

A posteriori error analysis for elliptic pdes on domains with complicated structures

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Summary. The discretisation of boundary value problems on complicated domains cannot resolve all geometric details such as small holes or pores. The model problem of this paper consists of a triangulated polygonal domain with holes of a size of the mesh-width at most and mixed boundary conditions for the Poisson equation. Reliable and efficient a posteriori error estimates are presented for a fully numerical discretisation with conforming piecewise affine finite elements. Emphasis is on technical difficulties with the numerical approximation of the domain and their influence on the constants in the reliability and efficiency estimates.

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1 Introduction

Porous media or advanced materials with microstructures provide examples for boundary value problems with small geometric details. Typically, those details cannot be completely resolved by the mesh of a finite element discretisation, but have to be involved. This work is devoted to the mathematical analysis for the Poisson equation on a domain with holes of a mesoscale:

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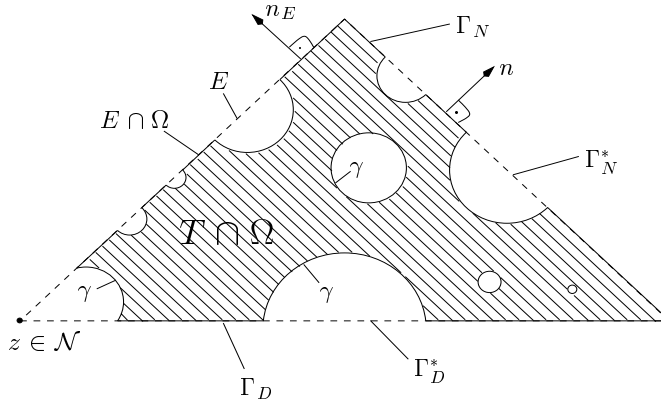


Fig. 1. Intersection of a triangle T with the domain Ω at the boundary

Large holes are resolved by the finite element mesh exactly, but holes of the diameter of the mesh-size and smaller are not, as illustrated in Fig. 1. Efficient and reliable a posteriori error estimates are derived for a conforming piecewise affine finite element scheme on a triangulation which covers a bigger domain Ω^* that includes the domain Ω with holes inside and on the surface.

For elliptic problems on complicated domains, the minimal dimension of any finite element space is huge since the finite element mesh has to resolve the geometry. Thus, from the viewpoint of accuracy and balancing of local errors, we cannot expect that the degrees of freedom of such a finite element space are distributed in a (nearly) optimal way. In [HS1], composite finite element spaces have been introduced where the minimal number of unknowns is independent of the size and number of geometric details. The combination of composite finite element spaces with an a posteriori error estimator (used as an error indicator) allows to design problem-adapted finite element spaces where the adaptation process starts from very coarse levels.

In addition, by using this a posteriori error estimator the finite element error can be estimated on discretisation levels where not all geometric details are resolved by the mesh (but taking them into account by using composite finite element functions).

Our paper is devoted to the definition and analysis of a reliable and efficient a posteriori error estimator. As a model problem we will study the Poisson problem $-\Delta u = f$ with mixed boundary conditions. We will consider a Lipschitz domain Ω which arises by removing from a polygonal domain Ω^* a possibly huge number of holes. “Holes” are simply connected domains which are collected in the set of holes \mathcal{H} . On the boundary of holes, we suppose homogeneous Neumann boundary conditions as the most simple mathemat-

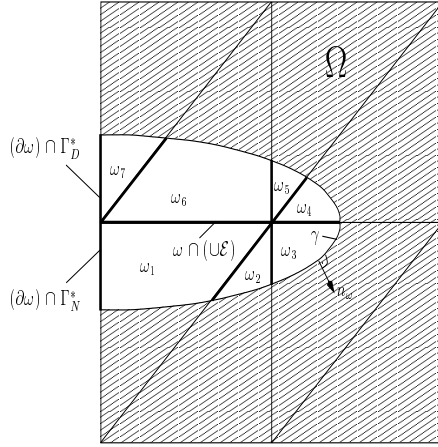


Fig. 2. Intersection of a hole ω with $U\mathcal{E}$

ical model. The generalisation to more general boundary conditions is the topic of current research.

The discretisation is based on a conforming triangulation \mathcal{T} of the *overlapping* domain Ω^* with continuous, piecewise affine finite elements and their restrictions to the domain Ω . In a first phase, a \mathcal{T} -piecewise affine discrete function U^* is computed on Ω^* while the approximation of the continuous solution u is given by $U := U^*|_\Omega$.

Our reliable a posteriori error estimator will be presented in Section 3. Besides error residuals, we obtain, for an interior hole $\omega \in \mathcal{H}$, the term

$$\eta_\omega^2 := h_\omega \int_{\partial\omega} |\partial U / \partial n_\omega|^2 ds$$

corresponding to $\partial u / \partial n_\omega = 0$ on $\partial\omega$ (and modifications for any hole which touches the boundary).

We carefully study the efficiency of this contribution where difficulties arise from the fact that ω may be intersected with edges of the triangulation \mathcal{T} in a quite arbitrary and complicated way; compare Fig. 2 for an illustration.

Our main result can be stated as follows. Suppose $u \in H^1(\Omega)$ denotes the exact solution and $U = U^*|_\Omega$ its discrete approximation. If all integrals are evaluated exactly (otherwise we obtain inconsistency error sources η_c), the error in energy norm $\|\nabla(u - U)\|_{L^2(\Omega)}$ is bounded by

$$\eta := \|h_{\mathcal{T}} f\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{1/2} [\partial U^* / \partial n_{\mathcal{E}}]\|_{L^2(\cup \mathcal{E})} + \left(\sum_{\omega \in \mathcal{H}} \eta_\omega^2 \right)^{1/2},$$

where h_T (resp. $h_{\mathcal{E}}, h_{\omega}$) is the local mesh-size (resp. edge-size, hole-size), and $[\partial U^*/\partial n_{\mathcal{E}}]$ is the jump of the normal components of two (\mathcal{T} -piecewise) discrete gradients across the edges (and standard modifications on the boundary). Theorem 3.1 implies (for exactly matched Dirichlet boundary conditions) the reliability of η in the sense of

$$\|\nabla(u - U)\|_{L^2(\Omega)} \leq c_1 \eta.$$

Theorem 6.1 shows efficiency, i.e., the converse inequality

$$c_2 \eta \leq \|\nabla(u - U)\|_{L^2(\Omega)} + \text{h.o.t.}$$

In the latter inequality, h.o.t. are known higher order terms and it holds in a local form. The constants c_1 and c_2 are independent of mesh-sizes or hole-sizes. They depend on some features of the geometry of holes. For instance, c_1 stays uniformly bounded if the holes are circular with diameter $h_{\omega} \leq c_3 h_T$ for neighbouring elements T of size h_T provided a separation condition is satisfied, namely, two distinct holes ω_1 and ω_2 have a distance $\text{dist}(\omega_1, \omega_2)$ with $h_{\omega_1} + h_{\omega_2} \leq c_4 \text{dist}(\omega_1, \omega_2)$. To bound c_2 from below, we will assume in addition that $h_{\omega} < c_4 \text{dist}(\omega, \Gamma^*)$ holds. It is stressed that c_1 and c_2 only depend mildly through the constants c_5 and c_{15} (cf. Section 4.1 and Assumption 6.7) on the way the edges intersect with holes and tiny pieces as well as entire edges may lie inside the hole. Emphasis is on the geometry of small holes and its qualitative influence in c_1 and c_2 .

The remaining part of the paper is organised as follows. The model problem will be posed and discussed in view of composite finite elements in Section 2. The a posteriori error estimator will be presented in Section 3. Constants in trace estimates and in bounds of appropriate extension operators on some geometry parameters will be investigated in Section 4. The reliability of the a posteriori error estimator (stated in Theorem 3.1) will be proved in Section 5. The conditions sufficient for the efficiency estimate of Theorem 6.1 may appear technical at first glance. Therefore, included examples illustrate the consequences of Assumptions 6.1 till 6.7. The proof in Section 7, however, clearly underlines that the assumptions posed are natural. The analysis of edge and volume contributions per se requires minor modifications of standard techniques [V] while the investigations of the hole contributions are more involved. We emphasise that, in contrast to [DR], where the effect of approximating the boundary of the domain is incorporated in the error estimator, our finite element spaces are defined on the true domain while the construction allows a low-dimensional discretisation even for very complicated domains.

2 Model problem

As a model problem we consider a domain $\Omega \subset \mathbb{R}^2$ which arises by removing holes from a polygonal domain. More precisely, let $\Omega^* \subset \mathbb{R}^2$ denote

a polygonal domain with boundary $\Gamma^* = \partial\Omega^*$ and let $\mathcal{H} = \{\omega_j : 1 \leq j \leq J\}$ be a countable set of simply connected Lipschitz domains ω_j , the ‘holes’, which have a positive distance from each other and are all (not necessarily compactly) contained in Ω^* . The physical domain $\Omega := \Omega^* \setminus \overline{\bigcup \mathcal{H}}$ is supposed to be Lipschitz (as a further assumption on the intersections of the hole boundaries $\partial\omega$ with Γ^*). Mixed boundary conditions are imposed on the boundary $\Gamma := \partial\Omega$, namely homogeneous Neumann boundary conditions on $\gamma := \Omega^* \cap \partial(\cup \mathcal{H})$, prescribed Neumann data g on $\Gamma_N := (\Gamma \cap \Gamma^*) \setminus \Gamma_D$, and prescribed Dirichlet data u_D on $\Gamma_D \subset \Gamma \cap \Gamma^*$ of positive length.

Remark 2.1 Note that $\Gamma_D \cup \Gamma_N = \Gamma^*$ if Ω^* includes all holes compactly. Otherwise, the inclusion $\Gamma_D \cup \Gamma_N \subset \Gamma^*$ can be strict.

The strong formulation of the continuous problem reads: *Given $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, and $u_D \in H^1(\Gamma_D)$, seek $u \in H^1(\Omega)$ satisfying*

$$(2.1) \quad \begin{aligned} \Delta u + f &= 0 & \text{in } \Omega, & & u &= u_D & \text{on } \Gamma_D, \\ \partial u / \partial n &= g & \text{on } \Gamma_N, & & \partial u / \partial n &= 0 & \text{on } \gamma. \end{aligned}$$

Since the normal derivative of u vanishes at interior boundaries of holes, the weak formulation reads: *Seek $u \in H^1(\Omega)$ with $u|_{\Gamma_D} = u_D$ and*

$$(2.2) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_N} g v \, ds$$

for all test functions $v \in H^1_D(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$.

The discretisation of the model problem is based on composite finite elements which are defined in three steps **(i)–(iii)** [HS1].

(i) The (overlapping) triangulation. The polygonal domain Ω^* is partitioned exactly by a regular triangulation \mathcal{T} into closed triangles $T \in \mathcal{T}$ in the sense of Ciarlet [BS, Ci], $\overline{\Omega^*} = \cup \mathcal{T}$. Two non-disjoint distinct triangles in \mathcal{T} share either a common edge E or a vertex z called *node*. The set of all edges resp. nodes is denoted by \mathcal{E} resp. \mathcal{N} . Edges on the boundary Γ^* (belong to only one triangle and) are collected in the set $\mathcal{E}_{\Gamma^*} = \mathcal{E}_D \cup \mathcal{E}_N$, where \mathcal{E}_{Γ^*} is split into edges of Dirichlet- and Neumann type as follows. Let $|E \cap \Gamma_D|$ and $|E \cap \Gamma_N|$ denote the one-dimensional measure of the sets $E \cap \Gamma_D$ and $E \cap \Gamma_N$, respectively, along the edge $E \in \mathcal{E}_{\Gamma^*}$; suppose that either $|E \cap \Gamma_D|$ or $|E \cap \Gamma_N|$ is positive (but not both of them). Then, set $\mathcal{E}_D := \{E \in \mathcal{E}_{\Gamma^*} : |E \cap \Gamma_D| > 0\}$ and $\mathcal{E}_N := \mathcal{E}_{\Gamma^*} \setminus \mathcal{E}_D$, $\mathcal{E}_{\Omega} := \mathcal{E} \setminus \mathcal{E}_{\Gamma^*}$, and $\Gamma_N^* := \cup \mathcal{E}_N$ resp. $\Gamma_D^* := \cup \mathcal{E}_D$.

