PAMM · Proc. Appl. Math. Mech. 7, 1026203–1026204 (2007) / DOI 10.1002/pamm.200700559

A posteriori error analysis for eigenvalue problems

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A new a posteriori error analysis for symmetric eigenvalue problems of the adaptive finite element method (AFEM) is presented. Based on an H^1 stable L^2 projection, we prove reliability of the edge contribution for P_1 finite element methods. Hence the AFEM proposed can ignore the volume contribution as well as oscillations.

1 Symmetric Eigenvalue Problem

The symmetric eigenvalue problem consist of the following problem. Seek eigenpair $(u, \lambda) \in V \times \mathbb{R}^+$ with $\|u\|_{L^2(\Omega)} = 1$ such that

$$-\Delta u = \lambda u \text{ in } \Omega \subset \mathbb{R}^d, \qquad u = 0 \quad \text{on } \partial\Omega.$$

The variational formulation of the symmetric eigenvalue problem reads as follows. Seek $u \in V := H_0^1(\Omega)$ with $b(u, u) = 1$ and $\lambda \in \mathbb{R}$ such that

$$a(u, v) = \lambda b(u, v) \quad \text{for all} \quad v \in V.$$

Here,

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \text{and} \quad b(u, v) = \int_\Omega uv \, dx.$$

The discrete problem seeks for the eigenpair $(u_h, \lambda_h) \in V_h \times \mathbb{R}^+$ with $b(u_h, u_h) = 1$ such that $a(u_h, v_h) = \lambda_h b(u_h, v_h)$ for all $v_h \in V_h$. The Min-max principle allows for ordering of the $\dim(V_h)$ discrete eigenvalues $\lambda_i \leq \lambda_{i,h}$. The finite element discretisation leads to the general matrix eigenvalue problem $A_h x_h = \lambda_h M_h x_h$. with stiffness-matrix A_h and mass-matrix M_h.

2 A Priori Error Estimates

Let $V_h := P_1(\mathcal{T}_h)$ be the first-order finite element space with respect to triangulation $\mathcal{T}_h$ and maximal mesh-size h sufficiently small. For simple eigenvalue λ with eigenvector u there exists discrete eigenpair (λ_h, u_h) with error $e := u - u_h$ such that

$$0 \leq h^{-2} b(e, e) + a(e, e) + \lambda_h - \lambda \lesssim h^2 \lambda^2.$$

The fact that λ occurs in the upper bound suggests poor approximation of large eigenvalues.

3 A Posteriori Error Estimates

Lemma 3.1 (Error Identity) $a(e, e) = \lambda b(e, e) + \lambda_h - \lambda$. *This implies* $0 \leq \lambda_h - \lambda \leq a(e, e)$.

Lemma 3.2 (Error Residual Identity) *Define* $\mathrm{Res} := \lambda_h b(u_h, \cdot) - a(u_h, \cdot) \in V^*$ *with* $V_h \subset$ *ker* Res.

$$a(e, e) = \frac{\lambda + \lambda_h}{2} b(e, e) + \mathrm{Res}(e).$$

Raviart and Thomas show $||e||_{L^2(\Omega)} \lesssim h^r |e|_{H^1(\Omega)}$, where r depends on the geometry of Ω.
Therefore, $b(e, e) \lesssim h^{2r} a(e, e) = \text{h.o.t.}$
R.G. Duran, C. Padra and R. Rodriguez derived 1999 the estimate

$$0 \leq \lambda_h - \lambda + a(e, e) \lesssim \eta_{DPR}^2 + h^{2r} a(e, e), \quad \eta_{DPR}^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \lambda_h^2 \|u_h\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}_h} h_E \Big\| \left[\frac{\partial u_h}{\partial \nu_E} \right] \Big\|_{L^2(E)}^2.$$

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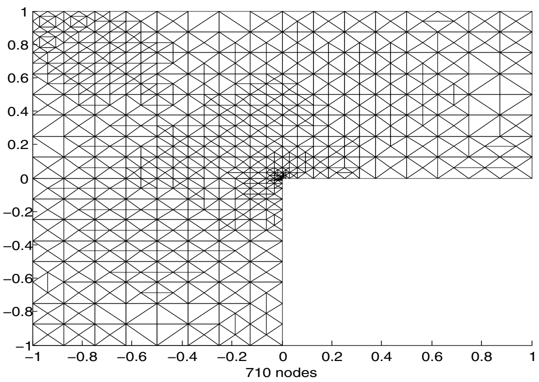

Fig. 1 Adaptively refined mesh for the L shaped domain.

