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Nonstandard Finite Element Methods

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Abstract. After just over half a century of use, finite element methods are ubiquitous in engineering (where they were first developed) and science. The mathematical analysis of standard conforming finite elements is very well advanced giving rise to highly efficient codes particularly for elliptic problems. More recently, to improve robustness, computational aspects or to provide extra properties (e.g. appropriate conservation properties) new finite element methods such as discontinuous Galerkin methods, generalized finite element methods and high order methods have been developed. These "non-standard" finite element methods are the subject of this Oberwolfach workshop. The extended abstracts here represent a snapshot of a varied and quickly evolving field.

Mathematics Subject Classification (2000): 65M50, 65M55, 65M60, 65N50, 65N55, 65N38.

Introduction by the Organisers

The workshop *Nonstandard Finite Element Methods*, organized by Susanne C. Brenner (Baton Rouge), Carsten Carstensen (Berlin) and Peter Monk (Newark) was held August 10 – 16, 2008. This meeting was well attended with over 40 participants with broad geographic representation.

Although Courant is often credited with the discovery of finite elements [2, 3], the first practical use of such methods in engineering seems to date back to the work of Argyris and Clough et. al. (apparently Clough coined the name "finite elements") in the 1950s [1, 4]. By the 1970s the finite element method was firmly established in engineering practice and the basic theory of conforming elements for elliptic problems was well understood [2]. Engineers and mathematicians have since expanded the use of finite elements to a wide variety of new applications and have improved the theoretical underpinnings of the method. In addition, efficient computational algorithms have been developed, such as the multigrid method,

that allow for the application of finite elements to sophisticated three dimensional problems.

Within current research, we can distinguish classical or standard finite element methods, and newer nonstandard finite elements The former include conforming elements applied to"standard" Galerkin discretizations of problems. Usually accuracy is obtained by mesh refinement. As the range of applications has increased, and to alleviate perceived shortcomings in standard conforming elements, numerical analysts and engineers have sought more general finite element methods based, for example, on mixed formulations, and non-conforming or even discontinuous Galerkin schemes. These methods may be superior to standard finite elements in particular applications because of enhancements to stability, robustness or conservation properties. We refer to them as non-standard finite element methods.

This Oberwolfach workshop was devoted to non-standard finite element methods and their analyses. Inevitability such a snapshot of the subject is biased, but we have tried to include non-standard methods in the broadest sense including mixed, discontinuous Galerkin, generalized, partition of unity and mortar FEM methods. All these schemes have in common that stability and convergence is not obvious and requires mathematical analysis. This is even more true for developing fast solvers, a posteriori error estimation and adaptive mesh design.

Through the week there were 29 presentations on various non-standard topics. These provided an overview of current research directions, new developments and open problems.

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Workshop: Nonstandard Finite Element Methods

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Abstracts

Computable error bounds for mixed *hp*-finite element approximation

Mark Ainsworth

We consider the model problem of a second-order linear elliptic problem in two dimensions governing the pressure in incompressible, irrotational flow in a porous medium with a piecewise constant local permeability tensor. The problem is expressed in mixed form whereby the flux is introduced as an additional independent unknown and the equation written as a first order system.

The approximation of such a system is often carried out based on a so-called Galerkin mixed finite element approximation where the pressure and flux are approximated independently using finite dimensional subspaces on a triangulation of the physical domain. It is well-known that the finite dimensional spaces must be chosen in a compatible fashion if the scheme is to be consistent and stable. The popular Raviart-Thomas mixed finite element scheme is based on seeking a flux from $H(div)$ which is locally to the polynomial space $\mathbb{P}_N^2 + \mathbf{x}\mathbb{P}_N$ in conjunction with a discontinuous pressure which locally belongs to the space $\mathbb{P}_N$. This scheme may be shown to be stable with a stability constant that is independent of the local order N of approximation and the mesh-size. Moreover, it is possible to vary the order N from element to element along with the local mesh-size provided that certain mild compatibility conditions are respected. This flexibility is vital if one is to efficiently resolve local features of the flow such as singularities.

We present a fully computable bound for the error in the flux variable obtained using the above scheme, measured in a least squares norm weighted with the permeability. The bound is obtained by post-processing the flux approximation locally over individual elements, and then performing an appropriate smoothing across element inter-faces to obtain a globally continuous approximation to the pressure variable. The difference between the corresponding flux and the finite element flux is shown to provide a computable bound for the error in the case when the source terms are piecewise polynomial. The technique is then extended to the case of non-polynomial source terms by augmenting the estimator with an additional (fully computable) term measuring the discrepancy from polynomial data. The resulting a posteriori error estimator that provides actual, guaranteed computable upper bounds on the error in the flux variable regardless of jumps in the material coefficients across interfaces. Moreover, the estimator is efficient in that it provides a local lower bound on the error up to a constant that is independent of the solution and the local mesh-size, although it is possible that the estimator will degenerate as the order N is increased. The estimator may be evaluated at virtually no additional cost compared to the evaluation of the finite element approximation itself, and generalises the technique presented in [1] to the case $N > 0$.

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Discretization of Saddle Point Problems without the LBB Condition

Constantin Bacuta

Based on spectral results for Schur complement operators we prove a convergence result for the inexact Uzawa algorithm on general Hilbert spaces. We prove that for any symmetric and coercive saddle point problem, the inexact Uzawa algorithm converges, provided that the inexact process for inverting the residual at each step has the relative error smaller than a computable fixed threshold. As a consequence, we provide a new type of algorithms for discretizing saddle point problems, which implement the inexact Uzawa algorithm at the continuous level as a multilevel algorithm. The discrete stability Ladyshenskaya-Babušca-Brezzi (LBB) condition might not be satisfied.

We present next the main result used in building and analyzing the new algorithms. We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual parings on $\mathbf{V}^* \times \mathbf{V}$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate operators
$\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle = a_0(\mathbf{u},\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{V}, \quad \text{and} \quad \langle \mathcal{C}p, q\rangle = (p,q) \quad \text{for all } \ p,q \in Q.$$

Next, we consider that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, satisfying the inf-sup condition. More precisely, we assume that

$$\inf_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},p)}{\|p\|\ |\mathbf{v}|} = m > 0, \quad \text{and} \quad \sup_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},p)}{\|p\|\ |\mathbf{v}|} = M < \infty.$$

For $f \in \mathbf{V}^*$, $g \in Q^*$, we consider the following variational problem:

Find $(\mathbf{u},p) \in \mathbf{V} \times Q$ such that

$$\begin{array}{rcll} \mathcal{A}\mathbf{u} & + & B^*p & = \mathbf{f}, \\ B\mathbf{u} & & & = g. \end{array}$$

It is known that the above system has a unique solution for any $f \in \mathbf{V}^*$, $g \in Q^*$, see [5, 7, 8]. Following the ideas in [4, 6], we will investigate the convergence of an abstract inexact Uzawa algorithm. In the standard Uzawa algorithm, the exact solve of the elliptic problem (the action of $\mathcal{A}^{-1}$) is replaced by an approximation process involving the residual of the first equation. We describe the approximate process as a map Ψ defined on a subset of $\mathbf{V}^*$, which for $\phi \in \mathbf{V}^*$, returns an approximation of ξ, the solution of $A\xi = \phi$. If $\mathbf{V}$ and Q are not finite dimensional spaces, then $\Psi(\phi)$ can be considered as a discrete Galerkin approximation of the elliptic problem $A\xi = \phi$. More precisely, the inexact Uzawa algorithm for approximating the solution $(\mathbf{u},p)$ is as follows.

Inexact Uzawa Method (IUM). Let $(\mathbf{u}_0, p_0)$ be any approximation for $(\mathbf{u}, p)$, and for $k = 1, 2, \cdots$, construct $(\mathbf{u}_k, p_k)$ by

$$\mathbf{u}_k = \mathbf{u}_{k-1} + \Psi(\mathbf{f} - \mathcal{A}\mathbf{u}_{k-1} - B^* p_{k-1}),$$
$$p_k = p_{k-1} + \alpha \mathcal{C}^{-1}(B\mathbf{u}_k - g).$$

For $k = 0, 1, \ldots$, let $e_k^{\mathbf{u}} := \mathbf{u} - \mathbf{u}_k$, $e_k^p := p - p_k$, $r_k = \mathbf{f} - \mathcal{A}\mathbf{u}_k - B^* p_k$, $e_k^{\mathbf{r}} := e_k^{\mathbf{u}} + \mathcal{A}^{-1} B^* e_k^p = A^{-1} r_k$, and let $S_0 := \mathcal{C}^{-1} B \mathcal{A}^{-1} B^* : Q \to Q$ be **the Schur complement on** Q.

The Main Result. Let $0 < \alpha < 2/M^2$ and assume that Ψ satisfies

$$\left|\Psi(r_k) - A^{-1} r_k\right|_{\mathbf{V}} \leq \delta \left|A^{-1} r_k\right|_{\mathbf{V}}, \quad k = 0, 1, \ldots,$$

with

$$\delta < \frac{2 - \alpha M^2}{2 + \alpha M^2}.$$

Then, the **IUM** converges. There exists $\rho = \rho(\alpha, \delta, m, M) \in (0, 1)$ such that

$$(\delta |e_k^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_k^p\|_{S_0}^2)^{1/2} \leq \rho^k \ (\delta |e_0^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_0^p\|_{S_0}^2)^{1/2} \quad k = 1, 2, \ldots.$$

The convergence factor $\rho = \rho(\alpha)$ is optimal (minimal) for

$$\alpha_{opt} := \frac{1 - \delta}{1 + \delta} \ \frac{2}{m^2 + M^2}.$$

The convergence result for the algorithm at the continuous level, combined with standard techniques of discretization and a posteriori error estimates leads to new and efficient algorithms for solving saddle point systems, see [1, 2, 3]. New applications of the Schur complements, including sharp estimates for Arrow-Hurwicz algorithms for non-symmetric saddle point systems are the focus of the author's work in progress.

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Main Ideas in Meshless Methods and the Generalized Finite Element Method

Uday Banerjee

In this talk, we presented the meshless method (MM) as a Galerkin method (to approximate the solution $u(x)$ of an elliptic PDE), where the construction of the shape functions $\{\phi_i(x)\}_{i=1}^n$ of the finite dimensional subspace $S_h = \text{span}\{\phi_i\}$, either does not depend on or depends minimally on a mesh. The functions $\{\phi_i\}$ are associated with a set of points $\{x_i\}$, called *particles*, and they have compact supports $\{\overline{\omega}_i\}$. The ϕ_i's are constructed such that they "reproduce" polynomials of degree k, i.e.,

$$\sum_i p(x_i)\phi_i(x) = p(x), \quad \forall p \in \mathcal{P}^k \tag{1}$$

These shape functions are not piecewise polynomials. A formula for $\phi_i(x)$ is often not available and it has to be "computed" for each x; it is one of the price of not using a mesh. The property (1) of the ϕ_i's dictates the approximation property of the subspace S_h [2, 6, 9]. There are many classes of such shape functions used in practice, e.g., the RKP shape functions [8].

But the approximation by a MM, as described above, does not incorporate any local information on the approximated function (the unknown solution $u(x)$), which is available in many situations. The Generalized Finite Element method (GFEM), which could be viewed as an extension of the MM, can incorporate the available local information on the unknown solution $u(x)$ [5, 10, 11]. In GFEM, we start with the functions $\{\phi_i\}$ satisfying (1); note that they form a partition of unity subordinate to the open sets $\{\omega_i\}$. For each i, we choose a finite dimensional space V_i of functions defined on ω_i. These functions mimic the known behavior of $u(x)$ in ω_i. The finite dimensional approximating space used in the GFEM is defined as $S_h \equiv \sum_i \phi_i V_i$. The accuracy of the GFEM solution depends on how well the functions in V_i approximates $u(x)$ in ω_i for each i. If we know that $u(x)$ is smooth in a particular ω_i, then one may choose $V_i = \mathcal{P}^k(\omega_i)$. But if $u(x)$ has a singularity of the type r^α in Ω, we include the singular function r^α in V_i. In other situations, it may be necessary to solve (numerically) a local auxiliary problem to obtain functions in V_i. Identifying the auxiliary problem is a challenge and there are a few ideas available in the literature [1, 7, 12].

Avoiding a mesh in the methods described above gives rise to various difficulties. One of them is the use of numerical integration to compute the elements of the stiffness matrix and the load vector. The effect of this "variational crime" is very different from the standard FEM and partial results in the context of meshless methods are available [4]. Finally, these the methods are "expensive" to implement and thus it is important to identify a class of problems where these methods are clearly superior to FEM.

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Approximation of Harmonic Maps and Wave Maps

Sören Bartels

Partial differential equations with a nonlinear pointwise constraint defined through a manifold N occur in a variety of applications: The magnetization of a ferromagnet can be described by a unit length vector field $m : \Omega \to S^2$ and the orientation of the rod-like molecules that constitute a liquid crystal is often modeled by a vector field that attains its values in the real projective plane $\mathbb{R}P^2$ thus respecting the head-to-tail symmetry of the molecules. Other applications arise in geometric modeling, quantum mechanics, and general relativity. Simple examples reveal that it is impossible to satisfy pointwise constraints exactly by lowest order finite elements. For two model problems we discuss the practical realization of the constraint and the efficient solution of the resulting nonlinear systems of equations.

Let $T > 0$, $\Omega \subset \mathbb{R}^m$, $m = 2, 3$, a bounded Lipschitz domain, and $N \subset \mathbb{R}^{n+1}$ a convex hypersurface, i.e., $N = \partial\mathcal{C}$ for a convex set $\mathcal{C}$. Let either $\widehat{X} = \Omega$ and g denote the standard Euclidean metric on $\mathbb{R}^m$ or $\widehat{X} = (0, T) \times \Omega$ and g denote the Lorentzian metric on $\mathbb{R}^{m+1}$. We then consider critical points of the functional

$$E_{g,\widehat{X}}(v) = \int_{\widehat{X}} \left|Dv\right|_g^2 \mathrm{d}\widehat{x}$$

among mappings $v : \widehat{X} \to N$ and subject to certain boundary condtions. If $\widehat{X} = \Omega$ then critical points $u : \Omega \to N$ are called *harmonic maps into* N and satisfy

$$-\Delta u \perp T_u N, \qquad u|_\Gamma = u_{\mathrm{D}}, \tag{1}$$

where $\Gamma = \partial\Omega$. If $\widehat{X} = (0,T) \times \Omega$ we look for critical points $u : (0,T) \times \Omega \to N$ called *wave maps into* N that solve the initial boundary value problem

$$\partial_t^2 u - \Delta u \perp T_u N, \quad \partial_n u = 0, \quad u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = v_0. \tag{2}$$

To approximate harmonic maps or wave maps we consider a regular triangulation $\mathcal{T}_h$ of Ω into triangles or tetrahedra and nodes (vertices of elements) contained in the set $\mathcal{N}_h$. We assume that $\mathcal{T}_h$ is weakly acute, i.e., $\int_\Omega \nabla\varphi_z \cdot \nabla\varphi_y \,\mathrm{d}x \le 0$ for distinct $z, y \in \mathcal{N}_h$ and the nodal basis $\big(\varphi_z : z \in \mathcal{N}_h\big)$ of the lowest order finite element space $\mathbb{V}_h$ subordinate to $\mathcal{T}_h$. We let $\mathcal{I}_h : C(\overline{\Omega}) \to \mathbb{V}_h$ denote the nodal interpolation operator. According to [2, 4] the triangulation $\mathcal{T}_h$ is weakly acute if and only if

$$\big\|\nabla \mathcal{I}_h\big[P \circ v_h\big]\big\| \le |P|_{W^{1,\infty}} \big\|\nabla v_h\big\| \qquad \forall v_h \in \mathbb{V}_h \ \forall P \in W^{1,\infty}(\mathbb{R}). \tag{3}$$

This fact implies the discrete maximum principle for the Dirichlet problem: If $u_h \in \mathbb{V}_h$ is minimal for $v_h \mapsto \|\nabla v_h\|^2$ among all $v_h \in \mathbb{V}_h$ subject to $v_h|_\Gamma = u_{\mathrm{D},h}$ then $\widetilde{u}_h := \mathcal{I}_h\big[P \circ u_h\big]$, for $P(s) := \min\{s, \overline{u}_{\mathrm{D},h}\}$ and $\overline{u}_{\mathrm{D},h} := \max u_{\mathrm{D},h}$, satisfies $\big\|\nabla\widetilde{u}_h\big\| \le \big\|\nabla u_h\big\|$. Thus $\widetilde{u}_h = u_h$ and $u_h \le \overline{u}_{\mathrm{D},h}$.

Motivated by work in [1, 2] we propose an iterative approximation of harmonic maps into N by a successive minimization of the Dirichlet energy subject to the linearized constraint about the current iterate and a subsequent projection of the update:

Alg. 1. *Let $u_h^0 \in \mathbb{V}_h^{n+1}$ such that $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h$ and $u_h^0|_\Gamma = u_{\mathrm{D},h}$:*

(1) *Compute $v_h^{j+1} \in \mathbb{V}_{h,0}^{n+1}$ such that $v_h^{j+1}(z) \in T_{u_h^j(z)}N$ for all $z \in \mathcal{N}_h$ and*

$$\big(\nabla\big[u_h^j + v_h^{j+1}\big], \nabla w_h\big) = 0$$

for all $w_h \in \mathbb{V}_{h,0}^{n+1}$ such that $w_h(z) \in T_{u_h^j(z)}N$ for all $z \in \mathcal{N}_h$.

(2) *Set*

$$u_h^{j+1} := \mathcal{I}_h\big[\pi_{\mathcal{C}} \circ \big(u_h^j + v_h^{j+1}\big)\big].$$

Here, T_pN denotes the tangent space of N at $p \in N$ and $\pi_{\mathcal{C}}$ is the orthogonal projection onto the convex set $\mathcal{C}$. Well posedness of the iteration is a consequence of the Lax-Milgram lemma and the fact that $\pi_{\mathcal{C}}$ globally well defined. Stability follows from choosing $w_h = v_h^{j+1}$, i.e.,

$$\big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|^2 - \big\|\nabla u_h^j\big\|^2 + \big\|\nabla v_h^{j+1}\big\|^2 = 2\big(\nabla\big[u_h^j + v_h^{j+1}\big], \nabla v_h^{j+1}\big) = 0$$

and the fact that owing to (3) we have $\big\|\nabla u_h^{j+1}\big\| \le \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|$.

The observation that $\partial_t u \perp T_u N$ holds for wave maps into N motivates a similar iteration for their approximation and has first been employed in [5]. We let $\tau > 0$ denote a time-step size and d_t the corresponding backward difference operator.

Alg. 2. *Let $u_h^0, v_h^0 \in \mathbb{V}_h^{n+1}$ such that $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h$:*

(1) *Compute* $v_h^{j+1} \in \mathbb{V}_h^{n+1}$ *such that* $v_h^{j+1}(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$ *and*

$$\big(d_t v_h^{j+1}, w_h\big) + \big(\nabla\big[u_h^j + \tau v_h^{j+1}\big], \nabla w_h\big) = 0$$

for all $w_h \in \mathbb{V}_h^{n+1}$ *such that* $v_h^j(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$.