Remark 2.2 For the ease of the discussion, we assume throughout the paper that the holes in \mathcal{H} are sufficiently small such that all triangles T

have non-empty intersection with Ω . Otherwise, those triangles have to be removed from the grid and Ω^* has to be replaced by the domain covered by the reduced grid.

(ii) The (overlapping) finite element space. Let \mathcal{S}^* denote the space of \mathcal{T} -piecewise affine finite elements and \mathcal{S}_D^* its subspace with vanishing traces on Γ_D^* , i.e.,

$$(2.3) \quad \begin{aligned} \mathcal{S}^* &:= \{V \in C(\Omega^*) : \forall T \in \mathcal{T}, V|_T \text{ is affine on } T\}, \\ \mathcal{S}_D^* &:= \{V \in \mathcal{S}^* : V|_{\Gamma_D^*} = 0\}. \end{aligned}$$

(iii) The (restricted) composite finite element space. The spaces \mathcal{S} and \mathcal{S}_D are given by

$$\begin{aligned} \mathcal{S} &:= \mathcal{S}^*|_{\Omega} := \{V \in C(\Omega) : \exists V^* \in \mathcal{S}^*, V = V^*|_{\Omega}\}, \\ \mathcal{S}_D &:= \mathcal{S}_D^*|_{\Omega}. \end{aligned}$$

Remark 2.3 Using $\Gamma_D \subset \Gamma^*$ we have

$$\mathcal{S}_D^* = \{V \in \mathcal{S}^* : \forall E \in \mathcal{E}_D, V|_E = 0\}.$$

Remark 2.4 Throughout the paper we write u, v , for functions lying in the infinite dimensional space $H^1(\Omega)$, $H_D^1(\Omega)$ and U, V, \dots for functions in the finite element space $\mathcal{S}, \mathcal{S}_D$. Functions on the extended domain Ω^* have a superscript \star , e.g., u^*, U^* . Approximations to the right-hand sides and the porosity (see below) are denoted with a tilde superscript as, e.g., \tilde{f}, \tilde{g} , while f^* is the extension of f from Ω to Ω^* by zero. If v and v^* appear in the same context, we understand $v = v^*|_{\Omega}$.

The intersection of an element with the domain and some notations are illustrated in Fig. 1. Note carefully that domains are open and connected and that $T \in \mathcal{T}$ and $E \in \mathcal{E}$ are closed sets in \mathbb{R}^2 ; the interior of T is denoted by $\text{int}(T)$.

To define a fully numerical discretisation, we need to approximate Dirichlet data $u_D \in H^1(\Gamma_D)$ by the trace of some function $U_D^* \in \mathcal{S}^*$. The finite element scheme requires a stiffness matrix where, for each element $T \in \mathcal{T}$ of area $|T|$, some constant $\tilde{\varrho}|_T$ approximates the ratio $\varrho_T := |\Omega \cap T|/|T|$, where $\tilde{\varrho} \in \mathcal{L}^0(\mathcal{T})$ and $\mathcal{L}^0(\mathcal{T})$ denotes the \mathcal{T} -piecewise constants. Furthermore, the computation involves approximate right-hand sides $\tilde{f}^* \in L^2(\Omega^*)$ and $\tilde{g}^* \in L^2(\Gamma_N^*)$. Then, the *discrete problem* reads: *Seek $U^* \in \mathcal{S}^*$ and $U := U^*|_{\Omega}$ satisfying $U^* = U_D^*$ on Γ_D^* and, for all $V^* \in \mathcal{S}_D^*$,*

$$(2.4) \quad \int_{\Omega^*} \tilde{\varrho} \nabla U^* \cdot \nabla V^* dx = \int_{\Omega^*} \tilde{f}^* V^* dx + \int_{\Gamma_N^*} \tilde{g}^* V^* ds.$$

Remark 2.5 Notice that the computational cost for solving the discrete problem (2.4) are small as the geometry is not resolved in detail (use of Ω^* , Γ_D^* , Γ_N^* instead of Ω , Γ_D , Γ_N) and holes are taken into account only via an approximate porosity. If the volume of the unresolved microstructure is not negligible, i.e., $|\Omega^* \setminus \Omega| = O(1)$, the choice $\tilde{q} \equiv 1$, in general, would lead to large approximation errors. Hence, it is necessary but, due the possible lack of regularity, not sufficient in general that the error estimator takes into account the contribution of these consistency errors.

Remark 2.6 The integrals over intersections $T \cap \Omega^*$ should be evaluated by using a composite quadrature rule employing a hierarchical resolution of the domain (cf., e.g., [HS2,OR]). The efficient implementation of the a posteriori controlled discretisation scheme presented here will be the topic of a forthcoming paper. Some ideas on the adaptive numerical integration can be found in the extended version of this paper [CCSAS] and [HS2].

3 A posteriori error estimate

Suppose $u \in H^1(\Omega)$ is the exact solution of (2.2) and let $U = U^*|_\Omega$ be the restriction of the discrete solution of (2.4). The ingredients of the error estimator for the energy error of $u - U$ are the volume residual η_Ω , the edge contributions η_E , the hole errors $\eta_{\mathcal{H}}$, the Dirichlet contribution η_D , and the consistency term η_c .

Let h_T (resp. $h_{\mathcal{E}}$) be a T -piecewise (resp. \mathcal{E} -piecewise) constant weight to measure the mesh-size (resp. edge-size) regarded as L^∞ -functions on Ω^* (resp. on $\cup \mathcal{E}$). Then, the volume contribution reads

$$(3.1) \quad \eta_\Omega := \|h_T f\|_{L^2(\Omega)}.$$

For each edge $E \in \mathcal{E}$, let n_E denote one unit normal on E with fixed orientation (oriented to the exterior if $E \in \mathcal{E}_{\Gamma^*}$) and $n_{\mathcal{E}} \in L^\infty(\cup \mathcal{E})$ denotes their composition, i.e., $n_{\mathcal{E}}|_E := n_E$. The T -piecewise constant discrete gradient ∇U has a jump across each inner edge $E \in \mathcal{E}_\Omega$ and the difference $(\nabla U)|_{T_+} - (\nabla U)|_{T_-}$ is denoted as $[\nabla U]$ on E ; T_+ and T_- are the two distinct elements which share the edge E such that n_E points into T_+ . Note that the jump of the normal components $[\nabla U] \cdot n_E =: [\partial U / \partial n_{\mathcal{E}}]|_E$ is independent of the chosen orientation of n_E . We regard $n_{\mathcal{E}}$ and $[\partial U / \partial n_{\mathcal{E}}]$ as \mathcal{E} -piecewise constant functions on the skeleton $\cup \mathcal{E}$ of edges; $\cup \mathcal{E}$ is the set of all points x on some boundary ∂T of some triangle $T \in \mathcal{T}$. We recall the definition of $\mathcal{E}_\Omega := \{E \in \mathcal{E} : |E \cap \Gamma^*| = 0\}$. Then, the edge contribution reads

$$(3.2) \quad \eta_{\mathcal{E}} := \left\| h_{\mathcal{E}}^{1/2} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\cup \mathcal{E}_\Omega)} + \left\| h_{\mathcal{E}}^{1/2} \left(g^* - \frac{\partial U^*}{\partial n} \right) \right\|_{L^2(\Gamma_N^*)}.$$

Owing to homogeneous Neumann conditions on γ , the analogue to the edge contribution defines the hole contributions,

$$(3.3) \quad \eta_{\mathcal{H}} := \left(\sum_{\omega \in \mathcal{H}} h_{\omega} \int_{\partial\omega \setminus \Gamma_D^*} |\partial U^* / \partial n_{\omega}|^2 ds \right)^{1/2}$$

(h_{ω} denotes the diameter of $\omega \in \mathcal{H}$). In case that u_D is the restriction of a smooth function on Γ_D^* , the Dirichlet contribution

$$(3.4) \quad \eta_D := \min\{\|\nabla v\|_{L^2(\Omega)} : v \in H^1(\Omega) \text{ such that } v = u_D - U \text{ on } \Gamma_D\}$$

is of higher order (and then may be neglected). Finally, the abstract consistency term is given by

$$(3.5) \quad \eta_c := \sup_{\substack{W^* \in \mathcal{S}_D^* \\ \|W^*\|_{H^1(\Omega^*)} = 1}} \left\{ \int_{\Omega^*} (f^* - \tilde{f}^*) W^* dx + \int_{\Gamma_N^*} (g^* - \tilde{g}^*) W^* ds - \int_{\Omega^*} (\varrho - \tilde{\varrho}) \nabla U^* \cdot \nabla W^* dx \right\}.$$

Theorem 3.1 *There exists an $(h_{\mathcal{T}}, h_{\mathcal{E}}, u, U, f, g)$ -independent positive constant c_1 such that*

$$\|\nabla(u - U)\|_{L^2(\Omega)} \leq c_1(\eta_{\Omega} + \eta_E + \eta_{\mathcal{H}} + \eta_D + \eta_c).$$

The constant c_1 depends on the domain Ω , Γ_D , Γ_N , and the shape of the elements in \mathcal{T} (such as their aspect ratio) but not on the number or size of the holes in \mathcal{H} and through the constant c_5 (only mildly) on the way they are hit by $\cup \mathcal{E}$ (cf. Remark 4.1 and 4.2).

Sections 4 and 5 are devoted to the proof of the theorem while efficiency will be studied in Sections 6 and 7.

4 Extension operators, trace theorems, and Poincaré inequalities on complicated domains

The discrete solution $U = U^*|_{\Omega}$ in (2.4) is defined as the restriction of the auxiliary solution U^* corresponding to a problem on the extended domain Ω^* . As a consequence, the proof of the reliability and the efficiency of the error estimator is based on estimates of the norm of certain extension and trace operators. In this section, we will show that their norms are moderately bounded for a broad class of domains which might contain a huge number of small holes.

4.1 Extension operators

Let $\Omega \subset \Omega^*$, Γ_D , Γ_D^* , etc. be as in Section 2 and $H_D^1(\Omega^*) := \left\{ v \in H^1(\Omega^*) : v|_{\Gamma_D^*} = 0 \right\}$. In this section, we will define an extension operator $\mathfrak{E}_D : H_D^1(\Omega) \rightarrow H_D^1(\Omega^*)$ so that the supremum

$$\sup_{v \in H_D^1(\Omega) \setminus \{0\}} \|\nabla \mathfrak{E}_D v\|_{L^2(\Omega^*)} / \|\nabla v\|_{L^2(\Omega)} =: c_5 < \infty.$$

is moderately bounded for a large class of domains, which may contain a huge number of geometric details.

Remark 4.1 The constant c_1 will depend on c_5 .

The extension operator \mathfrak{E}_D is constructed in three steps. Let $u \in H_D^1(\Omega)$. Since Ω is a Lipschitz domain it is well known that there exists an extension operator $\mathfrak{E}^{\text{Stein}} : H_D^1(\Omega) \rightarrow H^1(\Omega^*)$. Put $u_1^* := \mathfrak{E}^{\text{Stein}} u \in H^1(\Omega^*)$ and note that $u|_{\Gamma_D} = 0$. Next, we employ a function $m : \Omega^* \rightarrow [0, 1]$ satisfying $m \equiv 1$ on Ω and $m = 0$ on $\Gamma_D^* \setminus \Gamma_D$ to define $u_2^* := m u_1^* \in H_D^1(\Omega^*)$. Finally, the local Ritz-projections u_2^0 of u_2^* on $\omega \in \mathcal{H}$ is subtracted to end up with $u^* := u_2^* - u_2^0 =: \mathfrak{E}_D u$. The details of this construction along with illustrating examples will be discussed in this section.

Theorem 4.1 (Stein) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a continuous extension operator $\mathfrak{E}^{\text{Stein}} : H_D^1(\Omega) \rightarrow H^1(\Omega^*)$.*

For a proof, we refer to [S]. Theorem 4.1 neither implies that the operator norm of $\mathfrak{E}^{\text{Stein}}$ is moderately bounded (the bound might be very large for domains with a huge number of small geometric details) nor that the H^1 -seminorm of the extended function can be estimated by the H^1 -seminorm of the original function. To fulfil homogeneous Dirichlet boundary conditions on Γ_D^* , we assume the existence of an appropriate cutoff-function.