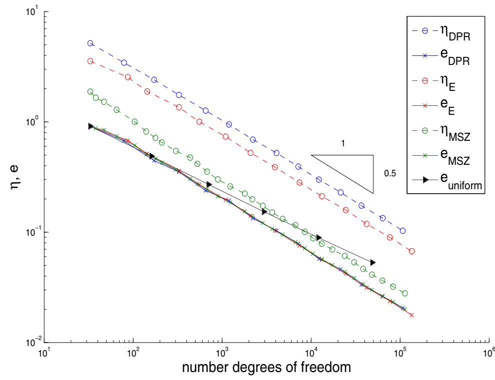

Fig. 2 Convergence history for the first eigenfunction of the L shaped domain.

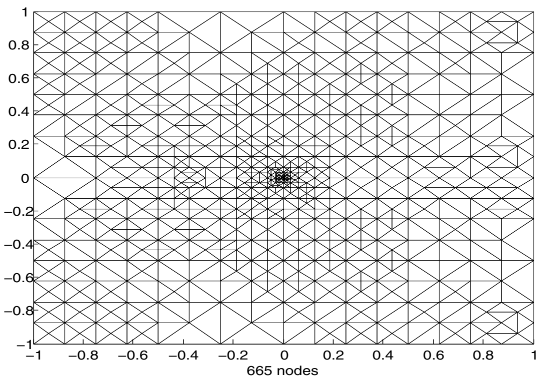

Fig. 3 Adaptively refined mesh for the slitted domain.

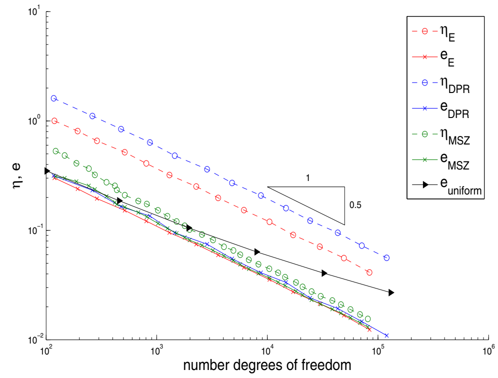

Fig. 4 Convergence history for the first eigenfunction of the slitted domain.

Considering the gradient recovery operator $G_h : V \to \mathcal{S}^1(\mathcal{T}_h)^d$, Mao, Shen and Zhou derived the estimate

$$0 \leq \lambda_h - \lambda + a(e,e) \lesssim \eta_{MSZ}^2 + h^{2r} a(e,e), \quad \eta_{MSZ}^2 = \sum_{T \in \mathcal{T}} \left(h_T^2 ||\lambda_h u_h + \mathrm{div}(G_h u_h)||^2_{L^2(T)} + ||G_h u_h - \nabla u_h||^2_{L^2(T)} \right).$$

Theorem 3.3 (New Edge Residual Estimator) $\| \,\mathrm{Res}\, \|^2_{V^*} \lesssim \eta_E^2 := \sum_{E \in \mathcal{E}_h} h_E \| \left[\frac{\partial u_h}{\partial \nu_E} \right] \|^2_{L^2(E)}$.

P r o o f. Let v_h be the L^2 projection of v in V_h which is H^1 stable for adaptive meshes.

$$\begin{aligned} \mathrm{Res}(v - v_h) &= \lambda_h b(u_h, v - v_h) - a(u_h, v - v_h) = -a(u_h, v - v_h) \\ &= \sum_{E \in \mathcal{E}_h} \int_E \left[\frac{\partial u_h}{\partial \nu_E} \right] (v - v_h)\, ds \lesssim \eta_E |v|_{H^1(\Omega)}. \end{aligned}$$

□

To sum up we now have the a posteriori estimate $0 \leq \lambda_h - \lambda + a(e,e) \lesssim \eta_E^2 + h^{2r} a(e,e)$.

References

[1] R. Duran, C. Padra, and R. Rodriguez, Math. Models Methods Appl. Sci. 9 **8**, 1165–1178 (1999).
[2] D. Mao, L. Shen, and A. Zhou, Advanced in Computational Mathematics **25**, 135–160 (2006).
[3] P. Raviart and J. Thomas, Introduction a l'Analyse Numerique des Equations aux Derives Partielles (Masson, 1983).
[4] G. Strang and G. Fix, An analysis of the finite element method (Prentice-Hall, 1973).
[5] T. Walsh, G. Reeseand, and U. Hetmanuik, Explicit a posteriori error estimates for eigenvalue analysis of heterogenous elastic structures, Tech. rep., Sandia Report, SAND2005-4237, 2005.