(2) *Set*

$$u_h^{j+1} := \mathcal{I}_h\big[\pi_{\mathcal{C}} \circ \big(u_h^j + \tau v_h^{j+1}\big)\big].$$

Upon choosing $w_h = v_h^{j+1}$ we deduce that

$$d_t\big\|v_h^{j+1}\big\|^2 + \tau\big\|d_t v_h^{j+1}\big\|^2 + \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|^2 - \big\|\nabla u_h^j\big\|^2 + \tau\big\|\nabla v_h^{j+1}\big\|^2 = 0.$$

Again the fact that $\big\|\nabla u_h^{j+1}\big\| \le \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|$ implies unconditional stability.

Convergence of the approximations to harmonic maps and wave maps as $h \to 0$ and $(h, \tau) \to 0$, respectively, can be established via weak compactness results. Necessary modifications of the algorithms for the case that N is nonconvex are discussed in [3, 4]. Nonstandard finite element methods are assumed to improve the performance of the algorithms and are currently under investigation.

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Mimetic finite differences and quadrilateral finite elements

Daniele Boffi

Recent results [1, 2] show that quadrilateral finite elements have to be handled with care. In this talk we are interested in the approximation properties of vector valued finite elements. A consequence of the results presented in [2] is that all commonly used finite elements for the approximation of $H(\mathrm{div})$ do not achieve optimal approximation properties on general quadrilateral meshes. This comment applies to Raviart–Thomas (RT), Brezzi–Douglas–Marini (BDM), and Brezzi–Douglas–Fortin–Marini (BDFM) elements. In particular, lowest order RT element does not achieve converge at all in the $H(\mathrm{div})$ norm.

As a model problem, we are considering the following eigenvalue problem associated with the **grad** div operator (we remark that this problem in 2D is actually equivalent to Maxwell's eigenvalue problem): find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in H_0(\mathrm{div})$ with $\mathbf{u} \not\equiv \mathbf{0}$ such that

$$(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathrm{div}).$$

It is well-known that for $\lambda \neq 0$ the problem we are considering is equivalent to a mixed approximation of the Neumann eigenvalue problem for Laplace operator: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in H_0(\mathrm{div})$ with $\mathbf{u} \not\equiv \mathbf{0}$ such that, for $p \in L^2_0(\Omega)$, it holds

$$\begin{cases} (\mathbf{u}, \mathbf{v}) + (\mathrm{div}\, \mathbf{v}, p) = 0 & \forall \mathbf{v} \in H_0(\mathrm{div}) \\ (\mathrm{div}\, \mathbf{u}, q) = -\lambda (p, q) & \forall q \in L^2_0. \end{cases}$$

It is also well-known that standard finite elements do not provide good approximation of eigenvalues/eigenfunctions of the problem we are interested in; more precisely, on unstructured meshes a lot of spurious eigenpairs are present so that it is not possible to pick out the correct ones and on particular structured meshes there are few spurious eigenpairs which might be difficult to distinguish from the correct ones. We refer the interested reader to [3, 4] for a rigorous analysis of this behavior. On the other hand mixed finite element for the approximation of $H(\mathrm{div})$ have good performances on affine meshes (triangles or parallelograms). The equivalence between the first eigenvalue problem and the mixed one is true at discrete level as well, provided the divergence of the finite elment discretization of $H(\mathrm{div})$ is contained in the discretization of L^2_0. This inclusion is generally satisfied on affine meshes (where it is actually an equality), but it fails on distorted quadrilateral meshes, since in this case the (non-polynomial) jacobian of the mapping enters the computation of the divergence of a Piola-mapped vector field.

Another consequence of the estimates presented in [2] is that Laplace eigenvalue problem is well approximated by standard mixed finite elements also on distorted quadrilateral meshes. On the other hand, the eigenvalue problem we are interested in, suffers from the lack of approximation properties of mixed elements (see also [5]).

In [6] a simple procedure has been presented for achieving optimal approximation properties on general quadrilateral meshes when using any order RT elements. It consists of a projection-based formulation which can be interpreted as a reduced integration strategy. For the lowest order case, for instance, it consists in using the midpoint rule for the evaluation of the integral on the left-hand side or, equivalently, in projecting the divergence onto piecewise constant functions. A rigorous mathematical analysis shows the optimality of the method.

The most recent result presented in this talk concerns a joint work with Lucia Gastaldi. Mimetic finite differences have become a very popular tool for the approximation of Darcy flow problem on very general geometries and meshes. We refer the interested reader to [7, 8] and to the references therein for an introduction to the method. In [9] we showed that a modification of lowest order Raviart–Thomas element (which does not increase the compuational cost) presented in [10] can be interpreted in the framework of mimetic finite differencces. This allows for a rigorous error analysis which proves the optimal performances of the method also on general distorted quadrilateral meshes. Such result looks promising since we hope that it can be extended to three-dimensional Raviart–Thomas element for which no satisfactory results are avaliable so far on distorted hexahedral meshes.

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A posteriori error estimates by the hypercircle method and nonstandard a priori error estimates

Dietrich Braess

(joint work with Joachim Schöberl)

The talk is concerned with a posteriori error estimates for finite element solutions of elliptic differential equations. Specifically we want upper estimates that have no generic constant in the main term. It turns out that we obtain in this way also a priori error estimates which are not known from classical finite element theory. Moreover for the *hp* method, the asymptotic behavior of the new estimator is better than that of residual estimators.

For convenience, we restrict ourselves to the Poisson equation $-\Delta u = f$ in a two-dimensional domain Ω, and to linear elements on a partition $\mathcal{T}_h$ of Ω into triangles. Here the mixed method for the Poisson equation will also be important, i.e., the system $\sigma = \nabla u$ and

$$\operatorname{div} \sigma + f = 0. \tag{1}$$

A flux σ which satisfies (1) is called *equilibrated*. The point of departure is the following theorem. Γ_D and Γ_N are the parts of the boundary with Dirichlet and Neumann boundary conditions, respectively. All norms without specification are L_2 norms.

Theorem of Prager and Synge (Two-Energies-Principle).
Let $\sigma \in H(\mathrm{div})$, $\sigma \cdot n = 0$ on Γ_N while $v \in H^1(\Omega)$, $v = 0$ on Γ_D, and assume that

(1) holds. Furthermore, let u be the solution of the Poisson equation. Then,

$$\|\nabla u - \nabla v\|^2 + \|\nabla u - \sigma\|^2 = \|\nabla v - \sigma\|^2. \tag{2}$$

There is much freedom in choosing v and σ. We also find the name *hypercircle method* in connection with the theorem. We emphasize that it is not restricted to the Poisson equation.

Let $v = u_h$ be a finite element solution for which an a posteriori error estimate is wanted. The crucial step is the construction of an equilibrated flux σ by a cheap postprocessing. Our construction is based on the observation that ∇u_h as well as the required flux σ_h belong to the *broken Raviart–Thomas space* of lowest order

$$\mathcal{RT}_{-1} := \{\tau \in L_2(\Omega);\ \tau|_T = a_T + b_T x,\ a_T \in \mathbb{R}^d,\, b_T \in \mathbb{R}\ \forall T\},$$

and the triangulation is the same as that for which the finite element solution was computed. The subspace of functions with continuous normal components is the usual space

$$\mathcal{RT}_0 := \mathcal{RT}_{-1} \cap H(\mathrm{div}).$$

The first step of the construction brings a separation of the *data oscillation* $ch\|f - \bar{f}\|$. Here, $\bar{f}$ is the L_2 projection of the given right-hand side of the given elliptic equation onto $\mathcal{M}^0$, i.e., the space of piecewise constant functions.

Now the finite element solution of the mixed method by Raviart–Thomas

$$\begin{aligned} (\sigma_h, \tau) + (\mathrm{div}\,\tau, w_h) &= 0 && \forall \tau \in \mathcal{RT}_0 \\ (\mathrm{div}\,\sigma_h, v) &= -(\bar{f}, v) && \forall v \in \mathcal{M}^0. \end{aligned} \tag{3}$$

provides an equilibrated flux. Moreover, we need not solve a global problem on the entire domain Ω, solutions on local patches are sufficient since the gradient ∇u_h provides additional information. The theorem of Prager and Synge provides the guaranteed upper estimate

$$\|\nabla u - \nabla u_h\| \le \|\sigma_h - \nabla u_h\| + ch\|f - \bar{f}\|\,. \tag{4}$$

The ingredients for the construction of σ_h are just the quantities found in residual estimators. Therefore, (4) is equivalent to the residual estimator, and the new estimator is not only reliable, but also efficient.

It follows from the theorem of Prager and Synge and the efficiency of (4) that we have $\|\sigma_h - \nabla u\| \le c\,\|\nabla u_h - \nabla u\| + ch\|f - \bar{f}\|$. This inequality applies to the solution σ_h of the mixed method by Raviart–Thomas. A byproduct is a comparison of different finite element families.

Classical results say that the error of the conforming P_1 element $u_h^{(1)}$, of the nonconforming P_1 element u_h^{CR}, and of the Raviart–Thomas element σ_h^{RT}, respectively, is $O(h)$. It is not excluded that one method is substantially better than the other ones for a special right-hand side f_1, while there is a different preference for some f_2. Now we get a more positive information by recalling that we have already used complementary spaces in (4). We also incorporate Ainsworth' application of the hypercircle method [1]. As usual, $A \preceq B$ means $A \le cB$ and $A \approx B$ that $A \preceq B$ and $B \preceq A$ holds.

Theorem. Assume that f is piecewise constant on the FE-mesh. Then

$$\|\nabla u_h^{(2)} - \nabla u\| \preceq |\nabla u_h^{CR} - \nabla u\|_{0,h} \approx \|\sigma_h^{RT} - \nabla u\| \preceq \|\nabla u_h^{(1)} - \nabla u\| .$$

Remark. Melenk and Wohlmuth [5] showed by theoretical and numerical investigations that the efficiency of residual estimators deteriorates as $O(p)$ when applied to the *hp* method. Numerical experiments, however, show efficiency factors not far from $\sqrt{2}$ for the hypercircle method. Indeed, p-efficiency could be proven for rectangular grids by the construction of uniformly bounded right inverses of the divergence operator in polynomial spaces. The main tool is a suitable interpolation on tensor products [3].

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Regge calculus as a non-standard finite element method

Snorre H. Christiansen

Regge calculus [1] is a rather old technique to construct discrete analogues of Einstein's equations of general relativity. Its representation of geometry appears natural and the method promises a structure preserving approach to these highly non-linear equations. Nevertheless few results exist on the numerical analysis and the method is not very popular in the present day numerical relativity community.

In this talk we show some results to the effect that the method can be interpreted as a (critically non-conforming) finite element method. It is our hope that they constitute a step towards efficient implementations and full convergence proofs.

In Regge calculus, spacetime is represented by a simplicial complex. Given this simplicial complex a finite dimensional space of metrics is defined. It consists of piecewise constant metrics which are continuous in the tangential direction across interfaces. We call such metrics Regge metrics. A functional defined on this space of metrics and mimicking the Einstein-Hilbert action, is provided. We call this functional the Regge action. A critical point of the Regge action on the space of Regge metrics is generally believed to be a good approximation to a true solution of Einsteins equations.

In [2] we relate the space of Regge metrics to Whitney forms. As remarked by Bossavit, Whitney forms correspond to lowest order mixed finite elements for which one has a relatively well developed convergence theory. We show that there is a natural basis for the space of Regge metrics expressed in terms of Whitney

forms and that second order differential operators restricted to Courant elements (scalar piecewise affine continuous functions) are in one to one correspondence with linear forms on Regge metrics, Nédélec's edge elements and Courant elements.

In [3] we show that in three dimensions, Regge elements can be inserted as the second space in a discrete version of the elasticity complex. The first space consists of piecewise linear affine vectorfields and the third and fourth spaces are constituted by tensor and vector valued measures which are in duality with the second and first spaces respectively.

Independently of any argument as to whether or not the Regge action is indeed an approximation of the Einstein Hilbert action, we also show that the first non-trivial term in the expansion of the Regge action in small perturbations of Euclidean metric is what it should be, namely a quadratic form associated with the curl T curl operator.

Finally we prove convergence of the eigenpairs of the curl T curl operator, discretized on Regge elements. The study is inspired by experience with the eigenvalue problem of electromagnetics. As in that case the operator has a large kernel, here constituted by the deformation tensors. The main additional difficulties are that there are eigenvalues of both signs and that Regge elements just fail to be in the natural Hilbert space where the quadratic form is continuous and non-degenerate.

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Error analysis for Raviart-Thomas interpolation on anisotropic triangular elements

Ricardo G. Durán

(joint work with Gabriel Acosta, Thomas Apel, Ariel Lombardi)

The original error analysis developed in [2, 3] for the Raviart-Thomas interpolation and its generalization to three dimensions are based on the regularity assumption on the elements and, therefore, narrow or anisotropic elements, which are very important in many applications, are excluded.

Several arguments have been introduced to prove error estimates valid under weaker hypotheses. A complete error analysis for the case of triangular and tetrahedral elements was developed recently in [1], where optimal error estimates under the maximum angle condition and its generalizations to three dimensions were obtained. Here we present the main ideas introduced in that paper. For the sake of

clarity we consider the case of triangular elements only and refer the reader to [1] for the rather technical details arising in the three-dimensional case.

Given k, $k = 0, 1, \cdots$, we denote with $\Pi_k \mathbf{u}$ the Raviart-Thomas interpolation of order k of a smooth enough vector field $\mathbf{u}$.

In the original proof given in [3] the authors obtained first the estimate

$$\|\Pi_k \mathbf{u}\|_{L^2(T)} \leq C \|\mathbf{u}\|_{H^1(T)}.$$

where T is the reference element with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Then, using the Piola transform for vector fields, combined with standard polynomial approximation results, they proved error estimates on a general triangle.

Observe that the complete H^1-norm appears on the right hand side of the estimate and this is the reason why the constant in the error estimate depends on the regularity of the elements (i.e., the constant blows up when the minimum angle goes to zero).

The main idea in [1] is to obtain sharper estimates for each component of $\Pi_k \mathbf{u}$ on the reference element. Consider, for example, the first component $\Pi_{k,1} \mathbf{u}$. Ideally, we would like to have

$$\|\Pi_{k,1} \mathbf{u}\|_{L^2(T)} \leq C \|u_1\|_{H^1(T)}$$

where u_1 is the first component of $\mathbf{u}$. But unfortunately this is not true, indeed, if $\mathbf{u} = (0, y^2)$, a simple computation shows that $\Pi_0 \mathbf{u} = \frac{1}{3}(x, y)$.

However, it is possible to obtain the following slightly weaker estimate:

$$\|\Pi_{k,1} \mathbf{u}\|_{L^2(T)} \leq C \left\{ \|u_1\|_{H^1(T)} + \|\operatorname{div} \mathbf{u}\|_{L^2(T)} \right\} \tag{1}$$

We will denote with ℓ_1 and ℓ_2 the edges of T contained in $\{x = 0\}$ and $\{y = 0\}$ respectively, and with ℓ_3 the other edge.

A key observation is that if $\mathbf{r} = (0, g(x))$ then $\Pi_{k,1} \mathbf{r} = 0$. Therefore, if

$$\mathbf{v} = (u_1(x, y), u_2(x, y) - u_2(x, 0))$$

we have that $\Pi_{k,1} \mathbf{v} = \Pi_{k,1} \mathbf{u}$. But, $\mathbf{v} \cdot \mathbf{n} = 0$ on ℓ_2 (where $\mathbf{n}$ denotes the unit outer normal on ∂T) and, therefore, the degrees of freedom defining Π_k associated with that edge vanish. On the other hand, defining now

$$\mathbf{w} = (u_1(x, y), u_2(x, y) - u_2(x, 0) - y q_{k-1}(x, y)),$$

with $q_{k-1} \in \mathcal{P}_{k-1}$, we still have that $\mathbf{w} \cdot \mathbf{n} = 0$ on ℓ_2 and $\Pi_{k,1} \mathbf{w} = \Pi_{k,1} \mathbf{u}$, because we have added to $\mathbf{v}$ a vector field which belongs to the Raviart-Thomas space of order k. But q_{k-1} can be chosen such that internal degrees of freedom corresponding to w_2 vanish. Indeed, by standard arguments it can be seen that there exists a unique $q_{k-1} \in \mathcal{P}_{k-1}$ (here we are assuming $k \geq 1$, the case $k = 0$ is simpler) such that

$$\int_T w_2 p_{k-1} = \int_T (u_2(x, y) - u_2(x, 0) - y q_{k-1}(x, y)) p_{k-1} = 0$$

for all $p_{k-1} \in \mathcal{P}_{k-1}$. Using this equality and that w_2 vanishes on ℓ_2, we can see that, for any $p_k \in \mathcal{P}_k$, we have

$$\int_{\ell_3} \mathbf{w} \cdot \mathbf{n}\, p_k = \int_{\ell_1} w_1\, p_k + \int_T \operatorname{div} \mathbf{w}\, p_k + \int_T w_1 \frac{\partial p_k}{\partial x}.$$

Therefore, the degrees of freedom defining $\Pi_k \mathbf{w}$ corresponding to ℓ_3 can be estimated in terms of $w_1 = u_1$ and $\operatorname{div} \mathbf{w}$. Summing up we can now apply standard arguments to obtain the estimate

$$\|\Pi_{k,1}\mathbf{u}\|_{L^2(T)} = \|\Pi_{k,1}\mathbf{w}\|_{L^2(T)} \leq C \left\{ \|u_1\|_{H^1(T)} + \|\operatorname{div} \mathbf{w}\|_{L^2(T)} \right\} \tag{2}$$

Therefore, we need to estimate $\|\operatorname{div} \mathbf{w}\|_{L^2(T)}$ in terms of $\mathbf{u}$. But, since

$$\operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{u} - \frac{\partial (y q_{k-1})}{\partial y}, \tag{3}$$

it is enough to estimate $\| \frac{\partial (y q_{k-1})}{\partial y} \|_{L^2(T)}$.

Let $p_{k-1} \in \mathcal{P}_{k-1}$. Using $\int_T w_2 p_{k-1} = 0$ and that w_2 vanishes on ℓ_2, and introducing the barycentric coordinate $\lambda_3(x, y) = 1 - x - y$, it follows by integration by parts that

$$\int_T \frac{\partial w_2}{\partial y} \, \lambda_3 \, p_{k-1} = 0$$

and then

$$\int_T \frac{\partial (y q_{k-1})}{\partial y} \, \lambda_3 \, p_{k-1} = \int_T \frac{\partial u_2}{\partial y} \, \lambda_3 \, p_{k-1}.$$

Taking now $p_{k-1} = \frac{\partial (y q_{k-1})}{\partial y}$, using the Schwarz inequality, and the equivalence of norms on $\mathcal{P}_{k-1}$ we obtain

$$\left\| \frac{\partial (y q_{k-1})}{\partial y} \right\|_{L^2(T)} \leq C \left\| \frac{\partial u_2}{\partial y} \right\|_{L^2(T)}.$$

Therefore, since

$$\left\| \frac{\partial u_2}{\partial y} \right\|_{L^2(T)} \leq \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2(T)} + \|\operatorname{div} \mathbf{u}\|_{L^2(T)},$$

(1) follows from (2) and (3).

Finally, let us mention that optimal order error estimates on a general triangle can be proved using (1) combined with the Piola transform and polynomial approximation results. The constants in the estimates obtained in this way depend only on k and on the maximum angle of the element.