Assumption 4.1 *There exists a function $m : \overline{\Omega^*} \rightarrow [0, 1]$ such that $m \equiv 1$ on Ω while $m = 0$ on $\Gamma_D^* \setminus \Gamma_D$ and, for all $v \in H^1(\Omega^*)$ with $v = 0$ on Γ_D , the product mv belongs to $H_D^1(\Omega^*)$. Set*

$$\mathfrak{M} : \{v \in H^1(\Omega^*) : v|_{\Gamma_D} = 0\} \rightarrow H_D^1(\Omega^*), \quad v \mapsto mv.$$

For holes $\omega \in \mathcal{H}$, which do not touch the exterior Dirichlet boundary $\Gamma_D^* \setminus \Gamma_D$, we may choose $m|_{\omega} \equiv 1$. The following example considers the characteristic model situation of Fig. 3.

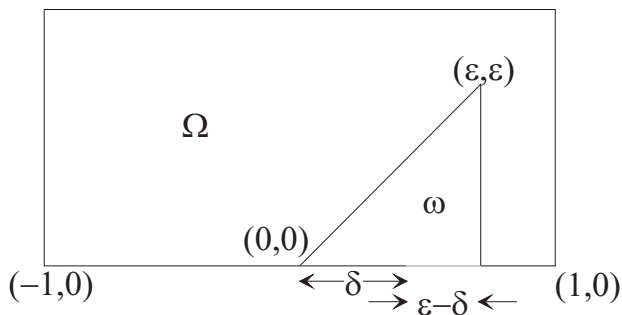


Fig. 3. Domain Ω with triangular hole ω and a change of the type of boundary conditions outside $\partial\Omega$ to illustrate Assumption 4.1

Example 4.1 Let $\Omega = \Omega^* \setminus \bar{\omega}$ where $\Omega^* = (-1, 1) \times (0, 1)$ and ω denotes the open triangle with vertices $(0, 0)$, $(\epsilon, 0)$, and (ϵ, ϵ) , where $0 < \epsilon < 1/2$.

Suppose $\Gamma_D^* := [-1, \delta] \times \{0\}$ for some $0 < \delta < \epsilon$, while $\Gamma_N^* := \Gamma^* \setminus \overline{\Gamma_D^*}$, $\Gamma_D = [-1, 0] \times \{0\}$. Let $(x, y) = r(\cos \alpha, \sin \alpha)$. We define the function m by

$$m(x, y) := \begin{cases} 1 & \text{if } x \in \bar{\Omega}, \\ \chi(x) + (1 - \chi(x)) \sin(2\alpha) & \text{if } x \in \omega, \end{cases}$$

where $\chi(x) = 0$ for $0 < x < \delta$ and $\chi(x) = (x - \delta)/(\epsilon - \delta)$ for $\delta \leq x \leq \epsilon$. It is easy to check that m is continuous in the open set Ω^* and $m = 0$ on $\Gamma_D^* \setminus \Gamma_D$. Given $v_1^* \in H^1(\Omega^*)$ with $v_1^*|_{\Gamma_D} = 0$, we define $v_2^* := mv_1^*$. The proof of

$$\|m\|_{L^\infty(\Omega^*)} = 1 \quad \text{and} \quad |\nabla m(x)| \leq \frac{2 + \sqrt{2}}{r(1 - \delta/\epsilon)} \quad \text{for all } x \in \Omega^*$$

is straightforward. Hence, Hardy’s inequality in the form of Theorem 1.4.44 in [G] (where $s = 1$, $p = 2$, $\alpha = 0$) yields $\mathfrak{M}v_1^* \in H^1(\Omega^*)$ and so eventually $v_2^* \in H_D^1(\Omega^*)$.

In the next step, we introduce the Ritz-projection of functions $H_D^1(\Omega^*)$ in the space $\mathcal{V} := \{v \in H^1(\Omega^*) : v|_{\Omega \cup \Gamma_D} = 0\}$.

Definition 4.1 The Ritz-projection $\mathfrak{R} : H_D^1(\Omega^*) \rightarrow \mathcal{V}$ is given for $v_1^* \in H_D^1(\Omega^*)$ by $v_2^* := \mathfrak{R}v_1^*$ where $v_2^* \in \mathcal{V}$ is the solution of

$$\int_{\Omega^*} \nabla v_2^* \cdot \nabla w = \int_{\Omega^*} \nabla v_1^* \cdot \nabla w \, dx \quad \text{for all } w \in \mathcal{V}.$$

Now, we have all ingredients for defining the extension operator \mathfrak{E}_D .

Definition 4.2 *The extension operator $\mathfrak{E}_D : H_D^1(\Omega) \rightarrow H_D^1(\Omega^*)$ is given by the composition*

$$(4.1) \quad \mathfrak{E}_D := (I - \mathfrak{R}) \mathfrak{M} \mathfrak{E}^{\text{Stein}}.$$

Our goal is to prove that the norm of \mathfrak{E}_D is moderately bounded for a broad class of domains which might contain a huge number of small holes. A sufficient condition for such domains is a certain separation criterion (see Section 1, (4.2), and [SW, (2.8)]) which requires compactly included holes. Examples 4.4 and 4.5 show that this condition is only necessary and the holes may hit also the boundary.

To reduce technicalities we focus on some characteristic examples and refer to [SW] and [OSY] for proofs and general considerations.

Example 4.2 Let $\Omega^* \subset \mathbb{R}^d$ denote a Lipschitz domain and $\{B_j\}_{j \in N}$, $N \subseteq \mathbb{N}$, be a family of balls with radius ε_j which are compactly included in Ω^* and satisfy the separation conditions

$$(4.2 \text{ a}) \quad \text{dist}(B_j, B_k) \geq c_6 \max\{\varepsilon_j, \varepsilon_k\},$$

$$(4.2 \text{ b}) \quad \text{dist}(B_j, \Gamma^*) \geq c_6 \varepsilon_j$$

for all distinct $j, k \in N$ and the global constant $c_6 > 0$. Let $\Omega := \Omega^* \setminus \overline{\bigcup_{j \in N} B_j}$. Choose $m \equiv 1$ in Ω^* (cf. Assumption 4.1) and define \mathfrak{E}_D as in (4.1). Then, the operator norm of \mathfrak{E}_D and its seminorm, i.e., the constant c_5 , is bounded independently of $\text{card}(N)$ or ε_j .

Example 4.3 For $\delta > 0$, let $\Omega^* = (-1 - \delta, 1 + \delta)^2$ and $\omega = (-1, 1)^2$. Then, there exists a constant $c_7 > 0$ so that the norm of every extension operator $\mathfrak{E} : H^1(\Omega^* \setminus \overline{\omega}) \rightarrow H^1(\Omega^*)$ can be estimated from below by

$$c_7 \delta^{-1/2} \leq \|\mathfrak{E}\|.$$

The following example shows that both parts of the separation condition (4.2) are not necessary in order to bound the norm of the minimal extension operator by a moderate constant.

Example 4.4 Let $\Omega^* = (-1, 1)^3$ and, for $j = 1, 2$, $\omega_j = B_j \times (-1, 1)$. Here, B_j denotes the open disc with radius ε about the points $(\pm 2\varepsilon, 0)^\top$. Then, the norm of the minimal extension operator $\mathfrak{E} : H^1(\Omega^* \setminus \overline{\omega_1 \cup \omega_2}) \rightarrow H^1(\Omega^*)$ is bounded uniformly as $\varepsilon \rightarrow 0$.

Finally, we revisit Example 4.1 and estimate the norm of the extension operator.

Example 4.5 Let Ω, Ω^*, ω , and the function m be defined as in Example 4.1. Then, the norm and the seminorm of the extension operator \mathfrak{E}_D as in (4.1) can be estimated from above by $C / (1 - \delta/\varepsilon)$. It is easy to see that this bound is sharp by considering the function $w \in H_D^1(\Omega)$, $w(x) := \max\{0, x_1\}$. The value of w^* is ε at $(\varepsilon, 0)$ and 0 at $(\varepsilon - \delta, 0)$.

Remark 4.2 This example indicates that the (semi-)norm of the extension operator \mathfrak{E}_D behaves critically if the ratio of the length of the Dirichlet portion $\partial\omega \cap \Gamma_D^*$ compared to the length of the outer boundary $\partial\omega \cap \Gamma^*$ tends to one.

4.2 Clément interpolation on complicated domains

The proof of the reliability of the error estimator makes use of the Clément approximation [Cl,V,CF] operator $P : H_D^1(\Omega^*) \rightarrow S_D^*$ such that, for all $T \in \mathcal{T}$ and $u^* \in H^1(\Omega^*)$,

$$(4.3) \quad \|u^* - Pu^*\|_{L^2(T)} + h_T |u^* - Pu^*|_{H^1(T)} \leq c_8 h_T |u^*|_{H^1(\omega_T)},$$

$$(4.4) \quad \|h_T^{-1}(u^* - Pu^*)\|_{L^2(\Omega^*)} + |u^* - Pu^*|_{H^1(\Omega^*)} \leq c_9 |u^*|_{H^1(\Omega^*)}.$$

Here, $\omega_T := \cup\{K \in \mathcal{T} : T \cap K \neq \emptyset\}$. The constants c_8 and c_9 depend merely on the aspect ratio of the elements. Their quantitative estimation is given in [CF]. With $u^* = \mathfrak{E}_D u$ we obtain that the right-hand side in (4.4) can be bounded from above by $c_{10} |u|_{H^1(\Omega)}$ where c_{10} depends on c_5 and c_9 .

4.3 Trace theorems

Traces of H^1 -functions along edges have to be estimated by their norms on adjacent triangles [Cl,CF]. For shape regular meshes, we have the local estimate for $u \in H^1(T)$ on the edge $E \subset \partial T, E \in \mathcal{E}, T \in \mathcal{T}$,

$$\|u\|_{L^2(E)}^2 \leq c_{11} \left(h_E^{-1} \|u\|_{L^2(T)}^2 + h_E |u|_{H^1(T)}^2 \right)$$

and a global version, for $u^* \in H^1(\Omega^*)$,

$$\|u^*\|_{L^2(\cup\mathcal{E})} \leq c_{12} \left(\|h_{\mathcal{T}}^{-1/2} u^*\|_{L^2(\Omega^*)} + \|h_{\mathcal{T}}^{1/2} \nabla u^*\|_{L^2(\Omega^*)} \right).$$

Non-resolved geometric details require further estimates.

Definition 4.3 Let $\omega \subset \mathbb{R}^2$ denote a Lipschitz domain of area $|\omega|$ and let $\gamma \subset \bar{\omega}$ be a Lipschitz curve of length $|\gamma|$. The trace constant $C(\gamma, \omega)$ is

$$(4.5) \quad C(\gamma, \omega) := \sup_{v \in H^1(\omega) \setminus \{0\}} \frac{\|v\|_{L^2(\gamma)}^2}{(|\gamma| / |\omega|) \|v\|_{L^2(\omega)}^2 + (|\omega| / |\gamma|) |v|_{H^1(\omega)}^2}.$$

Remark 4.3 Letting $v \equiv 1$ in (4.5) shows $1 \leq C(\gamma, \omega)$.

The trace constant $C(\gamma, \omega)$ is scaling-invariant.

Lemma 4.1 For ω and γ as in Definition 4.3 and $\varepsilon > 0$, define $\chi_\varepsilon : \omega \rightarrow \omega_\varepsilon$ by $\chi_\varepsilon(x) = \varepsilon x$ and $\omega_\varepsilon = \chi_\varepsilon(\omega)$, $\gamma_\varepsilon = \chi_\varepsilon(\gamma)$. Then, $C(\gamma_\varepsilon, \omega_\varepsilon) = C(\gamma, \omega)$.

Proof. Straightforward calculations yield $|v|_{H^k(\omega)} = \varepsilon^{k-1} |v \circ \chi_\varepsilon^{-1}|_{H^k(\omega_\varepsilon)}$, $k = 0, 1$, and $\|v\|_{L^2(\gamma)}^2 = \varepsilon^{-1} \|v \circ \chi_\varepsilon^{-1}\|_{L^2(\gamma_\varepsilon)}^2$ for each $v \in H^1(\omega)$ and $|\omega_\varepsilon| = \varepsilon^2|\omega|$ or $|\gamma_\varepsilon| = \varepsilon|\gamma|$ from which we deduce the assertion. \square

Example 4.6 (a) Let ω be a disc with boundary $\gamma = \partial\omega$. Then, $C(\gamma, \omega) \leq 3$.
 (b) Let ω be a parallelogram and γ one of its sides. Then, $C(\gamma, \omega) \leq 2$.

