PAMM · Proc. Appl. Math. Mech. 7, 1151203–1151204 (2007) / **DOI** 10.1002/pamm.200700878

Strong convergence for large bodies in micromagnetics*

Carsten Carstensen [1, **], **Dirk Praetorius**[2, ***], and **Wolfgang Boiger**[1, †]

[1] Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, D-10099 Berlin, Germany.
[2] Technische Universität Wien, Institut für Analysis und Scientific Computing, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria.

The convexified Landau-Lifshitz minimisation problem in micromagnetics leads to a degenerate variational problem. Therefore strong convergence of finite element approximations cannot be expected in general. This paper introduces a stabilised finite element discretisation which allows for the strong convergence of the discrete magnetisation fields with reduced convergence order for a uniaxial model problem.

1 Introduction

Numerical simulations of stationary micromagnetic phenomena are most frequently based on the mathematical model named after LANDAU and LIFSHITZ. In the case of vanishing exchange energy, the problem reads: Minimise

$$E(\mathbf{m}) := \int_\Omega \phi(\mathbf{m})\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{\mathbf{R^d}} |\nabla u|^2\,dx \text{ over } \mathcal{A} := \big\{\mathbf{m}\in L^\infty(\Omega;\mathbf{R^d})\,\big|\,|\mathbf{m}(\mathbf{x})| = \mathbf{1} \text{ a.e.}\big\}. \quad (1)$$

Here, $\Omega \subset \mathbf{R^d}$, $d = 2, 3$, is a bounded Lipschitz domain and $\phi \in \mathcal{C}(\mathbf{S};\mathbf{R_{\geq 0}})$ is an even function on the unit sphere $\mathbf{S} = \big\{\mathbf{x}\in\mathbf{R^d}\,\big|\,|\mathbf{x}| = \mathbf{1}\big\}$. For uniaxial materials, ϕ reads $\phi(x) = \frac{1}{2}\big(1-(x\cdot\mathbf{e})^2\big)$ with a fixed $\mathbf{e}\in\mathbf{R^d}$, $|\mathbf{e}| = 1$, called easy axis. The function $\mathbf{f}\in L^2(\Omega;\mathbf{R^d})$ models an exterior field. The potential u is given by $\nabla u \in L^2(\mathbf{R^d};\mathbf{R^d})$ in the magnetostatic Maxwell's equation $\operatorname{div}(-\nabla u + \mathbf{m}) = 0$ in $\mathbf{R^d}$ in the sense of distributions. Since the stray field ∇u is unique (see [2, Proposition 2.1]), $\mathcal{P}\mathbf{m} := \nabla u$ is well-defined.

The minimum of (1) is in general not attained, as infimizing sequences $(\mathbf{m}_j)$ develop finer and finer oscillations without strong limit. However, there exists a weak limit $\mathbf{m}$, which is a solution of the convexified problem:

$$\text{Minimise} \quad E^{**}(\mathbf{m}) \quad \text{over} \quad \mathcal{A}^{**} = \operatorname{conv}(\mathcal{A}) = \big\{\mathbf{m}\in L^\infty(\Omega;\mathbf{R^d})\,\big|\,|\mathbf{m}(\mathbf{x})| \leq \mathbf{1} \text{ a.e. }\big\} \quad (2)$$
$$\text{with} \quad E^{**}(\mathbf{m}) := \int_\Omega \phi^{**}(\mathbf{m})\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{\mathbf{R^d}} |\nabla u|^2\,dx$$

with $\phi^{**}(x)$ being the convex hull of $\phi(x)$ defined for $|x| \leq 1$. Problem (2) is the weak L^2-Γ-limit of the classical model by LANDAU and LIFSHITZ with vanishing exchange energy. Solutions of (2) are equivalently characterised by the corresponding Euler-Lagrange equations (see Theorem 4.2 in [4] or [2, (2.5)–(2.6)]): Find $(\lambda, \mathbf{m}) \in L^2(\Omega)\times L^2(\Omega;\mathbf{R^d})$ such that

$$\mathcal{P}\mathbf{m} + \nabla\phi^{**}(\mathbf{m}) + \lambda\mathbf{m} = \mathbf{f} \quad \text{with} \quad \lambda \geq 0,\ |\mathbf{m}| \leq 1,\ \lambda(1-|\mathbf{m}|) = 0 \quad \text{a.e. in } \Omega. \quad (3)$$

Existence of solutions is shown in Theorem 4.2 of [4], whereas Theorem 2.2 of [2] yields uniqueness of the quantities $\mathcal{P}\mathbf{m}$, $\nabla\phi^{**}(\mathbf{m})$ and $\lambda\mathbf{m}$. In the uniaxial case, even $\mathbf{m}$ is unique.

