h.o.t.}(u).$$

Here, $\mathrm{h.o.t.}(u)$ means of higher order when compared with $\|\nabla u - \nabla u_h\|$. For an unstructured grid $\mathcal{T}_H$, $H$ denotes the minimal mesh-size in $\mathcal{T}_H$ and $h$ denotes the maximal mesh-size in $\mathcal{T}_h$. The constant $c_{12}$ depends on the shape of the elements through an interpolation estimate, inverse inequality, and a stability estimate as seen in the proof.

Proof of Theorem 6. For $e := u - u_h$ and $v := u_H - u_{hH}$ with interpolant $v_h$ in $\mathcal{S}^1(\mathcal{T}_h)$, the Galerkin orthogonality implies

$$\|\nabla e\|^2 = \int_\Omega \nabla e \cdot \nabla(e - v)\, dx + \int_\Omega \nabla e \cdot \nabla(v - v_h)\, dx.$$

Since $e - v = (u_{hH} - u_h) + (u - u_H)$, a Cauchy inequality gives

$$|e|_{1,2}^2 \le \|\nabla e\| \left(\|\nabla u_h - \nabla u_{hH}\| + \|\nabla u - \nabla u_H\|\right) + \|h/H\, \nabla e\|\, \|H/h\, \nabla(v - v_h)\|.$$

The quadratic approximation of the smooth function $u$ leads to a higher order term,

$$\|u_h - u_{hH}\| + \|u - u_H\| = \eta + \mathrm{h.o.t.}(u).$$

An interpolation inequality for the nodal interpolation $v_h$ of $v$, an inverse inequality (as $v - v_h$ is a polynomial on each $T \in \mathcal{T}_h$ and $D_h^2 v$ denotes the piecewise second derivatives), and a stability estimate of the Galerkin-scheme $C_{\mathrm{stab}} = 1$ read

$$\|H/h \nabla(v - v_h)\| \le C_{\mathrm{interpol}} \left\|H\, D_h^2 v\right\|, \qquad \left\|H\, D_H^2 v\right\| \le C_{\mathrm{inv}} \|\nabla v\|, \quad \text{and} \quad \|\nabla v\| \le C_{\mathrm{stab}} \|\nabla e\|.$$

The combination of the aforementioned estimates proves reliability with $c_{12} = C_{\mathrm{interpol}} C_{\mathrm{inv}} C_{\mathrm{stab}}$. Efficiency follows with a triangle inequality

$$\eta \le \|\nabla(e - v)\| + \|\nabla(u - u_H)\| \le \|\nabla e\| + \mathrm{h.o.t.}(u). \qquad \square$$

As a consequence for $h/H \to 0$, the estimator $\eta$ is asymptotically exact if the patches in $\mathcal{T}_H$ become larger and larger compared to the finer mesh $\mathcal{T}_h$. This analysis gives no result for $\mathcal{T}_h = \mathcal{T}_H$ and $c_{12}$ should not be expected to be too small. However, depending on the shape of the elements in $\mathcal{T}_H$ there exist an integer $K$ such that $K$ red-refinements (each of which is a refinement of every element into four congruent sub-triangles) result in a mesh $\mathcal{T}_h$ which yields an efficient and reliable error estimator $\eta$.

**Remark 10.** In the same spirit, the smoothing steps within a multigrid method can be used to compute an averaged quantity with superconvergence properties and a corresponding error estimator [8].

**Remark 11.** Averaging techniques on larger and larger patches are suggested in [21] for the computation of $\|D^2 z\|$ within a duality approach for a posterior error control [13].

## 9 Conclusions

The use of averaging or gradient recovery techniques can be recommended as a simple and often very efficient tool to monitor the stress or flux error. For a large class of finite elements methods and their applications, reliability has been established and numerical evidence exists that the estimators $\eta_M \leq \eta_A$ are highly accurate.

The conditions for reliability are smoothness of coefficients and right-hand sides but not necessarily of the Lipschitz domain or the (unknown) exact solution. At the moment, time-depending problems and problems with accumulated errors or with polution are excluded. Moreover, the robustness of averaging estimators with respect to crucial parameters jumping or oscillating coefficients requires particular attention.

The practical performance of $\eta_A$ is often highly accurate but, owing to [24, 27, 28], seems to depend on local symmetry and local smoothness. Even for singular solutions, an appropriately adapted mesh balances the error and often improves the quality of the error estimations. It is conjectured but not (fully) understood that local superconvergence phenomena are responsible for the high accuracy.

Goal-oriented error control is excluded in this paper, but certainly very important from the conceptual and practical view point. Therein, the effective approximation of dual solutions [13] is another important field of further applications of averaging techniques in the future.

This paper aimed to mark the state of the art in the popular usage of averaging error estimators. One may conclude in that their efficiency and reliability is now understood. Their success in practical applications is observed and conjectured but not (fully) theoretically justified. This approach clearly has limitations (consistency term, non-smooth right-hand sides, etc.), but works perfectly fine if applied appropriately.

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