Proof. For the proof of (a) we refer to [BS, Sec. 1.6] and indicate the proof of (b) for rectangles (the case of a parallelogram is similar). Suppose $\omega = (0, a) \times (0, b)$. The mean value theorem guarantees that $f(\eta) = b^{-1} \int_0^b f(y) dy$ for $f \in H^1(0, b)$ and some $\eta \in (0, b)$. The fundamental theorem of calculus and Cauchy inequalities then show

$$\begin{aligned} f(0)^2 &= (b^{-1} \int_0^b f(y) dy - \int_0^\eta f'(y) dy)^2 \\ (4.6) \quad &\leq 2b^{-2} \|f\|_{L^1(0,b)}^2 + 2\|f'\|_{L^1(0,b)}^2 \leq 2b^{-1} \|f\|_{L^2(0,b)}^2 + 2b\|f'\|_{L^2(0,b)}^2. \end{aligned}$$

Replacing $f(y)$ in (4.6) by $v(x, y)$ (and prime by $\partial/\partial y$) and integrating with respect to x over $(0, a)$ we deduce (b). \square

In the sequel, we will frequently estimate functions on (subsets of) holes and appropriate neighbourhoods thereof. Lemma 4.2 provides the trace estimate for the local geometric situation arising in the proof of reliability.

Notation 4.1 For a set $A \subset \mathbb{R}^2$, the Chebyshev ball B_A is the minimal ball that contains A . The disc with radius $2 \text{diam } A$ about the midpoint of B_A is denoted by V_A and $V_A^{int} := V_A \cap \Omega$.

Remark 4.4 The definition of V_ω could be generalised by replacing the factor 2 in Notation 4.1 by any other factor which is larger than one or even to more generally shaped neighbourhoods V_ω , where ∂V_ω has positive distance to ω . Then, the constants in the estimates related to V_ω would depend on this factor and on the shape on V_ω .

Lemma 4.2 Let $\omega \subset \mathbb{R}^2$ be a domain with diameter h_ω and let \mathcal{T} be a shape regular triangulation of \mathbb{R}^2 . Then,

$$\|v\|_{L^2(\omega \cap (\cup \mathcal{E}))}^2 \leq 2c_{13} \left(h_\omega^{-1} \|v\|_{L^2(V_\omega)}^2 + h_\omega |v|_{H^1(V_\omega)}^2 \right) \quad \forall v \in H^1(V_\omega).$$

The constant c_{13} is the number of ω -intersections with edges.

Proof. Let

$$\mathcal{E}_\omega := \{\text{conv}(\omega \cap E) : E \in \mathcal{E} \wedge |\omega \cap E| > 0\},$$

where $\text{conv}(\cdot)$ is the convex hull of a set. For each $S \in \mathcal{E}_\omega$ define a rectangle $Q(S)$ with one side S and the other of length h_ω . Example 4.6.b shows

$$\frac{1}{2} \|v\|_{L^2(S)}^2 \leq h_\omega^{-1} \|v\|_{L^2(Q(S))}^2 + h_\omega |v|_{H^1(Q(S))}^2.$$

By definition, c_{13} is the number of overlaps of $Q(S) \subset V_\omega$ for all $S \in \mathcal{E}_\omega$. This leads to

$$\begin{aligned} \frac{1}{2} \|v\|_{L^2(\cup \mathcal{E}_\omega)}^2 &\leq h_\omega^{-1} \sum_{S \in \mathcal{E}_\omega} \|v\|_{L^2(Q(S))}^2 + h_\omega \sum_{S \in \mathcal{E}_\omega} |v|_{H^1(Q(S))}^2 \\ &\leq c_{13} h_\omega^{-1} \|v\|_{L^2(V_\omega)}^2 + c_{13} h_\omega |v|_{H^1(V_\omega)}^2. \quad \square \end{aligned}$$

4.4 Poincaré inequalities

The proof of reliability and efficiency of the error estimator requires Poincaré inequalities [N, PW]. We will show that the constants which are involved can be estimated by the norm of an appropriate extension operator.

Theorem 4.2 (Payne and Weinberger) *Let ω denote a convex domain in \mathbb{R}^2 with diameter h_ω . Then, for all $u \in H^1(\omega)$ and $u_\omega := \int_\omega u \, dx / |\omega|$,*

$$\|u - u_\omega\|_{L^2(\omega)} \leq h_\omega / \pi |u|_{H^1(\omega)}. \quad \square$$

For a nonconvex domain ω , we first extend u to a convex neighbourhood V_ω and then deduce

$$\begin{aligned} \|u - u_\omega\|_{L^2(\omega)} &\leq \|u - u_{V_\omega}\|_{L^2(\omega)} \leq \|u - u_{V_\omega}\|_{L^2(V_\omega)} \\ (4.7) \quad &\leq \text{diam}(V_\omega) / \pi |u|_{H^1(V_\omega)} \leq c_{14} h_\omega |u|_{H^1(\omega)}. \end{aligned}$$

5 Proof of reliability

Throughout the proof we write $a \lesssim b$ for $a \leq c b$, where the multiplicative constant $c > 0$ is independent of $h_{\mathcal{T}}, h_{\mathcal{E}}, u, U, f, g$ and may depend on $\Omega, \Gamma_D, \Gamma_N$, and on the shape of the elements or their aspect ratio. Furthermore, the estimates depend on the numbers

$$(5.1a) \quad \max_{\omega \in \mathcal{H}} \text{card} \{T \in \mathcal{T} : T \cap \omega \neq \emptyset\},$$

$$(5.1b) \quad \sup_{x \in \Omega^*} \text{card} \{\omega \in \mathcal{H} : x \in V_\omega\}.$$

The numbers in (5.1) are moderately small for a broad class of domains and finite element meshes as can be concluded from Remark 5.1.

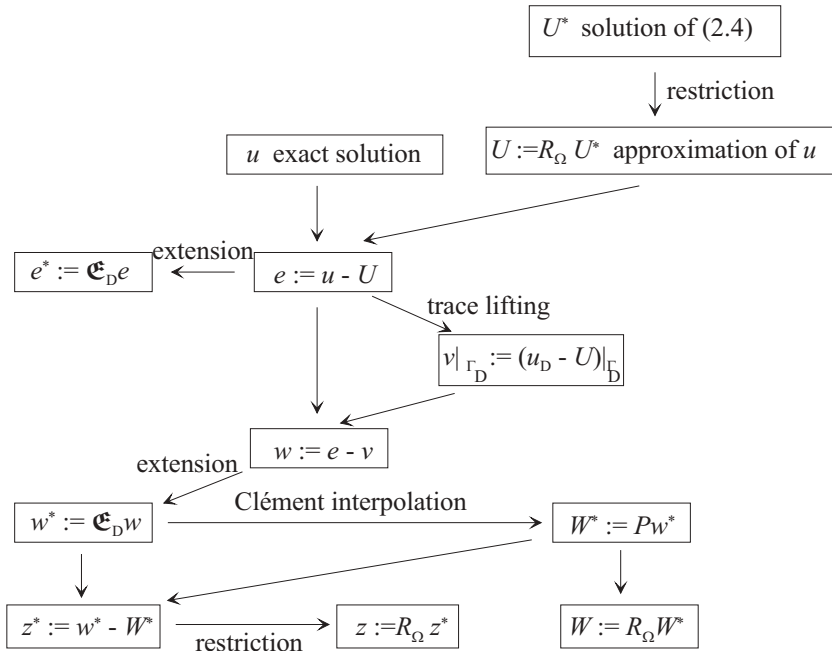


Fig. 4. Diagram illustrating the relationship of the functions $U, U^*, u, e, e^*, v, w, w^*, W^*, W, z^*$, and z

Remark 5.1 (a) Assume $h_\omega \lesssim h_T$ for all $T \in \mathcal{T}$ with $T \cap \omega \neq \emptyset$. Then, the maximum in (5.1a) is bounded by a constant which depends only on the shape regularity of the triangles.

Sketch of the proof: Take $x \in \omega$ and choose $T \in \mathcal{T}$ such that $x \in \bar{T}$. Define layers about x by the recursion $\mathcal{L}_x^{(0)} := \{T\}$ and for $i \in \mathbb{N}$ by $\mathcal{L}_x^{(i+1)} := \left\{ T \in \mathcal{T} : \bar{T} \cap \omega \cap \overline{\cup \mathcal{L}_x^{(i)}} \neq \emptyset \right\}$. Choose the minimal i_0 such that $\omega \subset \cup \mathcal{L}_x^{(i_0)}$. The shape regularity of the elements imply $\text{card } \mathcal{L}_x^{(i+1)} \lesssim \text{card } \mathcal{L}_x^{(i)}$. The condition $h_\omega \lesssim h_T$ yields $\omega \cap B_i(x) \subset \omega \cap (\cup \mathcal{L}_x^{(i)})$, where B_i is the disc, centred at x with radius $r_i \sim i h_\omega$. We conclude that $i_0 \lesssim 1$ and the assertion follows.

(b) As a simple example, consider a domain with holes satisfying the relaxed separation condition

$$\text{dist}(V_{\omega_1}, V_{\omega_2}) > 0 \quad \forall \omega_1, \omega_2 \in \mathcal{H}, \omega_1 \neq \omega_2$$

(cf. (4.2 a) and Notation 4.1). Then, the supremum in (5.1b) equals 1 (see Remark 4.4 for more general situations).

In the sequel, various functions arise with relationships illustrated in Fig. 4. Recall that $U^* \in \mathcal{S}^*$ solves (2.4) and $U := R_\Omega U^*$, where R_Ω is the restriction of a function $v : \Omega^* \rightarrow \mathbb{R}$ to Ω . Split the error $e := u - U \in H^1(\Omega)$ into $e - v$

and v , where $v \in H^1(\Omega)$ satisfies $v = u_D - U$ on Γ_D and $\eta_D = \|\nabla v\|_{L^2(\Omega)}$. Given $w := e - v \in H_D^1(\Omega)$ let $w^* := \mathfrak{E}_D w \in H_D^1(\Omega^*)$ and let $W^* := Pw^*$ denote the Clément approximation to w^* . Define $z^* := w^* - W^* \in H^1(\Omega^*)$ and $z := R_\Omega z^*$, $W := R_\Omega W^*$. Observe $z \in H_D^1(\Omega)$ and $z^* = 0$ on Γ_D^* . The H^1 -norm of W^* can be estimated by using a Friedrichs inequality for $w^* \in H_D^1(\Omega^*)$, the approximation properties of the Clément approximation (4.4) and the continuity of the extension operator \mathfrak{E}_D with respect to the H^1 -seminorm

$$\begin{aligned} \|W^*\|_{H^1(\Omega^*)} &\leq \|W^* - w^*\|_{H^1(\Omega^*)} + \|w^*\|_{H^1(\Omega^*)} \\ &\lesssim |w^*|_{H^1(\Omega^*)} \lesssim |e - v|_{H^1(\Omega)}. \end{aligned}$$

This, a triangle inequality with $|e^*|_{H^1(\Omega^*)} \lesssim |e|_{H^1(\Omega)}$, and $|U^* - v^*|_{H^1(\Omega^*)} \lesssim |e|_{H^1(\Omega)}$ at the end yield

$$\begin{aligned} &\|h_T^{-1} z^*\|_{L^2(\Omega^*)} + \|h_\mathcal{E}^{-1/2} z^*\|_{L^2(\cup \mathcal{E})} + |z^*|_{H^1(\Omega^*)} + \|W^*\|_{H^1(\Omega^*)} \\ (5.2) \quad &\lesssim |e - v|_{H^1(\Omega)} \lesssim |e|_{H^1(\Omega)} + \eta_D. \end{aligned}$$

The definition of v implies $\int_\Omega \nabla v \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in H_D^1(\Omega)$. The choice $\varphi = e - v$ leads to

$$\int_\Omega \nabla v \cdot \nabla e \, dx = \eta_D^2.$$

Hence, we obtain with $e = z + W + v$

$$(5.3) \quad |e|_{H^1(\Omega)}^2 = \int_\Omega \nabla e \cdot \nabla z \, dx + \int_\Omega \nabla e \cdot \nabla W \, dx + \eta_D^2.$$

The second term on the right-hand side in (5.3) is split into $\nabla u \cdot \nabla W$ and $\nabla U \cdot \nabla W$. Recall the definition of η_c as in (3.5). The concept of the porosity ϱ , the weak formulation (2.2), and (5.2) show (recall that f^* and g^* vanish outside Ω and Γ_N)

$$\begin{aligned} \int_\Omega \nabla e \cdot \nabla W \, dx &= \int_{\Omega^*} (f^* - \tilde{f}^*) W^* \, dx + \int_{\Gamma_N^*} (g^* - \tilde{g}^*) W^* \, ds \\ &\quad - \int_{\Omega^*} (\varrho - \tilde{\varrho}) \nabla U^* \cdot \nabla W^* \, dx \\ (5.4) \quad &\leq \|W^*\|_{H^1(\Omega^*)} \eta_c \lesssim |e|_{H^1(\Omega)} \eta_c. \end{aligned}$$

For the first term on the right-hand side in (5.3), an integration by parts on each $T \cap \Omega$ is performed. Careful account on the exact boundary conditions results in

$$\begin{aligned} \int_\Omega \nabla e \cdot \nabla z \, dx &= \int_{\cup \mathcal{E} \cap \Omega} [\partial U / \partial n_\mathcal{E}] z \, ds \\ (5.5) \quad &+ \int_{\Gamma_N} (g - \partial U / \partial n) z \, ds + \int_\Omega f z \, dx - \int_\gamma z \partial U / \partial n \, ds \end{aligned}$$

with jump terms on $\cup \mathcal{E} \cap \Omega := (\cup \mathcal{E}) \cap \Omega$ within Ω . Next, we will derive an appropriate representation of the last integral in (5.5). Figure 2 illustrates how edges and boundary pieces might hit a hole $\omega \in \mathcal{H}$.