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Evolving surface finite element method

Charlie Elliott

1. Introduction

In this talk we described the evolving surface finite element method (ESFEM) for transport and diffusion of a material quantity on an evolving surface in $\mathbb{R}^{n+1}$ (n=1,2). We have in mind a surface which not only evolves in the normal direction so as to define the surface evolution but also has a tangential velocity associated with the motion of material points in the surface which advects material quantities such as heat or mass. For our purposes here we assume that the surface evolution is prescribed. This is joint work with G. Dziuk, [2, 3] which has been extended to Eulerian approaches based on solving PDEs on implicitly defined surfaces,[4, 5].

1.1. **The advection diffusion equation.** Conservation of a scalar with a diffusive flux on an evolving hypersurface $\Gamma(t)$ leads to the diffusion equation

$$\dot{u} + u\,\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{D}_0 \nabla_\Gamma u) = 0 \tag{1}$$

on $\Gamma(t)$. Here $\dot{u}$ denotes the advective surface material derivative, v is the velocity of the surface and ∇_Γ is the tangential surface gradient. If $\partial\Gamma(t)$ is empty then the equation does not need a boundary condition. Otherwise we can impose Dirichlet or Neumann boundary conditions on $\partial\Gamma(t)$.

1.2. **Applications.** Such a problem arises, for example, in the modelling of the formation of nanoporosity by surface dissolution and phase separation, see [6], where we solve the nonlinear fourth order Cahn-Hilliard equation on a surface evolving by forced mean curvature flow.

1.3. **Evolving Surface Finite Element Method (ESFEM).** The finite element approximation is based on the variational form

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\dot{\varphi} \tag{2}$$

where φ is an arbitrary test function defined on the surface $\Gamma(t)$ for all t. This provides the basis of our evolving surface finite element method (ESFEM) which is applicable to arbitrary evolving n–dimensional hypersurfaces in $\mathbb{R}^{n+1}$ (curves in $\mathbb{R}^2$) with or without boundary. This is the extension of the method of Dziuk [1] for the Laplace-Beltrami equation on a stationary surface. The principal idea is to use a polyhedral approximation of Γ based on a triangulated surface. It follows that a quite natural local piecewise linear parameterisation of the surface is employed rather than a global one. The finite element space is then the space of continuous piecewise linear functions on the triangulated surface whose nodal basis functions enjoy the transport property

$$\dot{\phi}_j = 0.$$

The implementation is thus rather similar to that for solving the diffusion equation on flat stationary domains. For example, the backward Euler time discretization leads to the ESFEM scheme

$$\frac{1}{\tau}\left(\mathcal{M}(t^{m+1})\alpha^{m+1} - \mathcal{M}(t^m)\alpha^m\right) + \mathcal{S}(t^{m+1})\alpha^{m+1} = 0$$

where $\mathcal{M}(t)$ and $\mathcal{S}(t)$ are the time dependent surface mass and stiffness matrices and α^m is the vector of nodal values at time t^m. Here, τ denotes the time step size.

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Geometric decompositions and bases for spaces of piecewise polynomial differential forms

Richard S. Falk

(joint work with Douglas N. Arnold, Ragnar Winther)

We study the two primary families of spaces of finite element differential forms with respect to a simplicial mesh $\mathcal{T}$ on a domain $\Omega \subset \mathbb{R}^n$, for arbitrary n. The first family, which we denote by $\mathcal{P}_r\Lambda^k(\mathcal{T})$, consists of all elements of $H\Lambda^k(\Omega)$ (differential k forms ω on Ω, which are in L^2, and whose exterior derivative, $d\omega$, is also in L^2), which restrict to polynomial k-forms of degree at most r on each simplex T in the partition $\mathcal{T}$. The second family, which we denote $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, consists of spaces which sit between $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_{r-1}\Lambda^k(\mathcal{T})$. The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ are generalizations of the Raviart-Thomas spaces used to discretize $H(\mathrm{div})$ and $H(\mathrm{rot})$ in two space dimensions and the Nédélec edge and face spaces of the first kind, used to discretize $H(\mathrm{curl})$ and $H(\mathrm{div})$ in three space dimensions. The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ are generalizations of the Brezzi-Douglas-Marini spaces in two space dimensions and of the Nédélec edge and face spaces of the second kind in three space dimensions.

A key aim of the work, described in more detail in [3], is to explicitly construct geometric decompositions of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ for arbitrary values of $r \geq 1$ and $k \geq 0$, and an arbitrary simplicial partition $\mathcal{T}$ of a polyhedral

domain in an arbitrary number of space dimensions. More precisely, we will decompose the space into a direct sum with summands indexed by the faces of the mesh (of arbitrary dimension), such that the summand associated to a face is the image under an explicit extension operator of a finite-dimensional space of differential forms on the face. Such a decomposition is necessary for an efficient implementation of the finite element method, since it allows an assembly process that leads to local bases for the finite element space. The construction of explicit local bases, considered also by a number of other authors (e.g., see [1], [2], [4], [5], [6]) is the other key aim of this work.

The construction given leads to a generalization of the so-called Bernstein basis for ordinary polynomials, i.e., 0-forms on a simplex T in $\mathbb{R}^n$, and the corresponding finite element spaces, the Lagrange finite elements. A prominent role in the construction is played by the notion of a consistent family of extension operators, which expresses in an abstract framework, a sufficient condition for deriving a geometric decomposition of a finite element space leading to a local basis.

In the remainder of this abstract, we describe the basic ideas in the case of ordinary polynomials. Let T be a simplex in $\mathbb{R}^n$ with vertices $x_0, x_1, \ldots, x_n$ and $\{\lambda_i\}_{i=0}^n = \{\lambda_i^T\}_{i=0}^n \subset \mathcal{P}_1(T)$ the corresponding barycentric coordinates. For $r \geq 1$, the Bernstein basis for the space $\mathcal{P}_r(T)$ consists of all monomials of degree r in the variables λ_i, i.e., the basis functions are:

$$\{\, \lambda^\alpha = \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} \,:\, |\alpha| = r \,\}.$$

For $f \in \Delta(T)$ (subsimplex of T), $\mathcal{P}_r(f)$ has the corresponding basis:

$$\{\, (\lambda^f)^\alpha \,:\, |\alpha| = r,\ [\![\alpha]\!] \subseteq \mathcal{I}(f) \,\},$$

where $[\![\alpha]\!] = \{i \,|\, \alpha_i > 0\}$, and $\mathcal{I}(f)$ is the index set of f. From the Bernstein basis, one obtains a barycentric extension operator, $E = E^r_{f,T} : \mathcal{P}_r(f) \to \mathcal{P}_r(T)$, by simply replacing λ_i^f by λ_i^T in the basis and using linearity.

Let $\mathring{\mathcal{P}}_r(T)$ denote the subspace of $\mathcal{P}_r(T)$ of polynomials that vanish on ∂T, i.e., are divisible by the bubble function $\lambda_0 \cdots \lambda_n$ on T. Then

$$\mathring{\mathcal{P}}_r(T) = \operatorname{span}\{\lambda^\alpha \,:\, |\alpha| = r,\ [\![\alpha]\!] = \{0, \ldots, n\}\,\}.$$

Next associate to each face f, the subspace of $\mathcal{P}_r(T)$ spanned by λ^α, $[\![\alpha]\!] = \mathcal{I}(f)$. This subspace is precisely $E[\mathring{\mathcal{P}}_r(f)]$, i.e.,

$$E[\mathring{\mathcal{P}}_f(f)] = \operatorname{span}\{\, \lambda^\alpha \,:\, |\alpha| = r,\ [\![\alpha]\!] = \mathcal{I}(f)\,\}.$$

Thus, we are led to the following explicit geometric decomposition of $\mathcal{P}_r(T)$, which we call the *Bernstein decomposition* of $\mathcal{P}_r(T)$.

$$\mathcal{P}_r(T) = \bigoplus_{f \in \Delta(T)} E[\mathring{\mathcal{P}}_r(f)].$$

To obtain a geometric decomposition and basis for the finite element space $\mathcal{P}_r(\mathcal{T})$, we define a global extension operator $E_f^r : \mathring{\mathcal{P}}_r(f) \to \mathcal{P}_r(\mathcal{T})$ by

$$(E_f^r \mu)|_T = E^r_{f,T} \mu, \quad \text{if } f \subseteq T, \qquad (E_f^r \mu)|_T = 0, \quad \text{otherwise.}$$

Then

$$\mathcal{P}_r(\mathcal{T}) = \bigoplus_{f\in\Delta(\mathcal{T})} E_f^r[\mathring{\mathcal{P}}_r(f)].$$

Using the Bernstein bases for the spaces $\mathring{\mathcal{P}}_r(f)$, this formula gives a basis for the finite element space $\mathcal{P}_r(\mathcal{T})$.

We can also give a geometric characterization of E_f^r without barycentric coordinates. Let f^* denote the subsimplex of T opposite f (spanned by the vertices not in the face f) and

$$P_r(T,f) = \{\omega \in \mathcal{P}_r(T) : \omega \text{ vanishes to order } r \text{ on } f^*\},$$

i.e., $(\partial^\alpha\omega)(x) = 0$, $|\alpha| \le r-1$, for all $x \in f^*$. Then $P_r(T,f) = E_f^r[\mathcal{P}_r(f)]$ and for $\mu \in \mathcal{P}_r(f)$, $E_f^r\mu$ is characterized as the unique extension of μ to $P_r(T,f)$.

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Theory and applications of the DGFEM for nonstationary convection-diffusion problems

Miloslav Feistauer

The talk is concerned with the analysis of the discontinuous Galerkin finite element method (DGFEM) for the numerical solution of nonstationary convection-diffusion problems and applications to the simulation of compressible flow. We present results obtained in the cooperation with V. Dolejší and V. Kučera from the Department of Numerical Mathematics of the Faculty of Mathematics and Physics of Charles University Prague and V. Sobotíková from the Department of Mathematics of the Faculty of Electrical Engineering of Czech Technical University Prague.

The DGFEM is based on a piecewise polynomial approximation of the sought solution without any requirement on the continuity on interfaces between neighbouring elements. It is particularly suitable for the solution of conservation laws

with discontinuous solutions or singularly perturbed convection-diffusion problems with dominating convection, when solutions contain steep gradients.

The first part is devoted to the analysis of the DGFEM for the following nonstationary convection-diffusion problem: Find $u : Q_T = \Omega \times (0,T) \to I\!R$ such that

$$\begin{aligned}
&\text{a)} \quad \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \Delta u + g \quad \text{in } Q_T,\\
&\text{b)} \quad u\big|_{\partial\Omega_D \times (0,T)} = u_D,\\
&\text{c)} \quad \varepsilon \frac{\partial u}{\partial n}\big|_{\partial\Omega_N \times (0,T)} = g_N,\\
&\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega.
\end{aligned}$$

We analyze error estimates of the DGFEM applied to the space semidiscretization of the problem in $L^2(H^1)$- and $L^\infty(L^2)$-norm for linear and nonlinear problems. The full space-time DG discretization is applied to a linear problem. The optimality and uniformity of error estimates with respect to the diffusion coefficient ε tending to zero is discussed. The effect of numerical integration is also analyzed. The results are contained in [2], [3], [4], [5], [7], [9], [11], [12].

The second part is concerned with the DG solution of compressible flow described by the compressible Euler and Navier-Stokes equations. Our goal is to develop sufficiently accurate, efficient and robust numerical schemes allowing the solution of compressible flow for a wide range of Reynolds and Mach numbers. Our approach developed in [1], [6], [8] and [10] allows to solve gas flow with practically all Mach numbers (starting from Mach number $= 10^{-6}$ up to transonic and hypersonic regimes). The efficiency and accuracy of the method is demonstrated by computational results obtained for several test problems. The developed technique was combined with the ALE method and adapted to the solution of flow problems in time-dependent domains and the simulation of fluid-structure interaction.

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Hybridized methods for Stokes flow

Jay Gopalakrishnan
(joint work with B. Cockburn)

This talk is devoted to various hybridization techniques. We begin with an introduction to the simplest hybridized method, namely the hybridized Raviart-Thomas method. We present some old [1, 2] and a few new results [3, 4] on this topic, after which we concentrate on the application of similar techniques for Stokes flow [5, 6].

Consider the velocity-vorticity formulation of Stokes flow. One can write down a mixed variational formulation in which the pressure variable is absent, vorticity is in a curl-conforming space, and velocity is in a subspace of exactly divergence free functions. The latter space creates difficulties because it is not easy to obtain a basis for conforming finite element subspaces of it. Partly motivated by this difficulty, we propose a hybridization technique to overcome it. Since it is the continuity of the normal component that makes working with divergence free spaces difficult, we propose to move such continuity constraints from the finite element spaces to the system of equations defining the method. This gives rise to a hybridized method.

However, this first hybridization results in a system with a large number of unknowns. It is not easy to condense out any of the variables. Hence we perform one more hybridization, this time moving the tangential continuity constraints of the vorticity approximation from the finite element spaces to the system of equations. Once this hybridization is performed, the vorticity and velocity unknowns can be easily eliminated, resulting in a system for certain Lagrange multiplier unknowns on the interfaces of mesh elements. We thus essentially achieve a dimension reduction. An important computational consequence is the reduction in the number of globally coupled unknowns, for instance in three space dimension, from $O(p^3)$

to $O(p^2)$, where p is the degree of the polynomials used for approximation in each element. Hence this technique is especially suited for high order finite elements.

The reduced system of Lagrange multipliers has interesting properties. It can be thought of a new mixed method coupling approximations of 'tangential velocity' and 'pressure traces' on the interfaces of mesh elements [5, 6]. Once this mixed system is solved, it is possible to *locally* recover the following extra approximations:

(1) a globally divergence-free numerical approximation of the fluid velocity,
(2) an approximation of the vorticity whose tangential component is continuous across element interfaces, and
(3) a discontinuous numerical approximation of the pressure.

Numerical experiments in two space dimensions indicated optimal convergence rates for standard test cases. However, a theoretical *a* priori analysis of the error is, as yet, an open question for certain standard boundary conditions. The difficulty is that the spaces that result when using boundary conditions on (all components of) velocity, are not amenable to a direct application of the Babuška-Brezzi theory [2] of mixed methods.

One disadvantage of the above proposed hybridized method is that it requires fairly nonstandard (albeit local) mesh operations. The spaces where the Lagrange multipliers lie are spaces of normal and tangential jumps of certain finite element functions. To obtain a local basis for the space of tangential jumps, we resort to mesh objects (in three space dimensions) that we call "wedges", which are the union of two mesh faces that share an edge. The wedges are not standard data structures output by meshing programs, so the implementation of our method may be perceived to have an overhead that other methods do not.

In view of this, we attempt to design methods having the same structure (so as to continue making the elimination and dimension reduction possible), but with more standard spaces for the Lagrange multiplier unknowns. We are able to do this once we add some stabilization terms inspired by DG techniques. We conclude by conveying, very briefly, the potential for deriving exciting new methods, called hybridized DG (HDG) methods for Stokes flow [7]. The Stokes system has four transmission conditions, which are continuity constraints on (i) the tangential component of vorticity, (ii) the tangential component of velocity, (iii) the normal component of velocity, and (iv) the scalar pressure variable. Numerical methods contain discrete versions of these conditions, although at times they are not easy to spot. One can derive a series of HDG methods by relaxing any two of the above mentioned transmission conditions and using Lagrange multipliers for the other two. All these potential methods need further investigation.

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A new multiscale finite element method for high-contrast elliptic interface problems

Ivan G. Graham
(joint work with Chia-Chieh Chu and Thomas Y. Hou)

We introduce a new multiscale finite element method which is able to accurately capture solutions of elliptic interface problems with high contrast coefficients by using only coarse quasiuniform meshes, and without resolving the interfaces. A typical application would be the modelling of flow in a porous medium containing a number of inclusions of low (or high) permeability embedded in a matrix of high (respectively low) permeability, and modelled by the (weak form of) the elliptic PDE:

$$-\nabla.\alpha\nabla u = f \ . \tag{1}$$

Here the coefficient α is assumed piecewise constant and bounded away from zero, but may have large "contrast", i.e. the parameter $\widehat{\alpha} := \max_{x\in\Omega}\alpha(x)/\min_{x\in\Omega}\alpha(x)$ may be large.

Our method is H^1- conforming, with nodal degrees of freedom on a triangular mesh, and requires the solution of "subgrid problems" for the basis functions, on elements which straddle the coefficient interface. These involve solving the local version of (1) with $f = 0$ on such elements. A key point is the introduction of novel coefficient-dependent interior boundary conditions for the subgrid problems. For a given edge which is cut by an interface, the correct boundary condition depends not only on the contrast $\widehat{\alpha}$, but also on the angles of intersection of the interface with the edges of the elements adjoining the given edge. The method coincides with standard linear approximation on elements which are not cut by any interface.

The resulting multiscale finite element solution $u_H^{\rm MS}$ is defined by the usual Galerkin method for (1), and we show in [1] that, under appropriate assumptions, our method satisfies an error estimate of the form

$$|u - u_h^{\rm MS}|_{H^1(\Omega),\alpha} \ \leq \ Ch\left[h|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L^2(\Omega)}\right]^{1/2} \ , \tag{2}$$

where $|v|^2_{H^1(\Omega),\alpha} \ = \ \int_\Omega \alpha|\nabla v|^2$ is the usual energy norm and C is a generic constant which here is independent of h and of the contrast parameter $\widehat{\alpha}$. This should be

compared to the best result of $\mathcal{O}(h^{1/2-\epsilon})$ (with a hidden constant which generally depends on the contrast) for standard finite element methods, when the mesh does not resolve the interface. A non-standard duality argument shows that

$$\|u - u_h^{\mathrm{MS}}\|_{L_2(\Omega)} \leq Ch^2 \left[h|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L^2(\Omega)}\right]^{1/2} . \tag{3}$$

At present the theory is confined to 2D and the interfaces in α are assumed to be smooth simple curves although it is easy in principle to extend the method to 3D. Of key importance in the theory are new contrast-explicit results on the regularity of the solution u of (1), and the method for obtaining these has been kindly pointed out to us by I. Kamotski and V.P. Smyshlyaev (University of Bath).

The price to pay for the improved convergence rate of the method is the solution of subgrid problems on elements which straddle the interface and a slightly worse dependence than normal on the data f in the right hand sides of (2) and (3). The additional local subgrid problems can be solved as a preprocessing step before assembly and solution of the global finite element problem on the coarse mesh. The accuracy needed for these subgrid problems is also investigated numerically in [1]

The multiscale finite element method (in which basis functions are computed by solving local homogeneous PDEs subject to special boundary conditions) has a large literature. Most of the proofs are for the periodic homogenisation case where, in (1), $\alpha(x) = a(x/\epsilon)$, with ϵ a small parameter, and a a smooth positive-valued periodic function on a unit cell Y, and the analysis is geared to obtaining optimal convergence, robust with respect to the "oscillation parameter" $\epsilon \to 0$ (e.g. [4, 5]). However, the method itself is quite general and has been applied to non-periodic cases with considerable success. For example in [4] one finds an application to (1) for the case when α is a realisation of a random field, both in the isotropic and anisotropic cases and with highly contrasting media.

The theory in [1] makes no appeal to homogenisation theory in the proofs but nevertheless explains why "multiscale"-type basis construction can be beneficial in more general situations. It turns out that the new interior boundary conditions obtained in the present work are a genuine generalisation of the "oscillatory" boundary conditions of [4], in the sense that the two coincide if and only if the interfaces intersect the element edges orthogonally.