2 Discrete Model

In our efforts towards discretisation, we follow an approach studied in [3] and use a piecewise constant approximation: The side-constraint $|\mathbf{m}_h| \leq 1$ is replaced by a penalisation term, and a stabilisation term σ is added:

$$\text{Minimise} \quad E^{**}_{\varepsilon,h}(\mathbf{m}_h) \quad \text{over} \quad \mathcal{L}^0(\mathcal{T}) \quad \text{with} \quad (4)$$
$$E^{**}_{\varepsilon,h}(\mathbf{m}_h) := \int_\Omega \phi^{**}(\mathbf{m}_h)\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{m}_h\,dx + \frac{1}{2}\int_{\mathbf{R^d}} |\mathcal{P}\mathbf{m}_h|^2\,dx + \frac{1}{2}\int_\Omega \frac{1}{\varepsilon}(|\mathbf{m}_h|-1)_+^2\,dx + \frac{1}{2}\sigma(\mathbf{m}_h,\mathbf{m}_h)$$

Here $\mathcal{T}$ is a partition of Ω, $\mathcal{L}^0(\mathcal{T})$ denotes the space of all $\mathcal{T}$-elementwise constant functions, and $h \in \mathcal{L}^0(\mathcal{T})$ is the mesh-size function, i.e. $h|_T := h_T := \operatorname{diam}(T)$. The stabilisation σ is a positive semi-definite bilinear form and $(\cdot)_+^2 := \max\{\cdot, 0\}^2$.

Theorem 2.3 of [2] guarantees (4) has at least one solution $\mathbf{m}_h$. The quantities $\mathcal{P}\mathbf{m}$ and $\nabla\phi^{**}(\mathbf{m}_h)$ are unique among the solutions. In the uniaxial case, the solution is unique.

* Supported by DFG Research Center MATHEON ¨Mathematics for key technologies¨ in Berlin.
** E-mail: cc@math.hu-berlin.de, Phone: +49 30 2093 5489, Fax: +49 30 2093 5444
*** E-mail: dirk.praetorius@tuwien.ac.at, Phone: +43 1 58801 10150, Fax: +43 1 58801 10196
† E-mail: boiger@math.hu-berlin.de, Phone: +49 30 2093 5427

3 Convergence Result

Suppose that $\mathcal{T}$ is a regular triangulation in the sense of Ciarlet and let $\mathcal{S}^1(\mathcal{T}) = \{\varphi \in \mathcal{C}(\Omega) \,|\, \forall T \in \mathcal{T},\ \varphi|_T \in \mathcal{P}_1(\mathcal{T})\}$ denote the finite element space consisting of globally continuous and $\mathcal{T}$-piecewise affine functions. Let $\mathcal{A}_{\mathbf{h}} : L^2(\Omega;\mathbf{R}^{\mathbf{d}}) \to \mathcal{S}_{\mathbf{1}}(\mathcal{T})^{\mathbf{d}}$ be a linear operator which satisfies (with an h-independent constant $c_1 > 0$)

- $\|\mathcal{A}_{\mathbf{h}}\mathbf{m}\|_{L^2(\Omega)} \le c_1\|\mathbf{m}\|_{L^2(\Omega)}$ for all $\mathbf{m} \in L^2(\Omega;\mathbf{R}^{\mathbf{d}})$,
- $\|\mathbf{m} - \mathcal{A}_{\mathbf{h}}\mathbf{m}\|_{L^2(\Omega)} \le c_1\|hD\mathbf{m}\|_{L^2(\Omega)}$ for all $\mathbf{m} \in H^1(\Omega;\mathbf{R}^{\mathbf{d}})$ and $h \to 0$,
- $\|D(\mathbf{m} - \mathcal{A}_{\mathbf{h}}\mathbf{m})\|_{L^2(\Omega)} \le c_1\|D\mathbf{m}\|_{L^2(\Omega)}$ for all $\mathbf{m} \in H^1(\Omega;\mathbf{R}^{\mathbf{d}})$ and $h \to 0$,

Examples for such operators $\mathcal{A}_{\mathbf{h}}$ are the L^2-projection onto $\mathcal{S}_1(\mathcal{T})^d$ as well as the Clément interpolation operator. With a suitably chosen constant $c_2 > 0$ (see [2, (3.2)]), we define for $\mathbf{m}_h, \mathbf{n}_h \in \mathcal{L}^0(\mathcal{T})$ the following stabilisation term:

$$\sigma(\mathbf{m}_h, \mathbf{n}_h) = \frac{1}{c_2}\left\{\langle(\mathrm{id} - \mathcal{A}_{\mathbf{h}})\mathbf{m}_h\ ;\ (\mathrm{id} - \mathcal{A}_{\mathbf{h}})\mathbf{n}_h\rangle_{L^2(\Omega)} + \langle hD\mathcal{A}_{\mathbf{h}}\mathbf{m}_h\ ;\ hD\mathcal{A}_{\mathbf{h}}\mathbf{n}_h\rangle_{L^2(\Omega)}\right\}$$