Let us consider one hole $\omega \in \mathcal{H}$ with outer normal $n_\omega = -n_\Omega = -n$ and boundary $\partial\omega = \gamma_\omega \cup \gamma_D^* \cup \gamma_N^*$, where $\gamma_\omega := (\partial\omega) \cap \gamma$, $\gamma_D^* := (\partial\omega) \cap \Gamma_D^*$, and $\gamma_N^* := (\partial\omega) \cap \Gamma_N^*$. The edges cut ω into at most countably many connectivity components illustrated in Fig. 2, $\omega \setminus (\cup \mathcal{E}) = \cup\{\omega_1, \dots, \omega_I\}$. On each ω_j , ∇U^* is constant and equal to $\nabla U^*|_{\omega_j}$. The divergence theorem shows

$$(5.6) \quad \int_{\partial\omega_j} \partial U^* / \partial n_{\omega_j} ds = \nabla U^*|_{\omega_j} \cdot \int_{\partial\omega_j} n_{\omega_j} ds = 0.$$

Therefore, for any real constant c_ω , we obtain

$$\int_{\partial\omega_j} z^* \partial U^* / \partial n_{\omega_j} ds = \int_{\partial\omega_j} (z^* - c_\omega) \partial U^* / \partial n_{\omega_j} ds.$$

Note that ∇U^* is, in general, discontinuous across $\cup \mathcal{E}_\Omega$. Besides the situation in Fig. 2 it may happen that γ_ω has a positive intersection with the skeleton $\cup \mathcal{E}_\Omega$, $|\gamma_\omega \cap (\cup \mathcal{E}_\Omega)| > 0$. Even in this case, we have

$$\partial U / \partial n = -\partial U^* / \partial n_\omega - [\partial U^* / \partial n_E] \quad \text{on } E \cap \gamma_\omega, E \in \mathcal{E}_\Omega.$$

Therefore,

$$\begin{aligned} & \int_{\gamma_N^*} z^* \partial U^* / \partial n_\omega ds - \int_{\gamma_\omega} z \partial U / \partial n ds - \int_{(\partial\omega) \cap (\cup \mathcal{E}_\Omega)} z^* [\partial U^* / \partial n_E] ds \\ &= \int_{\partial\omega} z^* \partial U^* / \partial n_\omega ds \\ &= \sum_{j=1}^I \int_{\partial\omega_j} z^* \partial U_j^* / \partial n_{\omega_j} ds + \int_{(\cup \mathcal{E}_\Omega) \cap \omega} z^* [\partial U^* / \partial n_E] ds. \end{aligned}$$

We used $z^* = 0$ on γ_D^* and that the definition of the jumps $[\partial U^* / \partial n_E]$ does not depend on the underlying orientation of n_E . The combination of the last four identities shows

$$(5.7) \quad \begin{aligned} \int_{\gamma_\omega} z \partial U / \partial n ds &= - \int_{(\gamma_\omega \cup \omega) \cap (\cup \mathcal{E})} z^* [\partial U^* / \partial n_E] ds + \int_{\gamma_N^*} z^* \partial U^* / \partial n_\omega ds \\ &+ \sum_{j=1}^I \int_{\partial\omega_j} (c_\omega - z^*) \partial U^* / \partial n_{\omega_j} ds. \end{aligned}$$

A summation over all holes $\omega \in \mathcal{H}$ and a rearrangement of boundary pieces of $\cup\{\partial\omega_j : j = 1, \dots, I\}$ yield

$$\begin{aligned}
 & \int_{\gamma} z \frac{\partial U}{\partial n} ds \\
 &= - \int_{(\Omega^* \setminus \Omega) \cap (\cup \mathcal{E})} z^* \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] ds + \int_{\Gamma_N^* \setminus \Gamma_N} z^* \frac{\partial U^*}{\partial n} ds \\
 (5.8) \quad &+ \sum_{\omega \in \mathcal{H}} \left(\int_{\partial \omega} (c_{\omega} - z^*) \frac{\partial U^*}{\partial n_{\omega}} ds + \int_{\omega \cap (\cup \mathcal{E})} (c_{\omega} - z^*) \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] ds \right).
 \end{aligned}$$

Combining this representation with (5.5) leads to

$$\begin{aligned}
 & \int_{\Omega} \nabla e \cdot \nabla z dx \\
 &= \int_{\cup \mathcal{E}_{\Omega}} [\partial U^* / \partial n_{\mathcal{E}}] z^* ds \\
 &+ \int_{\Gamma_N^*} (g^* - \partial U^* / \partial n) z ds + \int_{\Omega^*} f^* z^* dx \\
 (5.9) \quad &+ \sum_{\omega \in \mathcal{H}} \left(\int_{\partial \omega} (z^* - c_{\omega}) \frac{\partial U^*}{\partial n_{\omega}} ds + \int_{\omega \cap (\cup \mathcal{E})} (z^* - c_{\omega}) \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] ds \right).
 \end{aligned}$$

The first three summands on the right-hand side of (5.9) can be estimated with standard arguments (e.g., from [V]) utilising (5.2) and Cauchy’s inequality. The last contribution of (5.9) is bounded by

$$(5.10) \quad \sum_{\omega \in \mathcal{H}} h_{\omega}^{-1/2} \| z^* - c_{\omega} \|_{L^2(\omega \cap (\cup \mathcal{E}))} h_{\omega}^{1/2} \left\| \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\omega \cap (\cup \mathcal{E}))}.$$

The trace inequality (cf. Lemma 4.2), (5.1), and a Poincaré inequality with proper c_{ω} (cf. Subsection 4.4) result in

$$\begin{aligned}
 \| z^* - c_{\omega} \|_{L^2(\omega \cap (\cup \mathcal{E}))}^2 &\lesssim h_{\omega}^{-1} \| z^* - c_{\omega} \|_{L^2(V_{\omega})}^2 + h_{\omega} |z^*|_{H^1(V_{\omega})}^2 \\
 (5.11) \quad &\lesssim h_{\omega} |z^*|_{H^1(V_{\omega})}^2.
 \end{aligned}$$

Its combination with (5.10) yields (recall the finite overlap of the neighbourhoods V_{ω} from (5.1)) and the boundedness of the extension operator)

$$\sum_{\omega \in \mathcal{H}} \int_{\omega \cap (\cup \mathcal{E})} (z^* - c_{\omega}) [\partial U^* / \partial n_{\mathcal{E}}] ds \lesssim \eta_{\mathcal{E}} |z^*|_{H^1(\Omega^*)}.$$

For the second last term on the right-hand side of (5.9), we consider first the case that $|\Gamma_D^* \cap \partial \omega| = 0$ and employ analogous arguments to obtain

$$\sum_{\omega \in \mathcal{H}} \int_{\partial \omega} (z^* - c_{\omega}) \frac{\partial U^*}{\partial n_{\omega}} ds \lesssim \eta_{\mathcal{H}} |z^*|_{H^1(\Omega^*)},$$

since $\partial\omega \setminus \Gamma_D^*$ equals $\partial\omega$ up to a set of measure zero (cf. (3.3)). If $|\Gamma_D^* \cap \partial\omega| > 0$ we set $c_\omega = 0$. Employing $z^* = 0$ on Γ_D^* and the trace theorem (cf. Section 4.3) yield

$$\int_{\partial\omega} (z^* - c_\omega) \partial U^* / \partial n_\omega ds \lesssim \|\partial U^* / \partial n\|_{L^2(\partial\omega \setminus \Gamma_D^*)} \left(h_\omega^{-1} \|z^*\|_{L^2(V_\omega)}^2 + h_\omega |z^*|_{H^1(V_\omega)}^2 \right)^{1/2}.$$

Since $|\partial\omega \cap \Gamma_D^*| > 0$ and using again $z^*|_{\Gamma_D^*} = 0$, a Friedrichs inequality leads to $\|z^*\|_{L^2(V_\omega)}^2 \lesssim h_\omega |z^*|_{H^1(V_\omega)}^2$ and

$$\sum_{\omega \in \mathcal{H}} \int_{\partial\omega} (z^* - c_\omega) \frac{\partial U^*}{\partial n_\omega} ds \lesssim \eta_{\mathcal{H}} |z^*|_{H^1(\Omega^*)}.$$

The combination of the above estimates concludes the proof of Theorem 3.1. □

6 Efficiency: geometric preliminaries and main result

This section is devoted to the presentation of sufficient assumptions for the converse (called efficiency) estimate of Theorem 3.1 and so to the sharpness of that (reliability) estimate. For the ease of this discussion, we assume throughout this section that all holes are compactly embedded in Ω^* . Otherwise, the hole boundaries $(\partial\omega) \cap \Gamma_D^*$ and $(\partial\omega) \cap \Gamma_N^*$ would require a special treatment, i.e., a modification of the extension operator \mathfrak{E}_D . The main part of this section is devoted to characterise a class of holes (of quite general geometry) that allows for an efficiency estimate. The main result is stated in Theorem 6.1 and proved in the subsequent section.

Definition 6.1 *Let*

$$\mathcal{T}_\omega := \{T \in \mathcal{T} : T \cap \partial\omega \neq \emptyset\}.$$

Assumption 6.1 (a) *For any hole $\omega \in \mathcal{H}$, the neighbourhood V_ω from Notation 4.1 is compactly included in Ω^* .*

(b) *For any hole $\omega \in \mathcal{H}$, we have $\text{card } \mathcal{T}_\omega \lesssim 1$.*

Note that Assumption 6.1(b) is satisfied under the conditions stated in Remark 5.1.

Definition 6.2 *For any ball B let ρ_B be the standard mollifier $\rho_B \in \mathcal{D}(B)$ with $0 \leq \rho_B \leq 1$, i.e., for the ball B about z with radius $r > 0$,*

$$\rho_B(x) = \exp(1/(|x - z|^2/r^2 - 1)) \quad \text{if } x \in B \text{ and } \rho_B(x) = 0 \text{ else.}$$

Definition 6.3 Let $d = 1, 2$. For any d -dimensional measurable set $V \subset \mathbb{R}^2$, let $|\omega \cap V|$ denote the d -dimensional measure of $\omega \cap V$ and set

$$\mathcal{H}_V := \{\omega \in \mathcal{H} : |\omega \cap V| > 0\}.$$

Assumption 6.2 Suppose that, for any edge $E \in \mathcal{E}$, there exists a ball $B_E \subset \Omega^*$ with $E \cap B_E = \emptyset$ and

- (a) $\text{diam}(B_E) \approx h_E$ (size control),
- (b) $\text{dist}(B_E, E) \lesssim h_E$ (distance control),
- (c1) $\int_{\Omega} \rho_{B_E} dx \approx h_E^2$ (porosity control),
- (c2) $\sum_{\omega \in \mathcal{H}_{\sigma_E}} h_{\omega}^2 \lesssim h_E^2$ (porosity control),

where σ_E denotes the union of B_E with all triangles $T \in \mathcal{T}$ with $E \subset T$.