Some of the arguments used in this paper have already been developed in the context of domain decomposition methods in [2, 3, 6].

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Partial evaluation of the discrete solution of elliptic boundary value problems

Wolfgang Hackbusch

Let $Ax = b$ be a large system of equations ($x, b \in \mathbb{R}^I$, I index set). In Linear Algebra one is used to consider only the (complete) solution vector $x \in \mathbb{R}^I$ as the solution of the system. Similarly, only the (full) inverse matrix $A^{-1} \in \mathbb{R}^{I\times I}$ seems to be the answer for the inverse mapping.

However, if $Ax = b$ is the discretisation of an elliptic boundary value problem $Lu = f$ in Ω with suitable boundary data, one is often not interested in $u(\xi)$ at *all* $\xi \in \Omega$. Instead, certain functionals are of interest. Examples of functionals are the boundary data $\partial u/\partial n$ at $\Gamma = \partial\Omega$ in the case of given Dirichlet data or only one integral $\int_{\Gamma_0} \partial u/\partial n \, \mathrm{d}\Gamma$ along a part $\Gamma_0 \subset \Gamma$ describing the flux through Γ_0, or u at a fixed point $\xi_0 \in \Omega$ or at several points.

A particular situation originates from differential operators $L = div a(\cdot) grad$ with strongly oscillatory coefficients (or coefficients with other small-scale behaviour). Since also the solution is strongly oscillatory, one is usually not interested in the complicated solution with all its details, but only in local mean values $\bar{u}$ representing the makroscopic behaviour. In the case of periodic coefficients $a(\cdot)$ one can apply homogenisation techniques leading to an approximation of $\bar{u}$. If the presuppositions for this technique do not hold, a numerical homogenisation is of interest.

We consider the boundary value problem $Lu_\Omega = f_\Omega$ in $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary condition $u_\Omega|_\Gamma = g_\Gamma$ on $\Gamma := \partial\Omega$, discretised by a finite element method with triangulation $T(\Omega)$.

Next we construct a binary domain decomposition tree G_Ω by nested dissection: Ω is the root, the sons of Ω are two subdomains ω_1 and ω_2 (unions of elements of $T(\Omega)$) with interior boundary $\gamma(\Omega) \subset \Omega$, i.e. $\overline{\gamma(\Omega)} = \partial\omega_1 \cap \partial\omega_2$. The subdomains ω_i are recursively devided in the same manner until the subdomain equals an element of the triangulation $T(\Omega)$.

For each subdomain $\omega \in G_\Omega$, the differential equation can be restricted onto ω: $Lu_\omega = f_\omega$ in ω with boundary data $u_\omega|_{\partial\omega} = g_{\partial\omega}$ on $\partial\omega$. Let $f_h(\omega)$ be the restriction of the finite element function f_h, while $g_h(\partial\omega)$ contains the nodal value of $g_{\partial\omega}$ on $\partial\omega$. The corresponding finite element solution is denoted by $u_h(\omega)$. This allows to define the trace mapping

$$\Phi_\omega : (f_h(\omega), g_h(\partial\omega)) \mapsto u_h(\omega)|_{\gamma(\omega)}$$

onto the interior boundary data $\gamma(\omega)$. Provided Φ_ω is given for all $\omega \in G_\Omega$ except the leaves. Then the recursive evaluation of Φ_ω starting from the root yields the complete solution $u_h = u_h(\Omega)$.

The computation of Φ_ω requires an auxiliary operator Ψ_ω. Given Φ_{ω_i} and Ψ_{ω_i} associated to the sons ω_1 and ω_2 of $\omega \in G_\Omega$, the mappings Φ_ω, Ψ_ω can be determined by solving a linear system. The linear mappings Φ_ω and Ψ_ω have full matrices. In order to reach a complexity of $O(n \log^* n)$, the exact arithmetic is replaced by the arithmetic of *hierarchical matrices* (cf. [1]).

In the *definition phase*, we determine the mappings Φ_ω for all domains $\omega \in G_\Omega \backslash \mathcal{L}(G_\Omega)$ of the domain decomposition tree which are no leaves. The algorithm starts from the leaves and ends in the root. The auxiliary mappings Ψ_ω for $\omega \in G_\Omega \backslash \{\Omega\}$ are determined only for intermediate use. Afterwards, the *evaluation phase* can be applied once or many times for different data $f_h(\Omega), g_h(\partial\Omega)$.

Now we assume that the differential equation contains coefficients with small-scale behaviour. For instance, the coefficient is highly oscillating, jumping or the domain Ω is complicated. We assume a fine grid resolution h, which is small enough to resolve all details. However, for the presentation of the solution one is often interested only in data of a coarser grid, say with size $H \gg h$. Define a domain decomposition $G_\Omega^{\text{coarse}} \subset G_\Omega$ such that the leaves of G_Ω^{coarse} are subdomains or size H. Note that the mappings Φ_ω need to be stored only for $\omega \in G_\Omega^{\text{coarse}}$ (without leaves). Performing the evaluation phase in the tree G_Ω^{coarse} we get the solution data on the boundaries of the leaves ω of G_Ω^{coarse} (size H) without any further error.

The described method has been implemented and tested for the 2D case in the thesis of Litvinenko [3].

This approach involves new questions to FEM: What is the appropriate a-posteriori error control for the discrete operator problem? Given the numerical solution operator evaluated in a coarse grid, how to derive a homogenised pde?

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Convergence Analysis of an Adaptive Interior Penalty Discontinuous Galerkin Method

Ronald H. W. Hoppe
(joint work with Guido Kanschat, Tim Warburton)

We are concerned with a convergence analysis of an adaptive symmetric Interior Penalty Discontinuous Galerkin (IPDG) method for second order elliptic boundary value problems.
As a model problem, for given $f \in L^2(\Omega), u^D \in H^{1/2}(\Gamma_D), u^N \in L^2(\Gamma_N)$ we consider Poisson's equation with inhomogeneous Dirichlet and Neumann boundary data whose variational formulation amounts to the computation of $u \in V := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = u^D\}$ such that

$$a(u,v) = (f,v)_\Omega + \langle u^N, v\rangle_{\Gamma_N} \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega) \; , \tag{1}$$

where $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$ and $(\cdot,\cdot)_\Omega$ refers to the L^2-inner product. For the discontinuous Galerkin approximation of (1) we assume that $\mathcal{T}_H(\Omega)$ is a simplicial triangulation of Ω which aligns with Γ_D, Γ_N on the boundary Γ. For $D \subseteq \overline{\Omega}$, we denote by $|D|$ the volume of D and by $\Pi_p(D), p \in \mathbb{N}_0$, the linear space of polynomials of degree p on D, and we refer to $\mathcal{E}_H(D)$ and $\mathcal{T}_H(D)$ as the sets of edges and elements in D. For $T \in \mathcal{T}_H(\Omega)$, h_T stands for the diameter of T, whereas for $E \in \mathcal{E}_H(\overline{\Omega})$ we denote by h_E the length of E. Setting $V_H := \prod_{T\in\mathcal{T}_H(\Omega)} \Pi_p(T), p \in \mathbb{N}$, the interior penalty method in its symmetric formulation amounts to the computation of $u_H \in V_H$ such that

$$a_H(u_H, v_H) = \ell(v_H) \quad , \quad v_H \in V_H \; . \tag{2}$$

Here, the bilinear form $a_H(\cdot,\cdot) : V_H \times V_H \to \mathbb{R}$ and the functional $\ell : V_H \to \mathbb{R}$ are given by

$$\begin{aligned} a_H(u_H, v_H) &:= \sum_{T\in\mathcal{T}_H(\Omega)} (\nabla u_H, \nabla v_H)_T \\ &\quad - \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \Big\{ (\{\nu_E \cdot \nabla u_H\}, [v_H])_E + ([u_H]_E, \{\nu_E \cdot \nabla v_H\})_E \Big\} \\ &\quad + \alpha \sum_{E\in\mathcal{E}_H(\overline{\Omega})} h_E^{-1} \, ([u_H]_E, [v_H]_E)_E \; , \\ \ell(v_H) &:= (f, v_H)_\Omega + \big(u^N, v_H\big)_{\Gamma_N} - \sum_{E\subset\Gamma_D} \big(u^D, \nu_{\Gamma_D} \cdot \nabla v_H - \alpha h_E^{-1} v_H\big)_E \, , \end{aligned}$$

where $\{\cdot\}$ and $[\cdot]$ stand for the averages and jumps of functions across edges (with appropriate modifications for edges located on the boundary Γ) and $\alpha > 0$ refers to a penalization parameter. For sufficiently large penalty parameter, $a_H(\cdot,\cdot)^{1/2}$ defines a mesh-dependent norm on V_H.
The residual-type a posteriori error estimator $\eta_H^2 := \sum_{T\in\mathcal{T}_H(\Omega)} \eta_T^2 + \sum_{E\in\mathcal{E}_H(\Omega)} \eta_E^2$ consists of element residuals $\eta_T := h_T \|f + \Delta u_H\|_T, T \in \mathcal{T}_H(\Omega)$ and edge residuals

$\eta_E^2 := \eta_{E,1}^2 + \eta_{E,2}^2 + \eta_{E,N}^2 + \eta_{E,D}^2$ as given by

$$\begin{aligned}
\eta_{E,1} &:= h_E^{1/2}\|\nu_E \cdot [\nabla u_H]\|_E \ , \ \eta_{E,2} := h_E^{-1/2}\|[u_H]\|_E \ , \ E \in \mathcal{E}_H(\Omega), \\
\eta_{E,N} &:= h_E^{1/2}\|u^N - \nu_E \cdot \nabla u_H\|_E \ , \ E \in \mathcal{E}_H(\Gamma_N), \\
\eta_{E,D} &:= h_E^{-1/2}\ \|u^D - u_H\|_E \ , \ E \in \mathcal{E}_H(\Gamma_D).
\end{aligned}$$

The convergence analysis further invokes the data oscillations $osc_H^2 := osc_H^2(f) + osc_H^2(u^D) + osc_H^2(u^N)$, where

$$\begin{aligned}
osc_H^2(f) &:= \sum_{T\in\mathcal{T}_H(\Omega)} osc_T^2(f) \ , \ osc_T(f) := h_T\|f - \hat{f}_T\|_T, \\
osc_H^2(u^D) &:= \sum_{E\in\mathcal{E}_H(\Gamma_D)} osc_E^2(u^D) \ , \ osc_E(u^D) := h_E^{-1/2}\|u^D - \hat{u}_E^D\|_E, \\
osc_H^2(u^N) &:= \sum_{E\in\mathcal{E}_H(\Gamma_N)} osc_E^2(u^N) \ , \ osc_E(u^N) := h_E^{1/2}\|u^N - \hat{u}_E^N\|_E.
\end{aligned}$$

Here, $\hat{f}_T$ and $\hat{u}_E^D, \hat{u}_E^N$ stand for the integral means of f and u^D, u^N on T and E, respectively. The selection of elements and edges for refinement follows the well-known Dörfler marking and bisection of marked elements and edges is used to create a refined geometrically conforming triangulation $\mathcal{T}_h$. The main convergence result states a reduction of the global discretization error in the mesh-dependent energy norm.

Theorem. Let $u \in V$ be the solution of (1) and suppose that $u_H \in V_H$ and $u_h \in V_h$ are the solutions of IPDG (2) with respect to the triangulation $\mathcal{T}_H(\Omega)$ and the adaptively refined triangulation $\mathcal{T}_h(\Omega)$. Assume that $osc_h^2 \le \rho_2 osc_H^2$ for some $\varrho_2 < 1$. Then, for sufficiently large penalization parameter α there exist positive constants $\varrho_1 < 1$ and C which only depend on α and the shape regularity of the triangulations such that for $e_H := u - u_H$ and $e_h := u - u_H$ there holds

$$\begin{pmatrix} a_h(e_h, e_h) \\ osc_h^2 \end{pmatrix} \le \begin{pmatrix} \varrho_1 & C \\ 0 & \varrho_2 \end{pmatrix} \begin{pmatrix} a_H(e_H, e_H) \\ osc_H^2 \end{pmatrix}.$$

For the proof we refer to [2]. Related results have been obtained in [1] and [3].

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New Finite Elements for Large-Scale Simulation of Optical Waves

Britta Heubeck
(joint work with Christoph Pflaum)

Trigonometric Finite Wave Elements (TFWE) (see [1, 2, 3]) were developed for solving wave equations as Helmholtz's equation $-\triangle u - k^2 u = 0$ and

$$-\triangle u - (k^2 - \mathrm{i}k\sigma)u = f \qquad \text{on } \Omega \tag{1}$$

where C_{med} is a real constant depending on the medium, $k : \Omega \to \mathbb{R}$, $k \in L^\infty(\Omega)$, is the so-called wave number, and Ω is a rectangular domain defined by $\Omega :=]0, L[\times]0, W[$. Let us assume homogeneous Dirichlet boundary conditions. Furthermore, we assume that k is smooth in y-direction and describes a grating structure in x-direction like that of a Distributed Feedback laser (DFB). For defining TFWE nodal basis functions, we have to discretize Ω by two mesh sizes h_x and h_y such that k is continuous on the grid cells $r_{ij} :=]x_{i-1}, x_i]\times]y_{j-1}, y_j]$ where $p_{ij} := (x_i, y_j) := (ih_x, jh_y)$ denotes the grid points and $\boldsymbol{h} := (h_x, h_y)$ the mesh size tuple. Now, let $k_{\boldsymbol{h}}$ be the interpolant of k at the midpoints of each cell r_{ij}. Let v_z^h be the 1-dimensional nodal basis function of linear finite elements on a grid of mesh size h. Then, we construct 2-dimensional TFWE at grid point p_{ij} in the following way:

$$\begin{aligned}
B_{ij}^{\cos}(x,y) &:= \cos\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big)v_{x_i}^{h_x}(x)v_{y_j}^{h_y}(y),\\
B_{ij}^{\sin}(x,y) &:= \sin\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big)v_{x_i}^{h_x}(x)v_{y_j}^{h_y}(y),\\
B_{ij}^{\text{mix}}(x,y) &:= \text{mix}\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big)v_{x_i}^{h_x}(x)v_{y_j}^{h_y}(y)
\end{aligned}$$

where

$$\text{mix}\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big) = \begin{cases} -\sin\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big) & \text{if } x \le x_i \\ \sin\big(k_{\boldsymbol{h}}(x,y)(x-x_i)\big) & \text{if } x > x_i \end{cases}.$$

Then, these TFWE span the following non-conforming finite element space $V_{\boldsymbol{h}}^{2D}$ with $\tilde{\Omega}_{\boldsymbol{h}} := \bigcup_{i=1}^{N_x}\bigcup_{j=1}^{N_y} \mathring{r}_{ij}$:

$$\begin{aligned}
V_{\boldsymbol{h}}^{2D} := \Big\{ \; & u \in H^1(\tilde{\Omega}_{\boldsymbol{h}}) \,\Big|\, u(x,y) = \sum_{j=0}^{N_y}\sum_{i=0}^{N_x} a_{ij}B_{ij}^{\cos}(x,y) + b_{ij}B_{ij}^{\sin}(x,y) \\
& + c_{ij}B_{ij}^{\text{mix}}(x,y), \quad \forall (x,y) \in \Omega, a_{ij}, b_{ij}, c_{ij} \in \mathbb{C},\ c_{0j} = c_{N_x j} = 0 \Big\}.
\end{aligned}$$

Let us consider the following weak problem derived from a time discretization of equation (1):
Find $u \in H^1(\Omega)$ such that $a(u,v) = \boldsymbol{f}(v)$, $\forall v \in H^1(\Omega)$, where

$$a(u,v) := \int_\Omega \big(\nabla u(x,y)\nabla\bar{v}(x,y) - \alpha u(x,y)\bar{v}(x,y) + \mathrm{i}\beta u(x,y)\bar{v}(x,y)\big)d(x,y), \tag{2}$$

$\alpha := k^2$, and $\beta_1 > \beta > \beta_0 > 0$.

Furthermore, we assume that the solution u of (2) satisfies the following Oscillation Assumption:

Assumption 1 (**Oscillation Assumption**). *Let $u \in H^2(\Omega) \cup \mathcal{C}(\Omega)$ be a function with an approximate local wavelength of size $\frac{2\pi}{k}$ where $k \in L^\infty(\Omega)$, $k > 0$, is a real-valued function. In mathematical notation, this means that u can be written as*

$$u(x,y) = u^+(x,y)\exp(\mathrm{i}kx) + u^-(x,y)\exp(-\mathrm{i}kx)$$

where $u^+(x,y)\exp(\mathrm{i}kx) \in H^2(\tilde{\Omega}_{\boldsymbol{h}})$, $u^-(x,y)\exp(-\mathrm{i}kx) \in H^2(\tilde{\Omega}_{\boldsymbol{h}})$,

$$\|\frac{\partial^2 u^+}{\partial x^2}\|_{L^2(\tilde{\Omega}_{\boldsymbol{h}})} \ll \|\frac{\partial^2 u}{\partial x^2}\|_{L^2(\Omega)}, \text{ and } \|\frac{\partial^2 u^-}{\partial x^2}\|_{L^2(\tilde{\Omega}_{\boldsymbol{h}})} \ll \|\frac{\partial^2 u}{\partial x^2}\|_{L^2(\Omega)}.$$

Then, the following approximation theorem holds where $I_{\boldsymbol{h}}^{2D\ osc} : H^2(\Omega) \longrightarrow V_{\boldsymbol{h}}^{2D}$ is a suitable interpolation operator [4], $h := \max\{h_x, h_y\}$, and $k_{max} := \max_{(x,y)\in\Omega} |k_h(x,y)|$, w.l.o.g. $k_{max} \geq 1$:

Theorem 1. *Let $u \in H^2(\Omega)$ satisfy Assumption 1. Then, we have*

$$\|u - I_{\boldsymbol{h}}^{2D\ osc}(u)\|_{H^1(\Omega)} \leq C((k_{max}+1)h+1)h\left(\|u^+\|_{H^2(\tilde{\Omega}_{\boldsymbol{h}})} + \|u^-\|_{H^2(\tilde{\Omega}_{\boldsymbol{h}})}\right)$$

where C can be chosen independently of h and k_{max}, if h, $|k|_{H^{1,\infty}(r_{ij})}$, and $|k|_{H^{2,\infty}(r_{ij})}$ are bounded from above.

Furthermore, the following convergence theorem holds:

Theorem 2. *Let u satisfy Oscillation Assumption 1, $\frac{\partial k}{\partial y} \in L^\infty(\Omega)$, and $u_{\boldsymbol{h}} \in V_{\boldsymbol{h}}^{2D}$ be the solution of the non-conforming TFWE discretization of (2). Then, we have*

$$\|u - u_{\boldsymbol{h}}\|_{H^1(\tilde{\Omega}_{\boldsymbol{h}})} \leq Ch(C_{up}C_{low}+1)\left(\|u^+\|_{H^2(\tilde{\Omega}_{\boldsymbol{h}})} + \|u^-\|_{H^2(\tilde{\Omega}_{\boldsymbol{h}})} + \left\|\frac{\partial u}{\partial y}\right\|_{H^1(\Omega)}\right)$$

where C is a constant, $C_{low} = 4\max\left(1, \frac{k_{max}^2}{|\beta_0|}\right)$, and $C_{up} = 1 + \frac{\beta_1}{k_{max}^2}$. Moreover, C can be chosen independently of h and k_{max}, if h, $|k_{ij}|^{-1}$, hk_{max}, $|k|_{H^{1,\infty}(\tilde{\Omega}_{\boldsymbol{h}})}$, and $|k|_{H^{2,\infty}(\tilde{\Omega}_{\boldsymbol{h}})}$ are bounded from above.