Denote $h_{\max} = \max_{T\in\mathcal{T}} h_T$ and $h_{\min} = \min_{T\in\mathcal{T}} h_T$. For $d = 2$ and the discrete energy $E^{**}_{\varepsilon,h}$ in (4) equipped with this stabilisation term, Thoerem 3.1 of [2] states the following: If $(\lambda, \mathbf{m})$ and $\mathbf{m}_h$ are the solutions of (3) and (4) for the uniaxial case, respectively, and if $\mathbf{m}, \lambda\mathbf{m} \in H^\alpha(\Omega;\mathbf{R}^{\mathbf{2}})$ for some $\alpha \in (0, 1]$ and $\varepsilon = \mathcal{O}(h^\alpha_{\max})$, then

$$\|\mathbf{m} - \mathbf{m}_h\|_{L^2(\Omega)} = \mathcal{O}(h^{3\alpha/2}_{\max}/h_{\min}).$$

This yields convergence for quasiuniform meshes and $\alpha > 2/3$.

4 Error Estimators

The following result provides error estimations of the stray field and the magnetisation in the direction of the easy axis of the uniaxial case. Theorem 3.3 of [1] shows without stabilisation ($\sigma = 0$), that with an h-independent constant $c_3 > 0$

$$\begin{aligned}&\|\mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h\|_{L^2(\mathbf{R}^{\mathbf{d}})} + \|\nabla\phi^{**}(\mathbf{m}) - \nabla\phi^{**}(\mathbf{m}_h)\|_{L^2(\Omega)} \le \\ &\le c_3\Big\{\langle(\mathbf{f} - \mathbf{f}_{\mathcal{T}}) - (\mathcal{P}\mathbf{m}_h - (\mathcal{P}\mathbf{m}_h)_{\mathcal{T}})\ ;\ \mathbf{m} - \mathbf{m}_{\mathcal{T}}\rangle_{L^2(\Omega)} \\ &\quad + \|(|\mathbf{m}_h| - 1)_+\,((\mathbf{f} - \mathbf{f}_{\mathcal{T}}) - (\mathcal{P}\mathbf{m}_h - (\mathcal{P}\mathbf{m}_h)_{\mathcal{T}}))\,\|_{L^2(\Omega)} + \|(|\mathbf{m}_h| - 1)_+\|_{L^2(\Omega)}\Big\}.\end{aligned}$$

The first term on the right hand side, though not being a posteriori, can be dominated with Hölders inequality. This estimate gives rise to several error estimators that are either reliable, or (expected to be) efficient, but not both. This phenomenon is called *Reliability-Efficiency-Gap* and illustrated in Figure 1.

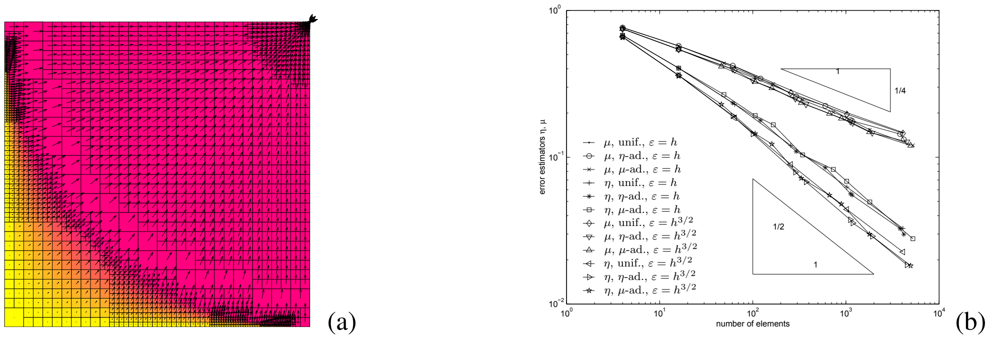

Fig. 1 Numerical solution of a problem with $\mathbf{m} \notin H^1$ (a) and Reliability-Efficiency-Gap of error estimators (b).

Acknowledgements The second author DP thankfully acknowledges the support of the Austrian Science Fund FWF under grant P15274. The third author WB thankfully acknowledges the support of the Deutsche Forschungsgemeinschaft (DFG) through the RTG 1128.

References

[1] C. Carstensen, D. Praetorius, *Numerical analysis for a macroscopic model in micromagnetics*, SIAM J. Numer. Anal., **42**, (2005).
[2] C. Carstensen, D. Praetorius, *Stabilization yields strong convergence of macroscopic magnetization vectors for micromagnetics without exchange energy*, IMA J. Numer. Anal., (to appear).
[3] C. Carstensen, A. Prohl, *Numerical analysis of relaxed micromagnetics by penalised finite elements*, Numer. Math., **90** (2005).
[4] A. De Simone, *Energy Minimizers for Large Ferromagnetic Bodies*, Arch. Rat. Mech. Anal., **125** (1993).