Note that Assumption 6.2(c2) is related to Remark 2.2.

Assumption 6.3 Suppose that, for any element $T \in \mathcal{T}$, there exists a ball $B_T \subset \Omega^* \setminus (\cup \mathcal{E})$ with

- (a) $\text{diam}(B_T) \approx h_T$ (size control),
- (b) $\text{dist}(B_T, T) \lesssim h_T$ (distance control),
- (c1) $\int_{\Omega} \rho_{B_T} dx \approx h_T^2$ (porosity control),
- (c2) $\sum_{\omega \in \mathcal{H}_{\sigma_T}} h_{\omega}^2 \lesssim h_T^2$ (porosity control),

where $\sigma_T := T \cup B_T$.

The local efficiency estimate for the contribution of the hole $\omega \in \mathcal{H}$ requires a set of additional assumptions.

First, we will mollify and extend the normal n along $\partial\omega$ (that points into $\omega; n = -n_{\omega}$) to a neighbourhood of $\partial\omega$. Some examples in Figure 5 illustrate Assumption 6.4.

Assumption 6.4 Suppose that, for each hole $\omega \in \mathcal{H}$, there exists $q = q(\omega) \in \mathbb{N}$, disjoint curves $\gamma_1, \dots, \gamma_q$, and open sectors of balls $B_1 = B_1(\omega), \dots, B_q = B_q(\omega) \subset \Omega$ so that

- (a) ω is a Lipschitz domain (global smoothness),
 - (b) $\partial\omega = \overline{\gamma_1 \cup \dots \cup \gamma_q}$ and $\gamma_1, \dots, \gamma_q$ are C^2 (local smoothness),
 - (c) an orthogonal projection $P : B \rightarrow \partial\omega$ (projection property),
can be defined on $B := \overline{B_1 \cup \dots \cup B_q}$ in a unique way
 - (d1) $\overline{B_j} \cap \gamma_j \neq \emptyset,$
 - (d2) $\overline{B_j} \cap \gamma_j$ has positive distance to $\partial\omega \setminus \gamma_j,$
 - (d3) $\overline{B_1}, \dots, \overline{B_q}$ are pairwise disjoint,
 - (d4) $h_{\omega} \sim |\gamma_j| \sim |\partial\omega| \sim |\overline{B_j} \cap V_{\omega}^{int}|^{1/2},$
- $\left. \begin{matrix} \text{(d1)} \\ \text{(d2)} \\ \text{(d3)} \\ \text{(d4)} \end{matrix} \right\} \begin{matrix} \text{(distortion)} \\ \text{control} \end{matrix}.$

Note that the projection P in Assumption 6.4 always exists. (Choose, for $1 \leq j \leq q$, the balls B_j centred at some point in γ_j so that B_j is sufficiently small and $B_j \cap \partial\omega = B_j \cap \gamma_j$.)

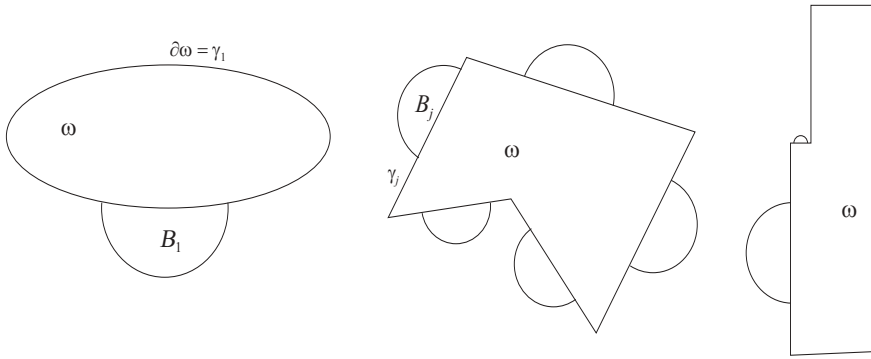


Fig. 5. The domains (a) and (b) satisfy Assumption 6.4 with moderate constants in the estimates while this is not the case for domain (c)

To mollify and to extend the normal field n to some neighbourhood of the hole boundaries we employ the ansatz (recall the definition of B and V_ω^{int} from Assumption 6.4 and Notation 4.1)

$$(6.1) \quad N := \begin{cases} \lambda n \circ P & \text{in } B, \\ 0 & \text{otherwise.} \end{cases}$$

The function λ is a generalization of bubble functions from the a posteriori error analysis [V] with an integral mean orthogonal to the direction $(\cos \alpha, \sin \alpha)$.

Assumption 6.5 *The function $\lambda = \lambda_\alpha \in C^\infty(\overline{\Omega})$ in (6.1) depends continuously on $\alpha \in [-\pi, \pi]$ and satisfies*

- (a) $\left. \begin{aligned} &\text{supp } \lambda_\alpha \subseteq \overline{B} \text{ and } 0 \leq \lambda_\alpha \leq 1 \text{ on } \partial\omega, \\ &|\lambda_\alpha|_{L^\infty(\overline{\Omega})} + h_\omega |\lambda_\alpha|_{W^{1,\infty}(\overline{\Omega})} \lesssim 1 \end{aligned} \right\} \quad (\text{cut-off function}),$
- (b) $\left. \begin{aligned} &\text{for any } \gamma_j \text{ there exists a sub-arc } \tilde{\gamma}_j \subset \gamma_j \\ &\text{with } |\gamma_j| \sim |\tilde{\gamma}_j| \text{ and } \lambda_\alpha \geq 1/2 \text{ on } \tilde{\gamma}_j \end{aligned} \right\} \quad (\text{positivity}),$
- (c) $\int_{V_\omega^{int}} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \cdot N \, dx = 0 \quad (\alpha\text{-orthogonality}).$

Remark 6.1 The compactness of $[-\pi, \pi]$ and the continuous dependence of λ on α imply an α -uniform estimate in Assumption 6.5.(a).

We need an abstract assumption on the hole boundaries.

Assumption 6.6 *The mollified and extended normal field N is of the form (6.1) where λ satisfies Assumption 6.5 and, for all $q \in \mathbb{R}^2$,*

$$(6.2) \quad \begin{aligned} &h_\omega^{-1} \|q \cdot N\|_{L^2(V_\omega^{int})}^2 + h_\omega |q \cdot N|_{H^1(V_\omega^{int})}^2 \\ &\lesssim \|q \cdot n\|_{L^2(\partial\omega)}^2 \lesssim \int_{\partial\omega} (q \cdot N) (q \cdot n) \, ds. \end{aligned}$$

We illustrate these abstract assumptions with two typical examples.

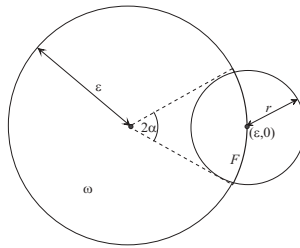


Fig. 6. Circular hole ω in Example 6.2

Example 6.1 (polygonal hole). Assume that ω is a polygonal hole satisfying Assumption 6.4 with straight lines γ_j orthogonal to n_j . Then,

$$\begin{aligned} \int_{V_\omega^{int}} |q \cdot N|^2 dx &= \sum_{j=1}^q \int_{B_j \cap V_\omega^{int}} |q \cdot \lambda n \circ P|^2 dx \\ &= \sum_{j=1}^q |q \cdot n_j|^2 \int_{B_j \cap V_\omega^{int}} \lambda^2 dx \lesssim \sum_{j=1}^q |q \cdot n_j|^2 |B_j \cap V_\omega^{int}| \\ &= \sum_{j=1}^q \frac{|B_j \cap V_\omega^{int}|}{|\gamma_j|} \int_{\gamma_j} |q \cdot n_j|^2 ds \lesssim h_\omega \int_{\partial\omega} |q \cdot n|^2 dx. \end{aligned}$$

The estimate of the H^1 -seminorm follows analogously by using

$$\int_{B_i \cap V_\omega^{int}} \|\nabla \lambda\|^2 dx \lesssim 1.$$

The second inequality in (6.2) follows from

$$\begin{aligned} \int_{\partial\omega} |q \cdot n|^2 dx &= \sum_{j=1}^q |q \cdot n_j|^2 \frac{|\gamma_j|}{|\tilde{\gamma}_j|} \int_{\tilde{\gamma}_j} 1 ds \\ &\lesssim \sum_{j=1}^q |q \cdot n_j|^2 \int_{\tilde{\gamma}_j} \lambda ds \leq \int_{\partial\omega} (q \cdot N) (q \cdot n) ds. \end{aligned}$$

Remark 6.2 The estimates in Example 6.1 are based on Assumptions 6.4 and 6.5 and so are the multiplicative constants hidden in the notation \lesssim .

Remark 6.3 The condition (6.2) is partly redundant as $\|q \cdot N\|_{L^2(V_\omega^{int})}$ is bounded by $\text{diam}(V_\omega^{int}) \|q \cdot N\|_{H^1(V_\omega^{int})}$ owing to a Friedrichs inequality.

The following example shows that Assumption 6.6 may hold with moderate constant for holes with curved boundary (cf. Fig. 6).

Example 6.2 (circle) If $\partial\omega$ is a circle and F is a sub-arc of $\partial\omega$ we find polar coordinates centred at the midpoint of ω and, without loss of generality,

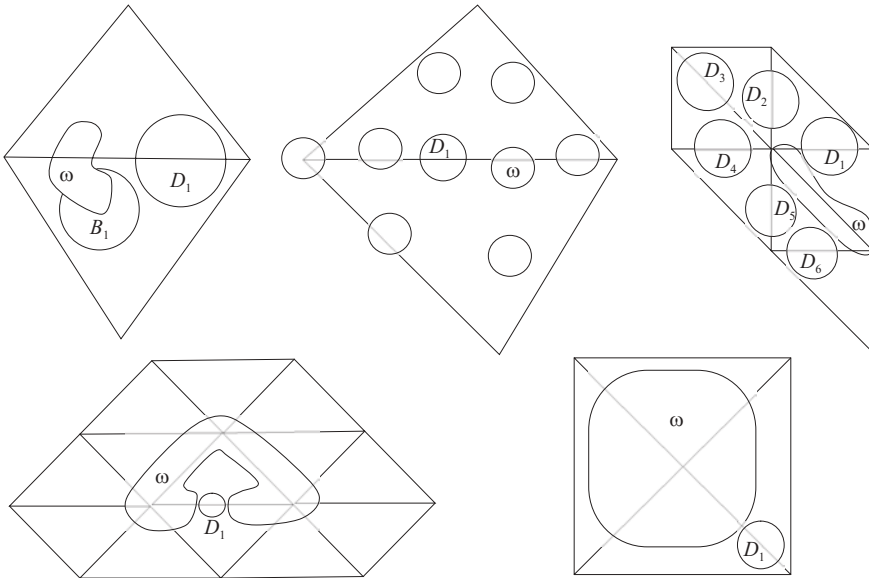


Fig. 7. Triangulations covering a hole ω . The discs of connecting edges are denoted by D_j while the segments of discs for each smooth component γ_j are denoted by B_j . The constant c_{15} is moderately bounded for the first three examples while this is not the case for the other examples

suppose $\omega = B(0, \varepsilon)$ and $F = \{\varepsilon(\cos(\varphi), \sin \varphi) : -\alpha < \varphi < \alpha\}$ for some $\varepsilon > 0$ and $0 < \alpha < \pi/2$. Let λ be a scaled mollifier with centre at $(\varepsilon, 0)$ and support $B := \overline{B((\varepsilon, 0), r)}$ for $r := \varepsilon \min\{1/2, \sqrt{2(1 - \cos \alpha)}\}$. Then, $N(r, \varphi) := \varrho_B(r, \varphi) (\cos \varphi, \sin \varphi)$ satisfies Assumption 6.6. The constant in (6.2) is independent of ε but degenerates if α is small.

Remark 6.4 The previous two examples illustrate that and how N can be constructed for a quite large class of piecewise smooth domains: Corners are cut-off. It is also clear that the support V_ω^{int} of N can be a subset of an arbitrary small neighbourhood of $\partial\omega$ on the expense of a large constant in (6.2).

Definition 6.4 A sequence $(T_j)_{j=0}^J$ of triangles in \mathcal{T} is edge-connected if, for $j = 1, 2, \dots, J$, the triangles T_{j-1}, T_j share a common edge E_j .

Some characteristic examples illustrating Assumption 6.7 are depicted in Fig. 7.