Proof of Theorem 1 and 2 can be found in [4].

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Nonconforming Discretisation of the Miscible Displacement Problem with Discontinuous Coefficients

Max Jensen
(joint work with Sören Bartels, Rüdiger Müller)

We study a system of nonlinear partial differential equations which models the single phase, miscible displacement of one fluid by another in a porous medium. With the assumption of incompressibility the system consists of an elliptic equation determining the pressure p and Darcy velocity u as well as of a parabolic equation which describes the evolution of the concentration c of one fluid in the mixture:

$$\phi\partial_t c - \operatorname{div}\big(\mathbb{D}(u)\nabla c\big) + u\cdot\nabla c + q^I c = \hat{c}q^I, \tag{1}$$

$$\operatorname{div} u = q^I - q^P, \tag{2}$$

$$u = -\frac{\mathbb{K}}{\mu(c)}\big(\nabla p - \rho(c)\, g\big), \tag{3}$$

subject to the boundary and initial conditions (n being the outward normal)

$$u\cdot n = 0,\ \big(\mathbb{D}(u)\nabla c\big)\cdot n = 0 \quad \text{on } (0,T)\times\partial\Omega; \qquad c(0,\cdot) = c_0 \quad \text{on } \{0\}\times\Omega. \tag{4}$$

We refer the reader to [2, 1] and references therein for detailed discussions concerning existence, uniqueness, and validity of a maximum principle for weak solutions of (1) to (4). A discussion of various generalisations of the mathematical model can be found in [3]. For a more detailed description of the system we refer to [4].

In order to deal with discontinuous coefficients and to control the time derivative of c_h we project $\mathbb{D}$ onto a space of piecewise polynomial functions:

$$\mathbb{D}_h : L^2(\Omega)^d \to \mathcal{S}^s(\mathcal{T}_c, \mathbb{R}^{d\times d}),\ v \mapsto \Pi_{\mathcal{T}}\circ\mathbb{D}(v,\cdot). \tag{5}$$

The diffusion term of the concentration equation is discretised by the symmetric interior penalty discontinuous Galerkin method: Given a $u_h \in \mathcal{S}_u^j$, we set

$$\begin{aligned} B_d(c_h, w_h; u_h) \;:=\; & \big(\mathbb{D}_h(u_h)\nabla_h c_h, \nabla_h w_h\big) - \big(n_{\mathcal{E}}\,[c_h], \{\mathbb{D}_h(u_h)\,\nabla_h w_h\}\big)_{\mathcal{E}_\Omega^j} \\ & - \big(n_{\mathcal{E}}\,[w_h], \{\mathbb{D}_h(u_h)\,\nabla_h c_h\}\big)_{\mathcal{E}_\Omega^j} + \big(\sigma^2[c_h], [w_h]\big)_{\mathcal{E}_\Omega^j} \end{aligned}$$

where, given a suitable $C_\sigma \in \mathbb{R}$, we set

$$\sigma^2 : \mathcal{E}_\Omega^j \to \mathbb{R},\ x \mapsto C_\sigma\, \max\{n_{\mathcal{E}}^{\mathsf{T}}\,\mathbb{D}_h(u_h^+, x)\, n_{\mathcal{E}},\ n_{\mathcal{E}}^{\mathsf{T}}\,\mathbb{D}_h(u_h^-, x)\, n_{\mathcal{E}}\}/h_{\mathcal{E}}.$$

The convection, injection and production terms are represented by

$$\begin{aligned} B_{cq}(c_h, w_h; u_h) := {}^1\!/_2 \Big(& \big(u_h\nabla_h c_h, w_h\big) - \big(u_h c_h, \nabla_h w_h\big) + \big((q^I + q^P)c_h, w_h\big) \\ & + \sum_{K\in\mathcal{T}^j} \big(c_h^+, (u_h\cdot n_K)_+[w_h]\big)_{\partial K\setminus\partial\Omega} - \big((u_h\cdot n_K)_-[c_h], w_h^+\big)_{\partial K\setminus\partial\Omega}\Big) \end{aligned}$$

where $(u_h\cdot n)_+ := \max\{u_h\cdot n, 0\}$ and $(u_h\cdot n)_- := \min\{u_h\cdot n, 0\}$. We consider the following method to solve the boundary value problem:

ALGORITHM (A^{dG}). *Choose* $c_h^0 \in \mathcal{S}_c^0$. *For* $1 \le j \le M$ *and* $c_h^{j-1} \in \mathcal{S}_c^{j-1}$ *find* $(u_h^j, p_h^j, c_h^j) \in \mathcal{S}_u^j \times \mathcal{S}_p^j \times \mathcal{S}_c^j$ *such that*

$$
\begin{aligned}
\big(\mu(c_h^j)\,\mathbb{K}^{-1}u_h^j, v_h\big) \;-\big(p_h^j, \operatorname{div} v_h\big) \;&= \big(\rho(c_h^j)\,g, v_h\big),\\
\big(q_h, \operatorname{div} u_h^j\big) \;&= \big(q^I - q^P, q_h\big)
\end{aligned}
\tag{6}
$$

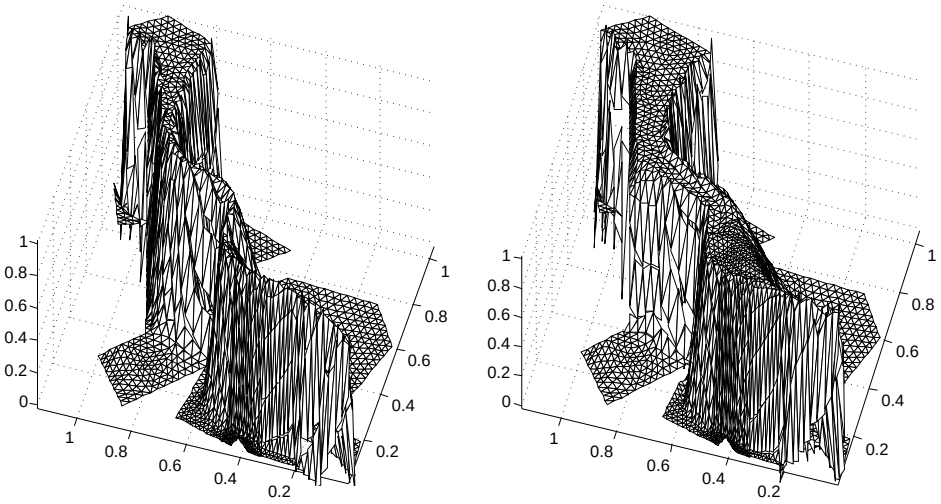
for all $(v_h, q_h) \in \mathcal{S}_u^j \times \mathcal{S}_p^j$ *and*

$$
\big(\phi\, d_t c_h^j, w_h\big) + B_d(c_h^j, w_h; u_h^j) + B_{cq}(c_h^j, w_h; u_h^j) = \big(\hat{c}q^I, w_h\big) \tag{7}
$$

for all $w_h \in \mathcal{S}_c^j$.

A detailed analysis of this method can be found in [4], where the convergence properties of the method under low regularity assumptions are studied.

NUMERICAL EXAMPLE. We consider the numerical solution of (1) to (4) on a polygonal domain. The permeability $\mathbb{K}$ is discontinuous on $\{(x,y) \in \mathbb{R}^2 : (x = 0.4 \text{ and } y \in [0.6, 1]) \text{ or } (x \in [0, 0.4] \text{ and } y = 0.6)\} \cap \Omega$. In particular the mesh is not aligned to the discontinuity in $\mathbb{K}$. Selecting the parameters as in the first numerical example of [4], we observe stable approximation by the proposed method.



Concentration c at times $t = 2$ and $t = 5$.

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A DGM-based fast sweeping method for Eikonal equations

Fengyan Li
(joint work with Chi-Wang Shu, Yong-Tao Zhang, Hongkai Zhao)

Hamilton-Jacobi equations provide an important mathematical model in many applications such as optimal control, geometric optics and computer vision. Efficient and accurate numerical algorithms for simulating these nonlinear equations are crucial in practice. One of the successful approaches to solve *static* Hamilton-Jacobi equations is the fast sweeping method. In the original fast sweeping method [1, 3], Gauss-Seidel iteration with alternating sweeping orderings is combined with upwind finite difference discretizations, and the method follows the causality of the equations along characteristics in a parallel way. This iterative method turns out to be stable, monotone, and first order accurate. It has optimal computational complexity in the sense that the number of iterations needed is independent of the number of the unknowns.

Our research effort focuses on the design of fast sweeping methods with higher order accuracy based on discontinuous Galerkin method (DGM). DG discretization is chosen due to its use of the compact stencil to achieve high order accuracy, and its flexibility in working with various approximations as well as in handling complicated geometries and different boundary conditions. As the initial progress, a second order DGM-based fast sweeping method is proposed [2] for solving an important family of static Hamilton-Jacobi equations, namely the Eikonal equations, on rectangular meshes. The causality of the equations is incorporated into the design of this method. The resulting local nonlinear problem in the Gauss-Seidel iteration is a simple quadratic system and can be solved explicitly. Extensive numerical experiments have been implemented to validate both the accuracy and the efficiency of the proposed method. Currently we are working towards the design of DGM-based fast sweeping methods with higher accuracy on unstructured meshes for more general static Hamilton-Jacobi equations. New methodologies need to be developed in order to achieve this goal.

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Difficulties Arising in GFEM, in the Patchwise RPP and RSP Meshfree Methods

Hae-Soo Oh

Oh et al ([8]) introduced the reproducing polynomial particle (RPP) shape functions that are smooth piecewise polynomial and satisfy the Kronecker delta property. We introduce RPPM (Reproducing Polynomial Particle Methods) that is the Galerkin approximation (meshfree) method associated with the use of the RPP approximation space ([3],[6]). The RPP shape functions and RPPM have the following features: (i) Integrals of Piecewise polynomial RPP functions are exact if the integral domain is divided into subdomains on which RPP functions are polynomials; (ii) Particles corresponding to RPP shape functions are uniformly distributed. (iii) The order of convergence in energy norm for smooth problems is the same as the reproducing order of approximation functions.

In order to make RPPM being applicable in nonuniform (adaptive) manner, like PUFEM([1],[4]), we consider a general patchwise RPPM by planting particles on the computation domain in a patchwise uniform (or nonuniform) manner. For this method, we divide the domain into disjoint patches and construct partition of unity subordinate to a covering that consists of neighborhoods of patches. Then we connect patchwise RPP shape functions defined on each patch together by using partition of unity functions.

For success of this approach, it is important to construct smooth partition of unity functions that could yield stable stiffness matrices with small condition numbers. For this purpose, we constructed a family of smooth PU functions with flat-top ([7],[9]) that are actually the convolution of characteristic functions of pathes with scaled window function.

Furthermore, to deal with problems containing singularity by meshless methods, we constructed Reproducing Singularity Particle (RSP) shape functions([10]). This method is similar to the Method of Auxiliary Mapping, introduced by Babuška and Oh ([2],[5]), in the framework of the p-FEM with $p \geq 2$. For adaptive approaches in meshless methods, we constructed PU functions with flat-top in adaptive manner ([6]).

However, difficulties arising in patchwise RPPM and adaptive RPPM include the followings:

- Condition numbers of resulting stiffness matrices, especially for problems on nonconvex domains, are very large even if PU functions with flat-top are used for the construction of global approximation functions.
- Even though approximation functions satisfy the Kronecker delta property at the particles, it is not easy to impose Dirichlet boundary conditions.
- It is difficult to construct smooth PU functions with flat-top for non convex domains. Adaptively constructed PU functions may have large gradients.
- Global approximation functions obtained by multiply PU function to patchwise uniform RPP functions are piecewise polynomials. However, they

consist of many polynomial pieces. Thus, longer computing times are required for accurate approximate solutions. Generally, the computing time for meshless methods are much longer than conventional FEM.

- Local error estimators for adaptive approaches in meshless method are different in structures from those in the conventional FEM.

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A Non-Conforming Finite Element Method for Convex Variational Problems

Christoph Ortner

(joint work with D. Praetorius)

Introduction. For a domain $\Omega \subset \mathbb{R}^n$, a strictly convex density $W : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ with $W(F) \geq c(|F|^p - 1)$ for some $p \in (1, \infty)$, and $f \in \mathrm{L}^{p'}(\Omega)^m$, we define the energy functional

$$\mathcal{J}(v) = \int_\Omega \Big(W(\nabla v) - f \cdot v \Big) \,\mathrm{dx}. \tag{1}$$

Furthermore, for given $g \in \mathrm{W}^{1,p}(\Omega)^m$ and $\Gamma_i \subset \partial\Omega$, $i = 1, \ldots, m$, we define the admissible set

$$\mathcal{A} = \big\{ v \in \mathrm{W}^{1,p}(\Omega)^m : v_i|_{\Gamma_i} = g_i|_{\Gamma_i}, i = 1, \ldots, m \big\}. \tag{2}$$

Throughout, we assume that the surface measure of the sets Γ_i is non-zero, for all $i = 1, \dots, m$. In that case, the direct method of the calculus of variations [3] establishes the existence of a minimizer of $\mathcal{J}$ in $\mathcal{A}$,

$$u = \operatorname{argmin} \mathcal{J}(\mathcal{A}), \tag{3}$$

which is even unique due to strict convexity. Since u, in general, is not accessible analytically, one aims at the numerical solution of (3).

Foss, Hrusa, and Mizel [4] showed that (3), even under the fairly strong conditions posed above, may exhibit the *Lavrentiev gap phenomenon*,

$$\inf \mathcal{J}(\mathcal{A} \cap \mathrm{W}^{1,\infty}(\Omega)) > \inf \mathcal{J}(\mathcal{A}), \tag{4}$$

which is the focus of the present paper.

Let $\mathcal{T}_h$ be a shape-regular simplicial triangulation of Ω with mesh-size h, and let $\mathrm{P}_1(\mathcal{T}_h)$ be the space of continuous, piecewise affine finite element functions. Then, the P_1-finite element discretization of (3) reads

$$u_h = \operatorname{argmin} \mathcal{J}\big(\mathcal{A} \cap \mathrm{P}_1(\mathcal{T}_h)^m\big). \tag{5}$$

In the presence of a Lavrentiev gap, however, the inclusion $\mathrm{P}_1(\mathcal{T}_h) \subset W^{1,\infty}$ immediately implies that $\mathcal{J}(u_h) \not\to \mathcal{J}(u)$, i.e., the P_1-FEM cannot converge to the "correct" limit. The purpose of this talk is to show that, by contrast, the nonconforming Crouzeix–Raviart finite element method can be employed for a successful discretization of (3).

Crouzeix–Raviart FEM for (3). Let $\mathcal{E}_h$ be the set of faces of elements and, for $E \in \mathcal{E}_h$, let m_E be the face midpoint. Let $\mathrm{CR}(\mathcal{T}_h)$ be the Crouzeix–Raviart finite element space, which consists of all piecewise affine functions, which are continuous in all face midpoints m_E. We assume that the mesh $\mathcal{T}_h$ respects the sets Γ_i, for all $i = 1, \dots, m$. We then discretize the admissible set $\mathcal{A}$ by

$$\mathcal{A}_h = \big\{\Pi_h v : v \in \mathcal{A}\big\} = \big\{v_h \in \mathrm{CR}(\mathcal{T}_h)^m : v_{h,i}(m_E) = \Pi_h g_i(m_E), \forall E \subset \Gamma_i\big\}, \tag{6}$$

where $\Pi_h : \mathrm{W}^{1,p}(\Omega)^m \to \mathrm{CR}(\mathcal{T}_h)^m$ denotes the Crouzeix-Raviart interpolant

$$\Pi_h v(m_E) = |E|^{-1} \int_E v \,\mathrm{ds} \qquad \forall E \in \mathcal{E}_h. \tag{7}$$

Extending the gradient operator to the space $\mathrm{CR}(\mathcal{T}_h)$, where it denotes the piecewise gradient, we also extend the function $\mathcal{J}$ to $\mathcal{A}_h$ in an obvious way. We thus can compute the *discrete minimizers*

$$u_h = \operatorname{argmin} \mathcal{J}(\mathcal{A}_h). \tag{8}$$

As for the continuous formulation (3), the direct method of the calculus of variations proves that the Crouzeix-Raviart FEM (8) has a unique solution.

It is crucial to observe that the Crouzeix-Raviart interpolant (7) satisfies

$$\mathcal{J}(\Pi_h v) \le \mathcal{J}(v) + C\, h\, \|f\|_{\mathrm{L}^{p'}(\Omega)} \|\nabla v\|_{\mathrm{L}^p(\Omega)} \quad \forall v \in \mathrm{W}^{1,p}(\Omega)^m. \tag{9}$$

Based on this estimate, the first main result of our presentation is the following theorem from [6], which proves *a priori* convergence of the Crouzeix-Raviart FEM even in the presence of the Lavrentiev gap phenomenon (4).

Theorem 1. *With u and u_h the solutions of* (3) *and* (8)*, respectively, there holds*

$$\left\{\begin{array}{rcll} u_h & \to & u & \text{strongly in } \mathrm{L}^p, \\ \nabla u_h & \to & \nabla u & \text{strongly in } \mathrm{L}^p, \\ \mathcal{J}(u_h) & \to & \mathcal{J}(u) & \end{array}\right. \tag{10}$$

as $h \searrow 0$. □

Adaptive Crouzeix–Raviart FEM for (3). The Lavrentiev gap phenomenon (4) is closely linked to the occurrence of singularities in the solution to the variational problem (3). It therefore seems to be natural to consider adaptive mesh refinement techniques.

In the following we replace the global mesh-size h by a local mesh-size function h_ℓ, where ℓ denotes the refinement level. We change the notation from $\mathcal{T}_h$ to $\mathcal{T}_\ell$, u_h to u_ℓ, and so forth. Let $[u_\ell]$ denote the jump of u_ℓ across internal faces, and denote $[u_\ell]_i = u_{\ell,i} - g_i$ on Γ_i, and $[u_\ell]_i = 0$ on $\partial\Omega \setminus \Gamma_i$, respectively.