Assumption 6.7 For any hole $\omega \in \mathcal{H}$ and any $K \in \mathcal{T}_\omega$ there is a sequence of edge-connected triangles $(K_j)_{j=0}^J$ in \mathcal{T} with $J = J(K)$ such that, for all edges $E_j = K_{j-1} \cap K_j$, there exists a ball D_j with radius r_j centred at $M_j \in E_j$ and

- (a) $r_j \geq c_{15}h_\omega$,
- (b) $D_j \subset \Omega$,
- (c) the endpoints of E_j have positive distance to $D_j \cap E_j$.

Remark 6.5 Assumption 6.7 can be generalized by allowing more general domains D_j for connecting neighbouring triangles with finite overlap.

The constants in the preceding assumptions of this section enter in the multiplicative constant in the efficiency estimate. Its proof is the contents of the next section. Recall the definition of σ_T and σ_E from Assumption 6.2 and 6.3 and the definition of the error estimator as in (3.1)–(3.5) and Theorem 3.1.

Theorem 6.1 *Under the Assumptions 6.1–6.7 and notation of Section 2 we have*

$$\begin{aligned} \eta_\Omega^2 + \eta_{\mathcal{E}}^2 + \eta_{\mathcal{H}}^2 &\lesssim \|\nabla e\|_{L^2(\Omega)}^2 + \sum_{E \in \mathcal{E}_N} h_E \min_{g_E \in \mathbb{R}} \|(g - g_E)\|_{L^2(E \cap \Gamma_N)}^2 \\ &+ \sum_{T \in \mathcal{T}} h_T^2 \min_{f_T \in \mathbb{R}} \|f - f_T\|_{L^2(\Omega \cap \sigma_T)}^2 + \sum_{E \in \mathcal{E}} h_E^2 \min_{f_E \in \mathbb{R}} \|f - f_E\|_{L^2(\Omega \cap \sigma_E)}^2 \\ &+ \sum_{\omega \in \mathcal{H}} h_\omega^2 \min_{f_\omega \in \mathbb{R}} \|f - f_\omega\|_{L^2(V_\omega^{int})}^2. \end{aligned}$$

Remark 6.6 Theorem 6.1 even holds in a more local form as shown in the proof in Section 7.

Remark 6.7 The last three terms are of higher order if the right-hand side f is smooth in the sense that it is the restriction of a function F in $H^1(\Omega^*)$. Indeed, $f = F|_\Omega$ yields

$$\min_{f_T \in \mathbb{R}} \|f - f_T\|_{L^2(\Omega \cap \sigma_T)} \leq \min_{F_T \in \mathbb{R}} \|F - F_T\|_{L^2(\sigma_T)} \lesssim h_T \|\nabla F\|_{L^2(\sigma_T)}$$

according to (4.7) and Assumption 6.3. Analogous estimates are derived for $\min_{f_E \in \mathbb{R}} \|f - f_E\|_{L^2(\Omega \cap \sigma_E)}$ and $\min_{f_\omega \in \mathbb{R}} \|f - f_\omega\|_{L^2(V_\omega^{int})}$.

The second term is of higher order if there exists $G \in H^1(\Gamma_N^*)$ such that $g = G|_{\Gamma_N}$. In this case, there holds (cf. [CB])

$$\sum_{E \in \mathcal{E}_N} h_E \min_{g_E \in \mathbb{R}} \|(g - g_E)\|_{L^2(E \cap \Gamma_N)}^2 \lesssim \|h_{\mathcal{E}}^{3/2} \partial G / \partial s\|_{L^2(\Gamma_N^*)}^2.$$

7 Proof of efficiency

The following results provide local estimates summarised in Theorem 6.1. The combination of Lemma 7.2, 7.4, 7.5, and 7.6 is the proof of the theorem.

Definition 7.1 For any edge E with ball B_E from Assumption 6.2 let β_E be the piecewise quadratic product of the two barycentric coordinates with $\beta_E = \lambda_1 \lambda_2$ on $T \in \mathcal{T}$ with $E \subset \partial T$ that vanishes on $\partial T \setminus E$ and equals $s(h_E - s)/h_E^2$ on E with respect to the arc-length s . Let

$$b_E := \beta_E - c_E \rho_{B_E} \quad \text{for} \quad c_E := \int_{\Omega} \beta_E dx / \int_{\Omega} \rho_{B_E} dx \in \mathbb{R}.$$

Lemma 7.1 Under the Assumption 6.2, the function b_E from Definition 7.1 equals $s(h_E - s)/h_E^2$ on E with respect to the arc-length s and satisfies $\text{supp } b_E \subset \overline{\sigma_E}$,

$$(7.1) \quad \int_{\Omega} b_E dx = 0, \quad \text{Lip}(b_E) \lesssim 1/h_E, \quad \text{and} \quad \|\nabla b_E\|_{L^2(\sigma_E)} \lesssim 1.$$

Proof. Since $E \cap B_E = \emptyset$ in Assumption 6.2, $b_E = \beta_E$ on E and $\int_{\Omega} b_E dx = 0$ follows by definition of c_E . The functions ρ_{B_E} and β_E are Lipschitz with Lipschitz constant $\lesssim 1/h_E$. Hence it remains to verify $0 \leq c_E \lesssim 1$. Assumption 6.2.(c) shows $c_E \lesssim 1$. The remaining estimate then follows from Assumption 6.2. \square

Lemma 7.2 Let Assumption 6.2 be satisfied. For all $E \in \mathcal{E}_{\Omega}$ and $f_E \in \mathbb{R}$, $\mathcal{E}' := \mathcal{E} \setminus \{E\}$, we have

$$\begin{aligned} & \|h_{\mathcal{E}}^{1/2} [\partial U^*/\partial n_E]\|_{L^2(E)}^2 \lesssim \|\nabla e\|_{L^2(\Omega \cap \sigma_E)}^2 + h_E^2 \|f - f_E\|_{L^2(\Omega \cap \sigma_E)}^2 \\ & + \sum_{\omega \in \mathcal{H}_{\sigma_E}} h_{\omega} \|\partial U^*/\partial n_{\omega}\|_{L^2(\partial \omega)}^2 + \sum_{\omega \in \mathcal{H}_E} h_{\omega} \|[\partial U^*/\partial n_{\mathcal{E}}]\|_{L^2(\omega \cap (\cup \mathcal{E}'))}^2. \end{aligned}$$

Proof. Let J_E denote the constant $[\partial U^*/\partial n_E]$ and notice that (since b_E reads $s(h_E - s)/h_E^2$ on E)

$$(7.2) \quad h_E \|J_E\|_{L^2(E)}^2 = h_E/3 J_E \int_E [\partial U^*/\partial n_E] b_E ds.$$

The last integral has a representation as in (5.9) with z^* replaced by b_E , namely,

$$\begin{aligned} & \int_E J_E b_E ds \\ & = \int_{\Omega} \nabla e \cdot \nabla b_E dx - \int_{\Omega} f b_E dx \\ (7.3) \quad & + \sum_{\omega \in \mathcal{H}} \left(\int_{\partial \omega} (c_{\omega} - b_E) \frac{\partial U}{\partial n_{\omega}} ds + \int_{\omega \cap (\cup \mathcal{E})} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] (c_{\omega} - b_E) ds \right). \end{aligned}$$

For holes $\omega \in \mathcal{H}$ with $\omega \cap E = \emptyset$, the function b_E equals zero on $\omega \cap (\cup \mathcal{E})$ and we set in these cases $c_\omega = 0$. Thus, the sum $\sum_{\omega \in \mathcal{H}}$ over the last integral in (7.3) reduces to a sum over all $\omega \in \mathcal{H}_E$, i.e.,

$$\sum_{\omega \in \mathcal{H}_E} J_E \int_{\omega \cap E} (c_\omega - b_E) ds + \sum_{\omega \in \mathcal{H}_E} \int_{\omega \cap (\cup \mathcal{E}')} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] (c_\omega - b_E) ds.$$

For the remaining holes $\omega \in \mathcal{H}_E$, we choose the constant c_ω such that $\int_{\omega \cap E} (c_\omega - b_E) ds$ vanishes. Then, c_ω equals $b_E(\xi)$ for some ξ in the convex hull of $\omega \cap E$ and for any $x \in \omega$ we have $|x - \xi| \lesssim h_\omega$. By Lemma 7.1, b_E is Lipschitz with $\text{Lip}(b_E) \lesssim 1/h_E$. This and a Cauchy inequality for the length $|\partial \omega|$ show

$$\begin{aligned} \int_{\partial \omega} |c_\omega - b_E| |\partial U^* / \partial n_\omega| ds &\leq \|c_\omega - b_E\|_{L^2(\partial \omega)} \|\partial U^* / \partial n_\omega\|_{L^2(\partial \omega)} \\ (7.4) \qquad \qquad \qquad &\lesssim |\partial \omega|^{1/2} h_\omega / h_E \|\partial U^* / \partial n_\omega\|_{L^2(\partial \omega)}. \end{aligned}$$

The sum $\sum_{\omega \in \mathcal{H}}$ for the second last integral in (7.3) reduces to $\sum_{\omega \in \mathcal{H}_{\sigma_E}}$. Assumption 6.2 implies $\sum_{\omega \in \mathcal{H}_{\sigma_E}} h_\omega^2 \lesssim h_E^2$ and, thus,

$$\sum_{\omega \in \mathcal{H}} \int_{\partial \omega} |c_\omega - b_E| |\partial U^* / \partial n_\omega| ds \leq \left(\sum_{\omega \in \mathcal{H}_{\sigma_E}} h_\omega \|\partial U^* / \partial n_\omega\|_{L^2(\partial \omega)}^2 \right)^{1/2}.$$

In the same fashion, we obtain by using $|\omega \cap (\cup \mathcal{E}')| \lesssim h_\omega$ and Cauchy's inequality the estimate

$$\begin{aligned} &\sum_{\omega \in \mathcal{H}_E} \int_{\omega \cap (\cup \mathcal{E}')} \left| \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] (c_\omega - b_E) \right| ds \\ &\leq \sum_{\omega \in \mathcal{H}_E} \frac{h_\omega}{h_E} \left\| h_\omega^{1/2} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\omega \cap (\cup \mathcal{E}'))} \\ &\leq \left(\sum_{\omega \in \mathcal{H}_E} \left\| h_\omega^{1/2} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\omega \cap (\cup \mathcal{E}'))}^2 \right)^{1/2}. \end{aligned}$$

By construction, $\int_\Omega b_E dx = 0$. Hence, $\int_\Omega f b_E dx$ can be replaced by $\int_\Omega (f - f_E) b_E dx$. Taking into account (7.1) we are led to

$$\begin{aligned} \int_E J_E b_E ds &\leq \|\nabla e\|_{L^2(\Omega \cap \sigma_E)} + \|h_T (f - f_E)\|_{L^2(\Omega \cap \sigma_E)} \\ &+ \left(\sum_{\omega \in \mathcal{H}_{\sigma_E}} \left\| \frac{\partial U^*}{\partial n_\omega} \right\|_{L^2(\partial \omega)}^2 \right)^{1/2} + \left(\sum_{\omega \in \mathcal{H}_E} \left\| h_\omega^{1/2} \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] \right\|_{L^2(\omega \cap (\cup \mathcal{E}'))}^2 \right)^{1/2}. \end{aligned}$$

This concludes the proof. □

Definition 7.2 For any triangle T with ball B_T from Assumption 6.3 let β_T be the cubic bubble function which is the product of the three barycentric coordinates, $\beta_T = \lambda_1 \lambda_2 \lambda_3$, on T that vanishes on ∂T . Let

$$b_T := \beta_T - c_T \rho_{B_T} \quad \text{for} \quad c_T := \int_{\Omega} \beta_T dx / \int_{\Omega} \rho_{B_T} dx \in \mathbb{R}.$$

Lemma 7.3 Under the Assumptions 6.3 and 6.4, the function b_T from Definition 7.2 satisfies $\text{supp } b_T \subset \overline{\sigma_T}$, and

$$\int_{\Omega} b_T dx = 0, \quad \text{Lip}(b_T) \lesssim 1/h_T, \quad \text{and} \quad \|\nabla b_T\|_{L^2(\sigma_T)} \lesssim 1.$$

Proof. The proof is analogous to that of Lemma 7.1 and so omitted. □

Lemma 7.4 Let Assumptions 6.3 and 6.5 be satisfied. We have, for all $T \in \mathcal{T}$ and $f_T \in \mathbb{R}$,

$$(7.5) \quad \begin{aligned} \|h_T f\|_{L^2(\Omega \cap T)}^2 &\lesssim \|\nabla e\|_{L^2(\Omega \cap \sigma_T)}^2 + h_T^2 \|f - f_T\|_{L^2(\Omega \cap \sigma_T)}^2 \\ &+ \sum_{\omega \in \mathcal{H}_{\sigma_T}} h_{\omega} \|\partial U^* / \partial n_{\omega}\|_{L^2(\partial \omega)}^2. \end{aligned}$$

Proof. Suppose that f_T is the integral mean of f over $\Omega \cap \sigma_T$ and calculate

$$(7.6) \quad \|h_T f\|_{L^2(\Omega \cap \sigma_T)}^2 - \|h_T(f - f_T)\|_{L^2(\Omega \cap \sigma_T)}^2 = \|h_T f_T\|_{L^2(\Omega \cap \sigma_T)}^2.$$

Assumption 6.3 implies that $|\sigma_T| \lesssim h_T^2$ and that (7.6) is bounded by

$$(7.7) \quad \begin{aligned} h_T^2 \|b_T^{1/2} f_T\|_{L^2(\Omega \cap \sigma_T)}^2 &\lesssim h_T^2 |f_T| \int_{\Omega \cap \sigma_T} b_T |f_T - f| dx \\ &+ h_T^2 |f_T| \int_{\Omega \cap \sigma_T} b_T f dx. \end{aligned}$$

This, Cauchy’s, and Young’s inequalities lead to

$$(7.8) \quad \begin{aligned} \|h_T f\|_{L^2(\Omega \cap T)}^2 &\leq \|h_T f\|_{L^2(\Omega \cap \sigma_T)}^2 \\ &\lesssim \|h_T(f - f_T)\|_{L^2(\Omega \cap \sigma_T)}^2 + h_T^2 \left(\int_{\Omega \cap \sigma_T} b_T f dx \right)^2. \end{aligned}$$

We focus on the estimate of $\int_{\Omega \cap \sigma_T} b_T f dx$. By choosing $z^* = b_T$ in (5.9), we obtain

$$(7.9) \quad \begin{aligned} &\int_{\Omega \cap \sigma_T} f b_T dx \\ &= \int_{\Omega \cap \sigma_T} \nabla e \cdot \nabla b_T dx \\ &- \sum_{\omega \in \mathcal{H}} \left(\int_{\partial \omega} (b_T - c_{\omega}) \frac{\partial U^*}{\partial n_{\omega}} ds + \int_{\omega \cap (\cup \mathcal{E})} (b_T - c_{\omega}) \left[\frac{\partial U^*}{\partial n_{\mathcal{E}}} \right] ds \right). \end{aligned}$$

Next, we choose the constants c_ω in (7.9). For all holes $\omega \in \mathcal{H}$ which are compactly included in σ_T , we choose $c_\omega := \int_{\partial\omega} b_T ds / |\partial\omega|$ and observe

$$\|b_T - c_\omega\|_{L^2(\partial\omega)} \lesssim |\partial\omega|^{1/2} h_\omega / h_T.$$

For all remaining holes, we choose $c_\omega = 0$ and, since b_T vanishes at some point of $\partial\omega$ in these cases, we get the estimate

$$\|b_T - c_\omega\|_{L^2(\partial\omega)} = \|b_T\|_{L^2(\partial\omega)} \lesssim |\partial\omega|^{1/2} h_\omega / h_T.$$

Thus, the last sum in (7.9) vanishes while the second last sum can be estimated from above by

$$\begin{aligned} \sum_{\omega \in \mathcal{H}} \int_{\partial\omega} \left| (b_T - c_\omega) \frac{\partial U^*}{\partial n_\omega} \right| ds &\leq \sum_{\omega \in \mathcal{H}} \|b_T - c_\omega\|_{L^2(\partial\omega)} \left\| \frac{\partial U^*}{\partial n_\omega} \right\|_{L^2(\partial\omega)} \\ &\lesssim \sum_{\omega \in \mathcal{H}_{\sigma_E}} |\partial\omega|^{1/2} \frac{h_\omega}{h_T} \left\| \frac{\partial U^*}{\partial n_\omega} \right\|_{L^2(\partial\omega)} \\ &\lesssim \left(\sum_{\omega \in \mathcal{H}_{\sigma_E}} \left\| h_\omega^{1/2} \frac{\partial U^*}{\partial n_\omega} \right\|_{L^2(\partial\omega)}^2 \right)^{1/2}, \end{aligned}$$

where Assumptions 6.3(c2), 6.4(d4) have been employed for the last estimate. The combination of those estimates concludes the proof. \square

Lemma 7.5 *Under Assumptions 6.1, 6.3, 6.4, 6.5, 6.6, 6.7 we have, for any $f_\omega \in \mathbb{R}$,*

$$h_\omega^{1/2} \left\| \partial U^* / \partial n \right\|_{L^2(\partial\omega)} \lesssim |e|_{H^1(V_\omega^{int})} + h_\omega \|f - f_\omega\|_{L^2(V_\omega^{int})}.$$

Proof. Abbreviate $q_T := \nabla U^*|_T \in \mathbb{R}^2$ for $T \in \mathcal{T}$ and obtain

$$\begin{aligned} \int_{\partial\omega} |\partial U^* / \partial n|^2 ds &\lesssim \sum_{T \in \mathcal{T}_\omega} \int_{T \cap \partial\omega} |q_T \cdot n|^2 ds \\ (7.10) \qquad \qquad \qquad &\lesssim \max_{T \in \mathcal{T}_\omega} \int_{T \cap \partial\omega} |q_T \cdot n|^2 ds \end{aligned}$$

from $\text{card } \mathcal{T}_\omega \lesssim 1$. Let the maximum on the right-hand side of (7.10) be attained for $K \in \mathcal{T}$. Hence,

$$(7.11) \qquad \int_{\partial\omega} |\partial U^* / \partial n|^2 ds \lesssim \int_{\partial\omega} |q_K \cdot n|^2 ds.$$

Let α denote the polar angle of $q_K \in \mathbb{R}^2$ and define $\lambda = \lambda_\alpha$ as in Assumption 6.5. Combining $\partial u / \partial n = 0$ on γ , the last condition in Assumption 6.5, and the divergence theorem we derive, for any $f_\omega \in \mathbb{R}$,

$$\begin{aligned}
 & \|q_K \cdot n\|_{L^2(\partial\omega)}^2 \\
 & \lesssim \int_{\partial\omega} (q_K \cdot N)(q_K \cdot n) \, ds \\
 & = \int_{\partial\omega} (q_K \cdot N)((q_K - \nabla u) \cdot n) \, ds \\
 & = \int_{V_\omega^{int}} ((\nabla(q_K \cdot N)) (q_K - \nabla u) + (q_K \cdot N) (f - f_\omega)) \, dx.
 \end{aligned}$$

With Assumption 6.6, we infer

$$\begin{aligned}
 (7.12) \quad & \|q_K \cdot n\|_{L^2(\partial\omega)}^2 \lesssim \|(q_K \cdot n)\|_{L^2(\partial\omega)} h_\omega^{-1/2} \|q_K - \nabla u\|_{L^2(V_\omega^{int})} \\
 & + \|(q_K \cdot n)\|_{L^2(\partial\omega)} h_\omega^{1/2} \|f - f_\omega\|_{L^2(V_\omega^{int})}.
 \end{aligned}$$

The combination of (7.12) with (7.10) yields

$$h_\omega^{1/2} \|\partial U^*/\partial n\|_{L^2(\partial\omega)} \lesssim \|q_K - \nabla u\|_{L^2(V_\omega^{int})} + h_\omega \|f - f_\omega\|_{L^2(V_\omega^{int})}.$$

It remains to consider the first term on the right-hand side. Since

$$(7.13) \quad \|q_K - \nabla u\|_{L^2(V_\omega^{int})} \leq \|\nabla e\|_{L^2(V_\omega^{int} \cap K)} + \|q_K - \nabla u\|_{L^2(V_\omega^{int} \setminus K)}$$

it is sufficient to investigate the last term in (7.13). Take the sequence of edge-connected triangles $(K_j)_{j=0}^J$ as in Assumption 6.7 and recall the definition of the discs D_j with radii r_j therein. Put $q_j := q_{K_j}$ and notice that continuity of U^* along E_j implies that $q_j - q_{j-1}$ is parallel to n_{E_j} . Then,

$$(7.14) \quad |q_j - q_{j-1}| = |[\partial U^*/\partial n_{E_j}]| \quad \text{for } j = 1, 2, \dots, J.$$

Define a bubble function b_j supported in D_j with

$$\begin{aligned}
 & \|b_j\|_{L^\infty(D_j)} + r_j |b_j|_{W^{1,\infty}(D_j)} \lesssim 1, \\
 & \int_{D_j \cap E_j} b_j \, ds \approx r_j, \quad \text{and} \quad \int_{D_j} b_j = 0.
 \end{aligned}$$

Assumption 6.7 implies $U^* = U$ on $D_j \subset V_\omega^{int} \subset \Omega$. An integration by parts shows, as in the proof of Lemma 7.2,

$$\begin{aligned}
 (7.15) \quad & h_\omega |[\partial U^*/\partial n_{E_j}]| \approx \left| \int_{D_j \cap E_j} b_j [\partial U/\partial n_{E_j}] \, ds \right| \\
 & = \left| \int_{D_j} (\nabla b_j \cdot \nabla e - f b_j) \, dx \right| \\
 & \lesssim |e|_{H^1(D_j)} + h_\omega \|f - f_\omega\|_{L^2(D_j)}.
 \end{aligned}$$

The combination of (7.14) and (7.15) results in

$$(7.16) \quad \begin{aligned} \|q_j - q_{j-1}\|_{L^2(V_\omega^{int})} &\lesssim h_\omega |[\partial U / \partial n_{E_j}]| \\ &\lesssim |e|_{H^1(D_j)} + h_\omega \|f - f_\omega\|_{L^2(D_j)}. \end{aligned}$$

Owing to $J \lesssim 1$, triangle inequalities lead to

$$(7.17) \quad \|q_K - \nabla u\|_{L^2(V_\omega^{int} \cap T)} \leq |e|_{H^1(V_\omega^{int})} + \sum_{j=1}^J \|q_j - q_{j-1}\|_{L^2(V_\omega^{int})}.$$

Utilizing (7.16)–(7.17) and summing the result for all $T \in \mathcal{T} \setminus \{K\}$ with $T \cap V_\omega^{int} \neq \emptyset$ we conclude

$$\|q_K - \nabla u\|_{L^2(V_\omega^{int} \setminus K)} \lesssim |e|_{H^1(V_\omega^{int})} + h_\omega \|f - f_\omega\|_{L^2(V_\omega^{int})}.$$

The combination with (7.13) concludes the proof. \square

Lemma 7.6 *Under Assumptions 6.1, 6.5, 6.4, 6.6, 6.7 we have, for any $f_\omega \in \mathbb{R}$,*

$$(7.18) \quad h_\omega^{1/2} \|[\partial U^* / \partial n_\mathcal{E}]\|_{L^2(\omega \cap (\cup \mathcal{E}))} \lesssim |e|_{H^1(V_\omega^{int})} + h_\omega \|f - f_\omega\|_{L^2(V_\omega^{int})}.$$

Proof. We adopt the notations of the previous proof. Consider any $E \in \mathcal{E}$ satisfying $|E \cap \omega| > 0$. The estimate

$$h_\omega |[\partial U^* / \partial n_E]| \lesssim |e|_{H^1(D_j)} + h_\omega \|f - f_\omega\|_{L^2(D_j)}$$

is derived as (7.15). By employing $\sqrt{h_\omega |\omega \cap E|} / h_\omega \lesssim 1$ we obtain

$$(7.19) \quad h_\omega^{1/2} \|[\partial U^* / \partial n_E]\|_{L^2(\omega \cap E)} \lesssim |e|_{H^1(V_\omega^{int})} + h_\omega \|f - f_\omega\|_{L^2(V_\omega^{int})}.$$

Since, for any hole $\omega \in \mathcal{E}$, the number of edges E with $|E \cap \omega| > 0$ is bounded by a (moderate) constant (cf. (5.1)), a summation of (7.19) over all $E \in \mathcal{E}$ leads to (7.18). \square

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