By careful examination of the proof of Theorem 1, one finds that the condition $h \searrow 0$ may be relaxed to

$$\|h_\ell f\|_{\mathrm{L}^{p'}(\Omega)} + \|h_\ell^{1/p}[u_\ell]\|_{\mathrm{L}^p(\cup\mathcal{E}_h)} \xrightarrow{\ell\to\infty} 0. \tag{11}$$

More precisely, (11) implies convergence of the Crouzeix-Raviart approximations u_ℓ to the solution u of (3) in the sense of Theorem 1 as $\ell \to \infty$. This motivates the definition of the convergence indicator

$$\eta_\ell = \sum_{E\in\mathcal{E}_\ell} \eta_\ell(E) = \sum_{E\in\mathcal{E}_\ell} \Big(\|h_\ell f\|^{p'}_{\mathrm{L}^{p'}(\omega_E)} + \|h_\ell^{1/p}[u_\ell]\|^p_{\mathrm{L}^p(E)} \Big), \tag{12}$$

where ω_E denotes the patch of a face $E \in \mathcal{E}_h$. This indicator is then used to steer the following adaptive algorithm:

0. INPUT: initial mesh $\mathcal{T}_0$, marking parameter $\theta \in (0, 1]$
1. COMPUTE: $u_\ell = \operatorname{argmin} \mathcal{J}(\mathcal{A}_\ell)$.
2. ESTIMATE: compute convergence indicators $\eta_\ell(E)$, for all $E \in \mathcal{E}_\ell$.
3. MARK: construct $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ so that
$$\sum_{E\in\mathcal{M}_\ell} \eta_\ell(E) \geq \theta\eta_\ell$$
4. REFINE: refine all faces $E \in \mathcal{M}_\ell$, so that the local mesh-size is reduced by a fixed ratio, to obtain a new mesh $\mathcal{T}_{\ell+1}$. Continue at 1.

Our second main result [7] states that this algorithm is convergent. Our argument is based on a localized version of (9) and thus is restricted to nonconforming finite elements. Contrary to preceding works on adaptive Crouzeix-Raviart FEM [2], our proof is neither based on the explicit use of the Euler-Lagrange equations nor on any kind of (generalized) Galerkin orthogonality. In particular, we note that the indicator η_ℓ from (12) does not provide a reliable upper bound for the error $u - u_\ell$, in general. This makes it even more surprising that one can prove convergence of the adaptive scheme.

Theorem 2. *The sequence of indicators computed in the above algorithm satisfies* $\eta_\ell \to 0$ *as* $\ell \to \infty$ *and thus guarantees* (11)*. In particular, there holds*

$$\text{(13)} \qquad \begin{cases} u_\ell & \to & u & \text{strongly in } \mathrm{L}^p, \\ \nabla u_\ell & \to & \nabla u & \text{strongly in } \mathrm{L}^p, \\ \mathcal{J}(u_\ell) & \to & \mathcal{J}(u), \end{cases}$$

as $\ell \to \infty$. □

The preceding theorem allows for a numerical verification of Lavrentiev gaps as follows: besides the adaptive Crouzeix-Raviart solution (11), we compute a conforming $\mathrm{P}_1(\mathcal{T}_h)$ approximation (5) of (3). Comparing the difference of the energies, one may see whether a Lavrentiev gap occurs or not.

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The perfectly matched layer approximation to elastic scattering problems

Joseph E. Pasciak

(joint work with J.E. Bramble and D. Trenev)

We consider a perfectly matched layer technique (PML) for approximating the elastic wave scattering problem. These problems are posed on the complement of a bounded domain $\Omega \subset \mathbb{R}^3$ (the scatterer). The boundary condition at infinity is given by the Kupradze-Sommerfeld radiation condition and involves different Sommerfeld conditions on different components of the field (the shear and compression waves). We shall see that the PML approach can be used to provide an artificial boundary condition which leads to an effective domain truncation strategy. This approach transparently deals with the different wave speeds and avoids the computational separation of shear and compression waves.

The PML technique on the infinite domain involves the definition of a new problem which has a solution which is identical to the original near the scatterer yet decays exponentially at infinity. Because of this decay, it is natural to truncate the PML problem to a bounded domain and subsequently apply finite element

approximation. We show that the truncated problem is stable provided that the size of the truncated domain is sufficiently large. We derive this result by perturbation from a coercive problem in the case when the PML strength is small. For the more general situation, we still derive stability but have to use a more complicated argument. As the general analysis is not based on perturbation, the classical techniques for analyzing non-symmetric and indefinite problems do not carry over for the finite element approximation. Accordingly, we cannot prove that finite element approximation remains stable for any PML strength. However, we show that it is possible to develop a stable and convergent negative norm least-squares method even in the general case. Details can be found in [1, 2, 3].

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Plane wave discontinuous Galerkin method for the Helmholtz equation

Ilaria Perugia

(joint work with C. Gittelson, R. Hiptmair, A. Moiola)

Consider for the following model problem for the Helmholtz equation:

$$\begin{aligned} -\Delta u - \omega^2 u &= f && \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} + i\omega\, u &= g && \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, Ω is a bounded polygonal Lipschitz domain in $\mathbb{R}^2$ and $\omega > 0$ is a fixed wave number (the corresponding wavelength is $\lambda = 2\pi/\omega$). The right hand side f is a source term in $L^2(\Omega)$, $\boldsymbol{n}$ is the outer normal unit vector to $\partial\Omega$, and i is the imaginary unit. Inhomogeneous first order absorbing boundary conditions in the form of impedance boundary conditions are used in (1), with $g \in L^2(\partial\Omega)$.

It is well-known that the oscillatory behavior of solutions to (1), along with numerical dispersion, renders standard finite element methods inefficient already in medium-frequency regimes. As an alternative, several ways to incorporate information from the equation into the discretization spaces have been proposed in the literature, giving rise to methods based on shape functions which are solutions to either the primal or the dual problem, among them the so-called "ultra weak variational formulation" (UWVF) introduced by Cessenat and Despres [3, 4, 5]. The UWVF is based on a domain decomposition approach and on the use of discontinuous piecewise plane wave basis functions. In [4], Cessenat and Despres managed to establish existence and uniqueness of solutions to the UWVF, but proved convergence only at the domain boundary. On the other hand, extensive

numerical experiments mainly conducted by P. Monk and collaborators indicate reliable convergence [9, 10] for a wide range of wave propagation problems (without volume sources). This carries over to the extension of the method to Maxwell's equations [3, 8].

In [1] and [7], the UWVF has been recast as a special discontinuous Galerkin (DG) method for (1) with trial and test spaces supplied by local plane wave spaces. This has made possible to derive global convergence properties by using the techniques of DG analysis. In [1], using estimates obtained in [13] and in [4], L^2–error estimates were derived for the original UWVF applied to the homogeneous Helmholtz equation. In [7], the convergence analysis has been performed by using the convergence theory of elliptic DG methods [2] combined with duality techniques. In order to do that, the classical UWVF has been slightly modified by introducing penalty parameters depending on the mesh size h and the wave number ω, in order to enhance the stability properties. We refer to this modified method as "plane wave discontinuous Galerkin" (PWDG) method. A priori h–asymptotic estimates in a mesh-dependent broken H^1–norm and in L^2–norm for the PWDG method applied to the inhomogeneous Helmholtz problem have been proved. Under the assumption that $h\omega^2\mathrm{diam}(\Omega)$ is sufficiently small, first order convergence in broken H^1–norm and second order convergence in L^2–norm have been derived. In order to do that, some theoretical tools which might be of interest in their own right have been developed: *(i)* construction of a basis for plane wave spaces that remains stable for small wave numbers; *(ii)* inverse estimates and projection error estimates for plane waves; *(iii)* new variants of duality arguments. The low order convergence, independently of the number of plane waves per element used in the approximation, might be disappointing. On the other hand, for general H^2 functions, the plane wave spaces only possess the approximation properties of linear finite elements. The same analysis framework has been applied in [12] in order to derive high order error estimates in broken H^1–norm and in L^2–norm for the PWDG method applied to the homogeneous Helmholtz equation. In order to do that, following [11], Vekua's theory has been applied in order to derive projection error estimates onto plane wave spaces in weighted Sobolev norms, for smooth functions satisfying $-\Delta v - \omega^2 v = 0$. By using $p = 2m + 1 \geq 3$ plane wave per element in the approximation space, under the same threshold condition as before, convergence rates equal to m and $m + 1$ in broken H^1–norm and in L^2–norm, respectively, have been proved. Numerical results reported in [7] and in [6] show that all these estimates are sharp and that the pollution effect actually takes place.

This latter aspect suggests that it is not advisable to try and improve accuracy by refining the mesh. Rather, the cell size should be linked to the wavelength and the number of plane wave directions should be increased. The investigation of the more interesting p–version of PWDG methods is an open problem.

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Convergence Analysis for Galerkin Discretizations of the Helmholtz equation

Stefan A. Sauter

(joint work with J. Markus Melenk)

The goal of these notes is the development of a general convergence theory for Galerkin methods for a model Helmholtz problem in $\mathbb{R}^d$, $d \in \{1,2,3\}$, which allows the construction and analysis of generalized finite element spaces (denoted by S). In particular, it is shown that the stability and asymptotic convergence rates are valid provided $\dim S \gtrsim k^d$. This is a substantial improvement compared, e.g., to the low order $\mathbb{P}_1$ finite element space where the threshold is $\dim S \gtrsim k^{2d}$.

This extended abstract is a summary of [1] and we refer for the proofs and references to this paper.

1. A Scattering Problem

Let $\Omega := B_R(0)$ be the ball with radius $R > 0$ about the origin. We consider the Helmholtz equation in the variational form

$$\text{find } u \in H^1(\Omega) \quad \text{s.t.} \quad a(u,v) = F(v) \quad \forall v \in H^1(\Omega), \tag{1}$$

where

$$a(u,v) := \int_\Omega \langle \nabla u, \nabla \overline{v} \rangle - k^2 u \overline{v} - \int_{\partial\Omega} (Tu)\,\overline{v} \quad \text{and} \quad F(v) := \int_\Omega f \overline{v}$$

and T denotes the Dirichlet-to-Neumann operator. In the following, we will generally assume that $k \geq k_0 > 0$ and $R \geq R_0 > 0$. We will derived stability and convergence estimates in the scaled H^1-norm $\|u\|_{\mathcal{H}} := \left(|u|^2_{H^1(\Omega)} + k^2 \|u\|^2_{L^2(\Omega)} \right)^{1/2}$. In the following, all constants might depend on k_0 and R_0.

Theorem 1. *There exists a constant $C(\Omega, k) > 0$ such that, for all $f \in \left(H^1(\Omega)\right)'$, problem (1) has a unique solution which satisfies*

$$\|u\|_{\mathcal{H}} \leq C(\Omega, k) \|f\|_{H^1(\Omega)'}.$$

If $f \in L^2(\Omega)$, then

$$\|v\|_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega)}.$$

There is a constant $C_c > 0$ which only depends on k_0 and R_0 such that

$$|a(u,v)| \leq C_c \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \qquad \forall u, v \in H^1(\Omega).$$

For a proof, we refer to [1].

2. Discrete Stability and Convergence Analysis

We introduce the acoustic volume potential

$$(N_k f)(x) := \int_\Omega G_k(x-y) f(y)\, dy \qquad \forall x \in \Omega,$$

where $G_k(z) := g_k(\|z\|)$ is the **fundamental solution** of the Helmholtz equation, e.g., $g_k(r) := \mathrm{e}^{ikr} / (4\pi r)$ for $d = 3$. The *key rôle* in the analysis plays the **adjoint approximation property**

$$\eta(S) := \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in S} \frac{\|N_k^\star f - v\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}}$$

where the concrete choice of the (generalized) finite element space S enters. $N_k^\star$ denotes the adjoint acoustic volume potential.

Theorem 2. *Assume that the finite element space S is chosen such that*

$$k\eta(S) \leq (4C_c)^{-1}. \tag{2}$$

Then, the discrete inf-sup constant (discrete stability) satisfies

$$\inf_{u \in S} \sup_{v \in S\setminus\{0\}} \frac{|a(u,v)|}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} \geq \frac{1/2}{1 + (2C_c)^{-1} + k}.$$

Let the stability condition (2) be satisfied. Then

$$\|u - u_S\|_{\mathcal{H}} \leq 2C_c \inf_{v \in S} \|u - v\|_{\mathcal{H}} \quad \text{and} \quad \|u - u_S\|_{L^2(\Omega)} \leq 2C_c^2 \eta(S) \inf_{v \in S} \|u - v\|_{\mathcal{H}}.$$

For the proofs of these theorems we refer to [1].

3. Estimate of the adjoint approximation property

The refined analysis for the Helmholtz problem is based on the following splitting of the regularity of the solution into a low frequency, low regularity part and a high frequency, analytic part (cf. [1]).

Lemma 3 (Decomposition Lemma)**.** *There exist constants C, γ depending only on* $\operatorname{diam}\Omega$*, k_0 such that, for $f \in L^2(\Omega)$, the solution of the adjoint Helmholtz problem $v = N_k^\star f$ can be decomposed as $v = v_{H^2} + v_{\mathrm{osc}}$ with*

$$\begin{aligned} \|v_{H^2}\|_{H^2(\Omega)} &\leq C_{\mathrm{reg}}^{H^2} \|f\|_{L^2(\Omega)}, \\ \|\nabla^p v_{\mathrm{osc}}\|_{L^2(\Omega)} &\leq C_{\mathrm{reg}}^{\mathrm{osc}} (\gamma k)^{p-1} \|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0. \end{aligned}$$

By using the decomposition lemma the adjoint approximation property can be split into two simpler approximation properties.

Definition 4. *For given $\gamma > 0$, and $k \geq k_0 > 0$ let*

$$\mathcal{H}^{\mathrm{osc}}(\gamma, k) := \left\{ v \in H^1(\Omega) : \|\nabla^p v\|_{L^2(\Omega)} \leq (\gamma k)^{p-1} \quad \forall p \in \mathbb{N}_0 \right\}.$$

The approximation properties for the oscillatory and the H^2-part are

$$\eta_{\mathrm{apx}}^{\mathrm{I}}(S, k) := \sup_{v \in \mathcal{H}^{\mathrm{osc}}(\gamma,k)} \inf_{w \in S} \|v - w\|_{\mathcal{H}} \text{ and } \eta_{\mathrm{apx}}^{\mathrm{II}}(S) := \sup_{\substack{v \in H^2(\Omega)\setminus\{0\} \\ \|v\|_{H^2(\Omega)}=1}} \inf_{w \in S} \|v - w\|_{\mathcal{H}}.$$

Corollary 5. *Let $d \in \{1, 2, 3\}$ and assume the hypotheses of the decomposition lemma. Let $S \subset H^1(\Omega)$ be a finite dimensional approximation space. Then*

$$\eta(S) \leq C_{\mathrm{reg}}^{\mathrm{osc}} \eta_{\mathrm{apx}}^{\mathrm{I}}(S, k) + C_{\mathrm{reg}}^{H^2} \eta_{\mathrm{apx}}^{\mathrm{II}}(S).$$

Hence, the estimate of the adjoint approximation property $\eta(S)$ is reduced to estimates of the approximation properties of a (generalized) finite element space S for a) H^2-functions and b) highly oscillatory, analytic functions.

4. Application to *hp*-FEM

We consider here an *hp*-finite element space based on a conforming quasi-uniform triangulation of $\mathcal{T}$. The mesh width h is the maximal triangle diameter and p denotes the local polynomial degree of approximation. Let $S_{\mathcal{T}}^p := \left\{ u \in C^0(\overline{\Omega}) \mid \forall \tau \in \mathcal{T} : u|_\tau \circ F_\tau \in \mathbb{P}_p \right\}$.

Theorem 6. *Let $kh/p \lesssim 1$. Then,*

$$k\eta_{\mathrm{apx}}^{\mathrm{II}}(S, k) = k \sup_{\substack{v \in H^2(\Omega) \\ \|v\|_{H^2(\Omega)=1}}} \inf_{w \in S} \|v - w\|_{\mathcal{H}} \leq C \frac{kh}{p},$$

$$k\eta_{\mathrm{apx}}^{\mathrm{I}}(S, k) = k \sup_{v \in \mathcal{H}^{\mathrm{osc}}(\gamma,k)} \inf_{w \in S} \|v - w\|_{\mathcal{H}} \leq C \left(\frac{1}{p} + \frac{kh}{p} \right) k \left(\frac{kh}{\sigma p} \right)^p.$$

Corollary 7. *Let the assumptions of the Theorem 6 be satisfied. Then, there exist constants c_1, c_2, c_3 independent of $k, h,$ and p such that*

$$kh/p \le c_1 \qquad \text{and} \qquad p \ge c_2 \ln k$$

implies the stability condition $k\eta(S) \le (4C_c)^{-1}$.

The minimal dimension of the corresponding $h - \log k$ finite element space satisfies $\dim S \gtrsim c_3 k^d$.

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Hybrid Finite Elements for the Helmholtz Equation

Joachim Schöberl
(joint work with Peter Monk, Astrid Sinwel)

In this talk we consider the Helmholtz equation in mixed form,

$$\begin{aligned} i\omega u + \operatorname{div} v &= f, \\ i\omega v + \nabla u &= 0, \end{aligned}$$

equipped with the impedance boundary condition

$$u + v_n = g.$$

Due to the boundary condition, the equation is uniquely solvable.

The mixed finite element method is to find $u_h \in U_h \subset L_2$ and $v_h \in V_h \subset H(\operatorname{div})$ such that

$$\begin{aligned} i\omega(u_h, \xi) + (\operatorname{div} v_h, \xi) &= (f, \xi) \qquad \forall\, \xi \in U_h \\ (u_h, \operatorname{div} \tau) - i\omega(v_h, \tau) + (v_n, \tau_n)_{\partial\Omega} &= (g, v_n)_{\partial\Omega} \qquad \forall\, \tau \in V_h. \end{aligned}$$

The canonical finite element spaces are discontinuous piecewise polynomials $U_h = P_k$, and normal-continuous p.w. polynomials for V_h, e.g.,. Raviart-Thomas finite elements RT_k. The discrete system is solvable for sufficiently fine meshes.

The classical hybridization method by Arnold and Brezzi, 86, is to break normal continuity of the vector finite elements, and to reinforce it by means of a Lagrange multiplier u^F defined on element interfaces called facets. This allows a local elimination of all element unknowns, and only the system for the Lagrange multiplier is remaining. The Lagrange multiplier u^F can be understood as discretization of the Dirichlet data on the facets. The local elimination procedure is to solve discretized Dirichlet problems on the elements. While these local problems are always solvable for elliptic pdes, when solving Helmholtz equation, one might hit a Dirichlet eigenvalue on the element rendering the local problem singular.

The idea is now to smuggle in the normal component v_n as a second variable v_n^F on the facets. For this, we add the consistent stabilization term $\langle v_n - v_n^F, \tau_n - \tau_n^F \rangle$,

where $\langle f, g\rangle := \sum_T (f,g)_{\partial T}$ denotes the sum of inner products on element boundaries.

The big, hybrid system is now to find $(u, v, u^F, v_n^F) \in X_h := P_k(T) \times RT_k(T) \times P_k(F) \times P_k(F)$ such that

$$B(u, v, u^F, v_n^F; \xi, \tau, \xi^F, \tau_n^F) = f(\xi, \tau, \xi^F, \tau_n^F) \qquad \forall (\xi, \tau, \xi^F, \tau_n^f) \in X_h$$

with the bilinear-form

$$\begin{aligned} B(u, v, u^F, v_n^F; \xi, \tau, \xi^F, \tau_n^F) &= i\omega(u, \xi) + (\operatorname{div} v, \xi) \\ &+ (u, \operatorname{div} \tau) - i\omega(v, \tau) + \langle v_n, \tau_n\rangle - \left\langle u^F + v_n^F, \tau_n\right\rangle \\ &- \left\langle v_n, \xi^F\right\rangle - (u^F, \xi^F)_{\partial\Omega} \\ &+ \left\langle -v_n + v_n^F, \tau_n^F\right\rangle \end{aligned}$$

and the linear-form

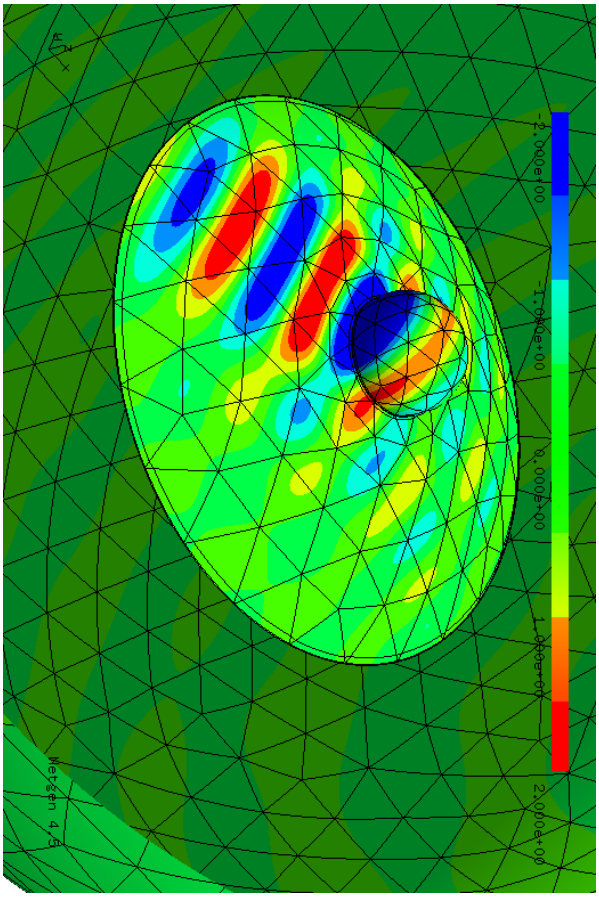
$$f(\xi, \tau, \xi^F, \tau_n^F) = (f, \xi) - (g, \xi^F)_{\partial\Omega}.$$

Due to the stabilization term, the equations for the local variables are now posed with impedance boundary conditions. At least on the continuous level, they are thus uniquely solvable, and the local elimination can be performed.

The boundary data for the local problem is $u^F + v_n^F$, which can be interpreted as incoming wave. The outgoing wave is $u^F - v_n^F$. In [1] we have shown that the incoming to outgoing map is an isometry, i.e., energy is preserved. This holds on the continuous level as well as on the finite element level.

In [1] we have proposed to add the additional consistent term $\left\langle u - u^F, \xi - \xi^F\right\rangle$ to the bilinear-form. The numerically observed effect is damping of unresolved waves.

The arising system matrix is complex symmetric. We have applied conjugate gradient iteration (with the non-hermitian scalar product), and block Jacobi preconditioners combining all unknowns on the individual facets. We observed quite low iteration numbers. The picture below shows the scattering of an incoming wave. The computational domain has a diameter of 15 wavelength. The mesh consists of about 14000 elements of 5^{th} order, which leads to about 5 million unknowns. The above mentioned solvers needs 204 iterations for a relative error reduction of 10^{-10}.



The computational results seem very promising. We consider a further, more detailed analysis of the discretization method as well as the iterative solver is a challenging task.

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A finite element/boundary element coupling method for strongly nonlinear transmission problems with contact

Ernst P. Stephan

(joint work with Heiko Gimperlein, Matthias Maischak, Elmar Schrohe)

The talk is split into two parts. First, we analyze an FE–BE coupling procedure for scalar elastoplastic interface problems involving friction, where a nonlinear uniformly monotone operator such as the p–Laplacian in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is coupled to the linear Laplace equation on the exterior domain Ω^c. In the second part we present a corresponding coupling formulation for a nonconvex double–well potential in Ω. In both cases the transmission problem is reduced to a boundary/domain variational inequality, which is solved by Galerkin's method with finite and boundary elements. The Galerkin approximations are

shown to converge in a suitable product of L^p– and L^2–Sobolev spaces.
The nonlinear frictional contact problem under consideration reads for $p \geq 2$: Given $f \in L^{p'}(\Omega)$, $u_0 \in W^{\frac{1}{2},2}(\partial\Omega)$, $t_0 \in W^{-\frac{1}{2},2}(\partial\Omega)$, $g \in L^\infty(\Gamma_s)$ and $a \in \mathbb{R}$ with $\int_\Omega f + \langle t_0, 1\rangle = 0$ for $n = 2$, find $u_1 \in W^{1,p}(\Omega)$, $u_2 \in W^{1,2}_{loc}(\Omega)$ such that

$$
\begin{aligned}
-\mathrm{div}\,(\varrho(|\nabla u_1|)\nabla u_1) &= f \quad \text{in } \Omega,\ -\Delta u_2 = 0 \quad \text{in } \Omega^c,\\
\varrho(|\nabla u_1|)\partial_\nu u_1 - \partial_\nu u_2 &= t_0 \quad \text{on } \partial\Omega,\ u_1 - u_2 = u_0 \quad \text{on } \Gamma_t,\\
-\varrho(|\nabla u_1|)\partial_\nu u_1(u_0 + u_2 - u_1) + g|(u_0 + u_2 - u_1)| &= 0,\ |\varrho(|\nabla u_1|)\partial_\nu u_1| \leq g \quad \text{on } \Gamma_s.\\
u_2(x) = a + o(1)\ \ (n = 2)\ \text{ or }\ u_2(x) &= \mathcal{O}(|x|^{2-n})\ \ (n > 2).
\end{aligned}
\tag{1}
$$

Here $\varrho \in C^1([0,\infty))$ satisfies

$$
0 \leq \varrho(t), (t\ \varrho(t))' \leq \varrho^*(1 + t^{p-2}) \quad \text{and} \quad \varrho(t) + t\min\{0, \varrho'(t)\} \geq \varrho_* t^{p-2} \quad (\varrho_* > 0).
$$

The interface $\partial\Omega = \overline{\Gamma_s \cup \Gamma_t}$ is divided into open components $\Gamma_s, \Gamma_t \neq \emptyset$. A naive variational formulation of the transmission problem (1) minimizes the functional

$$
\int_\Omega \varrho(|\nabla u_1|)(\nabla u_1)^2 + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle + \int_{\Gamma_s} g|u_2 - u_1 + u_0| \tag{2}
$$

over a suitable convex subset of $W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c)$. To reduce the exterior problem to $\partial\Omega = \partial\Omega^c$, we use the Steklov–Poincaré operator $S : W^{\frac{1}{2},2}(\partial\Omega) \to W^{-\frac{1}{2},2}(\partial\Omega)$ for the Laplacian on Ω^c, which satisfies $\partial_\nu u_2|_{\partial\Omega} = -S(u_2|_{\partial\Omega} - a)$. Hence, with $\lambda(u,v) = \langle t_0 + Su_0, u|_{\partial\Omega} + v\rangle + \int_\Omega fu$ and $\langle\cdot,\cdot\rangle$ denoting the pairing of $W^{\frac{1}{2},2}(\partial\Omega)$ and $W^{-\frac{1}{2},2}(\partial\Omega)$, the variational problem (2) is equivalent to minimizing

$$
J(u,v) = \int_\Omega \varrho(|\nabla u|)(\nabla u)^2 + \frac{1}{2}\langle S(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v\rangle + \int_{\Gamma_s} g|v| - \lambda(u,v) \tag{3}
$$

over a closed convex subset of $X = W^{1,p}(\Omega) \times \{v \in W^{\frac{1}{2},2}(\partial\Omega) : \mathrm{supp}\ v \subset \Gamma_s\}$. This translates into a domain/boundary variational inequality: Find $(\hat u, \hat v) \in X$ such that for all $(u,v) \in X$

$$
\int_\Omega \varrho(|\nabla\hat u|)\nabla\hat u\nabla(u-\hat u) + \langle S(\hat u|_{\partial\Omega} + \hat v), (u-\hat u)|_{\partial\Omega} + v - \hat v\rangle + \int_{\Gamma_s} g(|v| - |\hat v|) \geq \lambda(u-\hat u, v-\hat v). \tag{4}
$$

Theorem 1 ([2])**.** *The variational inequality is equivalent to the transmission problem (1) and has a unique solution.*

Let $X_h = H^1_h \times H^{\frac{1}{2}}_h$, $h \in I$, denote a family of finite dimensional subspaces of X. The Galerkin scheme of the FE–BE coupling for (4) reads: Find $(\hat u_h, \hat v_h) \in X_h$ s.t.

$$
\begin{aligned}
\int_\Omega \varrho(|\nabla\hat u_h|)\nabla\hat u_h\nabla(u_h - \hat u_h) + \langle S_h(\hat u_h|_{\partial\Omega} + \hat v_h), (u_h - \hat u_h)|_{\partial\Omega} + v_h - \hat v_h\rangle\\
+ \int_{\Gamma_s} g(|v_h| - |\hat v_h|) \geq \lambda_h(u_h - \hat u_h, v_h - \hat v_h)
\end{aligned}
\tag{5}
$$

for all $(u_h, v_h) \in X_h$, where S_h is a discrete version of the Steklov–Poincaré operator ([4]) and λ_h is obtained from λ by replacing S by S_h.

Theorem 2 ([2]). *a) There exists an $h_0 > 0$ such that for all $h < h_0$ the Galerkin scheme has a unique solution $(\hat{u}_h, \hat{v}_h) \in X_h$.*
b) For the solutions $(\hat{u}, \hat{v}) \in X$ of (4) and $(\hat{u}_h, \hat{v}_h) \in X_h$ of (5) there holds

$$\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|_X^p \lesssim \inf_{(u_h, v_h) \in X_h} \left\{ \|\hat{u} - u_h, \hat{v} - v_h\|_X^2 + \|\hat{v} - v_h\|_{L^2(\Gamma_s)} \right\} + \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)(\hat{u} + \hat{v} - u_0), W^{-\frac{1}{2},2}(\partial\Omega))^2.$$

The last term on the RHS in (6) results from the difference $S - S_h$ ([5]). Here the operators V and K denote the single, resp. double, layer potential of the Laplacian.

Remark 1 ([2]). *The above procedure carries over to transmission problems in nonlinear elasticity with a Hencky material in Ω and the Lamé equation in Ω^c.*

Next, we consider an FE–BE coupling for transmission problems with microstructure and Signiorini contact, thus generalizing the pure FEM method for boundary value problems in [1]. Our starting point in [3] is the relaxed energy functional

$$\Phi^{**}(u_1, u_2) = \int_\Omega W^{**}(\nabla u_1) + \frac{1}{2} \int_{\Omega^c} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega} \rangle, \tag{6}$$

where W^{**} is the convex envelope of the double–well potential $W(F) = |F - F_1|^2 \, |F - F_2|^2$ for $F_1 \neq F_2 \in \mathbb{R}^n$. Given $f \in L^{4/3}(\Omega)$, $t_0 \in W^{-\frac{1}{2},2}(\partial\Omega)$, $u_0 \in W^{\frac{1}{2},2}(\partial\Omega)$, we consider Φ^{**} over the convex set

$$\{(u_1, u_2) \in W^{1,4}(\Omega) \times W^{1,2}_{loc}(\Omega^c) : (u_1 - u_2)|_{\Gamma_t} = u_0, \, (u_1 - u_2)|_{\Gamma_s} \leq u_0, \\ \Delta u_2 = 0 \text{ in } W^{-1,2}(\Omega^c) + \text{radiation condition}\}.$$

The minimization problem for Φ^{**} corresponds to the variational inequality: Find $(\hat{u}, \hat{v}) \in \mathcal{A} = \{(u, v) \in W^{1,4}(\Omega) \times W^{\frac{1}{2},2}(\partial\Omega) : v|_{\Gamma_s} \geq 0, \, \langle S(u|_{\partial\Omega} + v - u_0), 1 \rangle = 0 \text{ if } n = 2\}$ such that

$$\int_\Omega DW^{**}(\nabla \hat{u}) \nabla(u - \hat{u}) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v} \rangle \geq \lambda(u - \hat{u}, v - \hat{v}) \tag{7}$$

for all $(u, v) \in \mathcal{A}$. In [3], we show that the stress $DW^{**}(\hat{u})$, a certain projection $\mathbb{P}\nabla\hat{u}$ of the gradient, the region of microstructure and the boundary value $u|_{\partial\Omega} + v$ are independent of the minimizer. Discretizing as above, we obtain:

Theorem 3 ([3]). *Let $(\hat{u}, \hat{v}) \in \mathcal{A}$ solve (7) and $(\hat{u}_h, \hat{v}_h) \in \mathcal{A}_h$ its Galerkin approximation. Then*

$$\|DW^{**}(\hat{u}) - DW^{**}(\hat{u}_h)\|^2_{L^{4/3}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\ \lesssim \inf_{(u_h, v_h) \in \mathcal{A}_h} \left\{ \|\hat{u} - u_h\|_{W^{1,4}(\Omega)} + \|(\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h)\|_{W^{\frac{1}{2},2}(\partial\Omega)} \right\}.$$

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Nonconforming Maxwell Eigensolvers

Li-yeng Sung
(joint work with Susanne C. Brenner and Fengyan Li)

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider the Maxwell eigenproblem of finding $\lambda \in \mathbb{R}$ such that there exists a nontrivial vector field $\boldsymbol{u}$ satisfying

$$\nabla \times (\nabla \times \boldsymbol{u}) = \lambda \boldsymbol{u} \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$\boldsymbol{n} \times \boldsymbol{u} = 0 \qquad \text{on } \partial\Omega, \tag{1c}$$

where $\boldsymbol{n}$ is the outer unit normal. It is known that the eigenvalue λ is nonnegative and that $\lambda > 0$ for a simply connected Ω.

A commonly used variational formulation for (1) is to find $\lambda \in \mathbb{R}$ such that there exists a nontrivial $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega)$ satisfying

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) = \lambda(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega), \tag{2}$$

where $(\cdot,\cdot)$ is the L_2 inner product. The rationale behind this approach is that $H(\mathrm{curl};\Omega)$ conforming subspaces can be constructed by using edge elements.

Note that for $\lambda > 0$ the divergence-free condition (1b) can be recovered from (2) by taking $\boldsymbol{v} = \nabla\varphi$ for an arbitrary $\varphi \in H_0^1(\Omega)$. Therefore the positive eigenvalues defined by (1) and (2) agree. However, 0 is an eigenvalue of (2) whose eigenspace contains the infinite dimensional space$\nabla H_0^1(\Omega)$. Hence 0 is a spurious eigenvalue with infinite multiplicity. Furthermore, the space $H_0(\mathrm{curl};\Omega)$ is not a compact subspace of $[L_2(\Omega)]^2$. Consequently the analysis of Maxwell eigensolvers based on (2) is quite delicate [9, 2, 8].

Another variational formulation for (1) is to find $\lambda \in \mathbb{R}$ such that there exists a nontrivial $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ satisfying

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) = \lambda(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega). \tag{3}$$

The advantage of this formulation is that there are no spurious eigenvalues and that $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ is a compact subspace of $[L_2(\Omega)]^2$. On the other hand, $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ conforming finite element spaces are also $H^1(\Omega)$ conforming, and such finite elements are not suitable for electromagnetic problems on nonconvex domains [10].

Recently, we discovered [3, 4, 5] that by combining the classical nonconforming P_1 element [11] with techniques from discontinuous Galerkin methods, it is possible

to construct convergent nonconforming finite element methods for the the following source problem: Find $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ such that

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega), \tag{4}$$

where $\boldsymbol{f} \in [L_2(\Omega)]^2$.

Let $T : [L_2(\Omega)]^2 \longrightarrow H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega) \subset\subset [L_2(\Omega)]^2$ be the solution operator of (4), and $T_h : [L_2(\Omega)]^2 \longrightarrow V_h$ be the solution operator of the nonconforming finite element method. We have obtained the L_2 error estimate

$$\|T - T_h\|_{[L_2(\Omega),L_2(\Omega)]} \leq C_\epsilon h^{2-\epsilon} \tag{5}$$

for these nonconforming methods [3, 4, 5], where $\epsilon > 0$ can be arbitrarily small.

The uniform estimate (5) implies that these nonconforming methods for the source problem can also be applied to the eigenproblem (3), and the performance of these eigensolvers can be analyzed in a straight-forward fashion by the classical spectral approximation theory [1]. The following theorem is established in [6].

Theorem Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the Maxwell eigenvalues, $\lambda = \lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+m-1}$ be a Maxwell eigenvalue with multiplicity m, and $Z_\lambda \subset H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ be the corresponding m dimensional eigenspace. Let $0 \leq \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots$ be the eigenvalues obtained by one of the nonconforming eigensolvers. Then as $h \downarrow 0$, we have

$$|\lambda_{h,l} - \lambda| \leq C_{\lambda,d_\lambda,\epsilon} h^{2-\epsilon} \qquad l = j, j+1, \cdots, j+m-1,$$

where d_λ is the distance from λ to the other Maxwell eigenvalues and ϵ is an arbitrary positive number. Furthermore, if $Z_{h,\lambda}$ is the space spanned by the discrete Maxwell eigenfields corresponding to $\lambda_{h,j}, \cdots, \lambda_{h,j+m-1}$, then the gap between Z_λ and $Z_{h,\lambda}$ goes to zero at the rate of $C_{\lambda,d_\lambda,\epsilon} h^{2-\epsilon}$ in the L_2 norm and at the rate of $C_{\lambda,d_\lambda,\epsilon} h^{1-\epsilon}$ in the energy norm.

Numerical examples illustrating the performance of the nonconforming eigensolvers can be found in [6], and generalization to higher order elements is discussed in [7].

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Plane wave discretizations of the Helmholtz equation with Lagrange multipliers, and a domain decomposition method for resulting systems

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(joint work with Charbel Farhat, Radek Tezaur)

The oscillatory behavior of the solution of the Helmholtz equation necessitates a fine discretization. Methods employing plane waves in construction of basis functions usually require fewer unknowns to reach a given accuracy than standard finite element methods. The discontinuous enrichment method (DEM) [3, 4], is a hybrid discontinuous multiscale technique that enriches a continuous polynomial solution with homogeneous solutions of the PDE, in this case with planar waves. The inter-element continuity is enforced using Lagrange multipliers. We can and will drop the polynomial part of the approximation, thereby transforming DEM into a non-standard discontinuous Galerkin method (DGM).

We consider the following truncated exterior Helmholtz problem in a domain Ω in $\mathbb{R}^n$, $n = 2, 3$: Find $u \in H^1(\Omega)$ such that

$$(1) \quad -\Delta u - k^2 u = 0 \text{ in } \Omega, \quad \frac{\partial}{\partial \boldsymbol{\nu}} u = -\frac{\partial}{\partial \boldsymbol{\nu}} g \text{ on } \Sigma_1, \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = M(u) \text{ on } \Sigma_2,$$

where k is the wavenumber, Σ_1 is the boundary of a sound-hard scatterer, and Σ_2 is the far-field boundary. The operator M defines an absorbing boundary condition and $\boldsymbol{\nu}$ denotes the unit outward normal. The function g gives the incident field.

We enforce the inter-element continuity in a weak form using Lagrange multipliers adopting the DEM formulation in [1]. Let Ω_e be a nonoverlapping element partition of Ω. A space $\mathcal{V}$ for the primal variable u and a space $\mathcal{W}$ for Lagrange multiplier λ are

$$\mathcal{V} = \left\{ v \in L^2(\widetilde{\Omega}) \ : \ v|_{\Omega_e} \in H^1(\Omega_e) \right\} \qquad \text{and} \qquad \mathcal{W} = \prod_{e=1}^{n_e} \prod_{e'=1}^{e-1} H^{-1/2}(\Gamma_{e,e'}),$$

where $\widetilde{\Omega} = \bigcup_{e=1}^{n_e} \Omega_e$ and $\Gamma_{e,e'} = \partial\Omega_e \cap \partial\Omega_{e'}$.

The hybrid variational formulation of DGM reads: Find $(u, \lambda) \in \mathcal{V} \times \mathcal{W}$ such that

$$\begin{aligned} a(u, v) + b(\lambda, v) &= r(v) \qquad \forall v \in \mathcal{V} \\ b(\mu, u) \qquad\quad &= 0 \qquad \forall \mu \in \mathcal{W}, \end{aligned}$$

where the bilinear and linear forms are defined as

$$a(u, v) = \int_{\widetilde{\Omega}} (\nabla u \cdot \nabla v - k^2 uv) d\Omega - \int_{\Sigma_2} M(u) v d\Gamma,$$

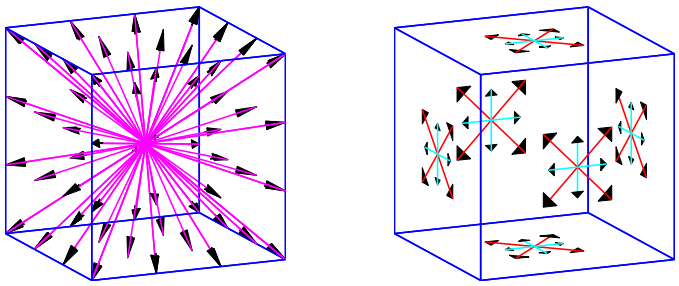

FIGURE 1. The left-hand side vectors give the directions $\boldsymbol{\theta}_p$ for the H-56-8 elements and the right-hand side describe the coefficients c_p and the directions $\boldsymbol{\varphi}_p$ for it.

$$b(\lambda, v) = \sum_{e=1}^{n_e} \sum_{e'=1}^{e-1} \int_{\Gamma_{e,e'}} \lambda \left(v|_{\Omega_{e'}} - v|_{\Omega_e} \right) d\Gamma, \qquad \text{and} \qquad r(v) = -\int_{\Sigma_1} \frac{\partial g}{\partial \boldsymbol{\nu}}\, v\, d\Gamma.$$

Our discrete approximation of the space $\mathcal{V}$ is

$$\mathcal{V}_{n_\theta} = \left\{ u \in \mathcal{V} \ : \ u(\mathbf{x}) = \sum_{p=1}^{n_\theta} e^{ik\boldsymbol{\theta}_p^T \mathbf{x}} u_{e,p}, \ \ \mathbf{x} \in \Omega_e, \ \ u_{e,p} \in \mathbb{C} \right\},$$

where $\boldsymbol{\theta}_p$ are the unit wave propagation directions. The discretization of the Lagrange multiplier space $\mathcal{W}$ should approximate the normal derivatives of $\mathcal{V}_{n_\theta}$ on the element interfaces, but it cannot be too rich in order to satisfy the *inf-sup* condition. Our discretization of $\mathcal{W}$ is

$$\mathcal{W}_{n_\lambda} = \left\{ \lambda \in \mathcal{W} \ : \ \lambda(\mathbf{x}) = \sum_{p=1}^{n_\lambda} e^{ikc_p \boldsymbol{\varphi}_p^T \boldsymbol{\tau}_{e,e'}(\mathbf{x})} \lambda_{e,e',p}, \ \ \mathbf{x} \in \Gamma_{e,e'}, \ \ \lambda_{e,e',p} \in \mathbb{C} \right\},$$

where c_p are given coefficients, $\boldsymbol{\varphi}_p^T$ are unitary directions in $\mathbb{R}^{n-1}$, and $\boldsymbol{\tau}_{e,e'}(\mathbf{x})$ is $\mathbf{x}$ projected on local (orthogonal) coordinate(s) on the edge/face $\Gamma_{e,e'}$. The richness of the $\mathcal{W}_{n_\lambda}$ space is controlled by the number of waves n_λ on each edge/face. For the two-dimensional and three-dimensional problems $\mathcal{V}_{n_\theta}$ and $\mathcal{W}_{n_\lambda}$ pairs have been considered in [3] and [4], respectively. The H-56-8 hexahedral element [4] with $n_\theta = 56$ and $n_\lambda = 8$ is shown in Figure 1.

With the above discretization, direct solvers have been employed exclusively to solve the resulting systems like in [3, 4]. We propose a domain decomposition method for the iterative solution of these problems. For a description of the method and numerical results, see [2].

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Accelerating the Discontinuous Galerkin Time-Domain Method

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(joint work with Thomas Hagstrom, Lucas C. Wilcox)

There is a growing literature of local time stepping methods for discontinuous Galerkin (DG) methods that allow for variable rates of time stepping throughout the computational domain (see for instance [1, 2, 3, 4, 5, 6]). We consider here an alternative and simpler class of multi-rate Adams-Bashforth time stepping methods [7] for use with DG spatial discretizations of Maxwell's equations. This special combination of time and space discretizations allows for flexible domain discretization in space with full non-confornimg h and p adaptivity in space and more importantly local time steps can be chosen uniquely in each element.

We assume the homogeneous material, non-dimensionalized Maxwell's equations are being solved in a domain Ω with a cover $\Omega = \bigcup_{k=1}^{k=K} D^k$ of K elements that are typically but not limited to simplices, tensor-product elements or even general polygons and polyhedra.

The semi-discrete DG variational equations for Maxwell's equations demand that we find $\mathbf{Q} = \begin{pmatrix} \mathbf{H} \\ \mathbf{E} \end{pmatrix} \in X^h \times Y^h$ such that in the k'th element:

$$\begin{aligned} \left(\varphi, \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E}\right)_{D^k} &= \left(\varphi, -\mathbf{n} \times \left(\mathbf{E}^* - \mathbf{E}^-\right)\right)_{\partial D^k}, \\ \left(\psi, \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H}\right)_{D^k} &= \left(\psi, +\mathbf{n} \times \left(\mathbf{H}^* - \mathbf{H}^-\right)\right)_{\partial D^k}. \end{aligned}$$

for all $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X^h \times Y^h$. We have allowed the variational spaces for the solution fields to be quite general, but in practice we use product spaces of broken polynomial spaces on each element:

$$X^h = Y^h = \bigoplus_{k=1}^{k=K} \left(P^N\left(D^k\right)\right)^3.$$

Furthermore, we did not specify the extension states $\mathbf{H}^*, \mathbf{E}^*$ for the distributional derivatives. These are typically set to be an average or upwind combination of the '-' boundary trace of the solution from within the k'th element and the '+' boundary trace of the solution from the elements sharing parts of the boundary

of the k'th element, ∂D^k. See [8] for further details for determining the extension states and general time stepping approaches.

After choosing a basis for the test and trial space we obtain semidiscrete equations for the unknown solution $\mathbf{Q}_N = \begin{pmatrix} \mathbf{H}_N \\ \mathbf{E}_N \end{pmatrix}$, given by $\frac{d\mathbf{Q}_N}{dt} = \mathcal{L}\mathbf{Q}_N$, where $\mathcal{L}$ corresponds to a matrix representing the spatial derivatives discretized with the discontinuous Galerkin distributional derivatives.

If all elements were to advance with the same dt we might choose an Adam-Bashforth (AB) time integrator then the fully discrete equations are

$$\mathbf{Q}_N^{n+1} = \mathbf{Q}_N^n + dt\left(a_0\mathcal{L}\mathbf{Q}_N^n + a_1\mathcal{L}\mathbf{Q}_N^{n-1} + a_2\mathcal{L}\mathbf{Q}_N^{n-2}\right),$$

with coefficients for the first to third order AB schemes given in Table 1.

TABLE 1. Coefficients for Adams-Bashforth time integrator: **a** and for the half Adams-Bashforth time integrator: **b**

Order	a_0	a_1	a_2	b_0	b_1	b_2
1	1	0	0	1	0	0
2	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{5}{4}$	$\frac{-1}{4}$	0
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	$\frac{17}{12}$	$\frac{-7}{12}$	$\frac{2}{12}$

Invoking basic properties of the polynomial bases we can estimate that there is an upper limit for the time step given by $dt < C\frac{h}{N^2}$ where the constant depends on the order of the AB scheme. Here h is the minimum element size of *all* the elements in the mesh covering the domain. Unfortunately this is a global restriction and dt may be made small by a tiny minority of elements with high aspect ratio or small size driven by small scale domain geometry features.

To remedy the global time step restriction we consider using a multirate extension to the basic AB schemes. We consider a partition of the domain into "coarse" and "fine" parts denoted by Ω_C and Ω_F respectively. Denoting the solution restricted to the fine and coarse domains as $\mathbf{Q}_F$ and $\mathbf{Q}_C$ respectively and reordering the degrees of freedom we write

$$\frac{d}{dt}\begin{pmatrix} \mathbf{Q}_F \\ \mathbf{Q}_C \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{FF} & \mathcal{L}_{FC} \\ \mathcal{L}_{CF} & \mathcal{L}_{CC} \end{pmatrix}\begin{pmatrix} \mathbf{Q}_F \\ \mathbf{Q}_C \end{pmatrix}$$

For simplicity we assume here that the largest stable time step for the coarse mesh, dt, is twice as large as the largest stable time step for the fine mesh, dt_F. We now describe the multirate AB scheme in stages. We first time march the fine mesh by $dt/2$ with

$$\mathbf{Q}_F^{n+1/2} = \mathbf{Q}_F^n + \frac{dt}{2}\begin{pmatrix} \mathcal{L}_{FF}\left(a_0\mathbf{Q}_F^n + a_1\mathbf{Q}_F^{n-1/2} + a_2\mathbf{Q}_F^{n-1}\right) \\ +\mathcal{L}_{FC}\left(a_0\mathbf{Q}_C^n + a_1\mathbf{Q}_C^{n-1/2} + a_2\mathbf{Q}_C^{n-1}\right) \end{pmatrix}.$$

Note that the half time step values for $\mathbf{Q}_C$ are only actually required at the positive trace of the boundary of the fine mesh where shared with the coarse mesh by the locality of the flux terms in the DG formulation. We can evaluate this coarse boundary data by performing half AB time steps

$$\mathbf{Q}_C^{n+1/2} = \mathbf{Q}_C^n + \frac{dt}{2}\left(\begin{array}{c} \mathcal{L}_{CC}\left(b_0\mathbf{Q}_C^n + b_1\mathbf{Q}_C^{n-1} + b_2\mathbf{Q}_C^{n-2}\right) \\ +\mathcal{L}_{CF}\left(b_0\mathbf{Q}_F^n + b_1\mathbf{Q}_F^{n-1} + b_2\mathbf{Q}_F^{n-2}\right) \end{array} \right),$$

with b coefficients given in Table 1. We stress that it is only necessary to evaluate this at the coarse-fine interface. With this data in hand we can update the fine mesh one more half step

$$\mathbf{Q}_F^{n+1} = \mathbf{Q}_F^{n+1/2} + \frac{dt}{2}\left(\begin{array}{c} \mathcal{L}_{FF}\left(a_0\mathbf{Q}_F^{n+1/2} + a_1\mathbf{Q}_F^n + a_2\mathbf{Q}_F^{n-1/2}\right) \\ +\mathcal{L}_{FC}\left(a_0\mathbf{Q}_C^{n+1/2} + a_1\mathbf{Q}_C^n + a_2\mathbf{Q}_C^{n-1/2}\right) \end{array} \right).$$

Finally we can update the coarse mesh data with available data

$$\mathbf{Q}_C^{n+1} = \mathbf{Q}_C^n + dt\left(\begin{array}{c} \mathcal{L}_{CC}\left(a_0\mathbf{Q}_C^n + a_1\mathbf{Q}_C^{n-1} + a_2\mathbf{Q}_C^{n-2}\right) \\ +\mathcal{L}_{CF}\left(a_0\mathbf{Q}_F^n + a_1\mathbf{Q}_F^{n-1} + a_2\mathbf{Q}_F^{n-2}\right) \end{array} \right).$$

This completes a full time step for both coarse and fine meshes. In practical computations a limited amount of additional book keeping allows for more general ratios of time steps between sub-domains. Each linear combination on the right hand sides of these update equations represents an integration of a temporal interpolation of the solution and as such this notation also suggests alternative means of evaluating the interpolants with different coefficients and temporal samples, to be presented in later publications with numerical and theoretical support.

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Equilibrated a posteriori error estimators for variational inequalities

Barbara Wohlmuth

Recently, locally defined a posteriori error estimators based on equilibrated fluxes are getting more and more popular. Here we present an error estimator for a Signorini type problem. The error estimator is defined in terms of $H(\mathrm{div})$-conforming stress approximations and equilibrated fluxes. To handle the variational inequality, we introduce a mixed formulation where the discrete Lagrange multiplier on the contact boundary approximates the contact stress. No additional terms due to the variational inequality enter. A residual based error estimator for obstacle problems can be found in [3]. In the case of a one-sided contact problem or matching meshes, the error is bounded by our estimator with a constant one plus a higher order data oscillation term plus a term arising from the contact that is shown numerically to be of higher order. Several numerical tests demonstrate the performance of the estimators and confirm the reliability and efficiency.

It is well known that a Signorini problem can be rewritten as a constrained minimization problem or equivalently as a variational inequality on a convex set. Although our discretization is based on a pure displacement approach, the analysis and the definition of the error estimator use a mixed formulation. Introducing the surface traction on the contact boundary Γ_C as additional unknown, the variational inequality can be rewritten as a saddle point problem: find the displacement $\mathbf{u}_h \in \mathbf{V}_{h;D}$ and the contact stress $\boldsymbol{\lambda}_h \in \mathbf{M}_h^+$ such that

$$\begin{aligned} \int_\Omega \mathcal{C}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h)\, dx + \int_{\Gamma_C} \mathbf{v}_h \boldsymbol{\lambda}_h\, ds &= f(\mathbf{v}_h)\,, & \mathbf{v}_h &\in \mathbf{V}_h\,, \\ \int_{\Gamma_C} \mathbf{u}_h(\boldsymbol{\mu}_h - \boldsymbol{\lambda}_h)\, ds &\le 0\,, & \boldsymbol{\mu}_h &\in \mathbf{M}_h^+\,, \end{aligned}$$

where $(\mathbf{V}_{h;D}, \mathbf{M}_h)$ form a mortar stable pair of low order conforming finite elements on a simplicial triangulation in two dimensions. Here the linearized strain tensor is defined by $\varepsilon(\mathbf{v}) = 1/2\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right)$ and the stress tensor is given in terms of Hooke's tensor $\mathcal{C}$ by $\sigma(\mathbf{v}) = \mathcal{C}\varepsilon(\mathbf{v})$. The space $\mathbf{V}_h$ is associated with zero boundary values on Γ_D, and $\mathbf{V}_{h;D}$ includes Dirichlet boundary conditions on Γ_D. The convex set $\mathbf{M}_h^+$ is defined by $\mathbf{M}_h^+ = \{\boldsymbol{\mu}_h \in \mathbf{M}_h \,:\, \boldsymbol{\mu}_h = \sum_i \alpha_i \psi_i \mathbf{n}, \alpha_i \ge 0\}$, where $\{\psi_i\}_i$ forms a biorthogonal set of basis functions with local support, and $\mathbf{n}$ is the outer unit normal.

As it is quite standard, the global error estimator for the energy norm of the error in the displacement is given in terms of element-wise contributions,

$$\eta^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2, \quad \eta_T^2 := \|\mathcal{C}^{-1/2}(\sigma_h - \sigma(\mathbf{u}_h))\|_{0;T}^2\,,$$

where σ_h is a $H(\mathrm{div})$-conforming stress approximation in terms of Arnold–Winther finite elements [2]. These elements can be defined in terms of vertex, edge and element degrees of freedom. As value of the edge degrees of freedom, we use equilibrated fluxes. These fluxes can be computed locally by solving vertex based low dimensional systems, see [1, 4]. It can be shown that η provides up to higher order terms an upper bound and local lower bounds for the error. We refer to [5, 6] for details including the role of $\boldsymbol{\lambda}_h$.

To get contact problems with low regularity, we push a squared shaped elastic body onto a triangular shaped rigid obstacle. Here, the regularity of the solution depends on the angle α of the triangle. In our tests, we use $\alpha = \pi/6$, $\alpha = \pi/4$ and $\alpha = \pi/3$. The material parameters of the square are given by $E = 200, \nu = 0.3$. Due to the decreasing regularity for increasing α the adaptive meshes differ for the different values of α, see Figure 1. We observe that for $\alpha = \pi/3$, the mesh is much more locally refined compared to $\alpha = \pi/6$.

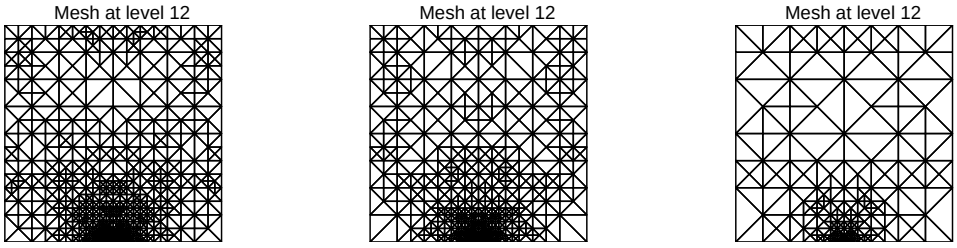

FIGURE 1. Adaptively refined mesh for different angles $\alpha = \pi/6, \pi/4, \pi/3$.

We observe numerically a convergence rate of approximatively $\mathcal{O}(h^{0.7})$ for $\alpha = \pi/6$, $\mathcal{O}(h^{0.5})$ for $\alpha = \pi/4$ and $\mathcal{O}(h^{0.3})$ for $\alpha = \pi/3$, see Figure 2. However, using adaptive mesh refinement an optimal convergence rate can be recovered in all cases.

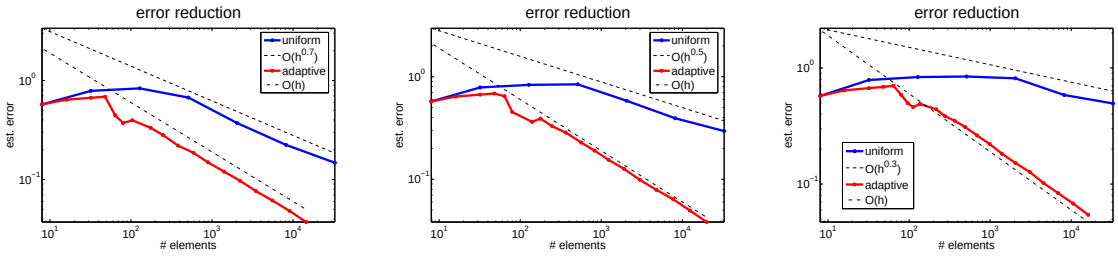

FIGURE 2. Estimated error decay for adaptive and uniform refinement.

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