ERROR REDUCTION IN ADAPTIVE FINITE ELEMENT APPROXIMATIONS OF ELLIPTIC OBSTACLE PROBLEMS

DIETRICH BRAESS *, CARSTEN CARSTENSEN †, AND RONALD H.W. HOPPE ‡

Abstract. We consider an adaptive finite element method (AFEM) for obstacle problems associated with linear second order elliptic boundary value problems and prove a reduction in the energy norm of the discretization error which leads to R-linear convergence. This result is shown to hold up to a consistency error due to the extension of the discrete multipliers (point functionals) to H^{-1} and a possible mismatch between the continuous and discrete coincidence and noncoincidence sets. The AFEM is based on a residual-type error estimator consisting of element and edge residuals. The a posteriori error analysis reveals that the significant difference to the unconstrained case lies in the fact that these residuals only have to be taken into account within the discrete noncoincidence set. The proof of the error reduction property uses the reliability and the discrete local efficiency of the estimator as well as a perturbed Galerkin orthogonality. Numerical results are given illustrating the performance of the AFEM.

AMS subject classifications. 65N30, 65N50

Key words. Adaptive finite element methods, convergence analysis, error reduction, elliptic obstacle problems

1. Introduction. Adaptive finite element methods (AFEMs) for partial differential equations based on residual- or hierarchical-type estimators, local averaging techniques, the goal-oriented dual weighted approach, or the theory of functional-type error majorants have been intensively studied during the past decades (see, e.g., the monographs [1, 3, 4, 16, 25, 33] and the references therein). As far as elliptic obstacle problems are concerned, we refer to [2, 5, 7, 8, 14, 19, 23, 26, 27, 31].

More recently, substantial efforts have been devoted to a rigorous convergence analysis of AFEMs, initiated in [15] for standard conforming finite element approximations of linear elliptic boundary value problems and further investigated in [24]. Using techniques from approximation theory, under mild regularity assumptions optimal order of convergence has been established in [6, 29]. Nonstandard finite element methods such as mixed methods, nonconforming elements and edge elements have been addressed in [11, 12, 13]. A nonlinear elliptic boundary value problem, namely for the p-Laplacian, has been treated in [32]. The basic ingredients of the convergence proofs are the reliability of the estimator, its discrete local efficiency, and a bulk criterion realizing an appropriate selection of edges and elements for refinement.

For elliptic obstacle problems, the issue of error reduction in the energy functional associated with the formulation of the obstacle problem as a constrained convex minimization problem has been studied in [9] and [28]. The approach in [28] relies on techniques from nonlinear optimization, whereas the convergence analysis in [9] is restricted to the case of affine obstacles.

In this paper, we focus on the error reduction property with respect to the energy norm for general obstacles. The error estimator is of residual type and consists of element and edge residuals. The a posteriori error analysis reveals that in contrast to the unconstrained case the

*Faculty of Mathematics, Ruhr University Bochum, D-44780 Bochum, Germany. *dietrich.braess@ruhr-uni-bochum.de*

†Institute of Mathematics, Humboldt University of Berlin, D-10099 Berlin, Germany. *cc@math.hu-berlin.de* The work has been supported by the German Research Association (DFG) within the DFG Research Center MATHEON "Mathematics for Key Technologies" Project C13

‡Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA, *rohop@math.uh.edu*, and Institute of Mathematics, University at Augsburg, D-86159 Augsburg, Germany. *hoppe@math.uni-augsburg.de* The work has been supported by the NSF under Grant No. DMS-0511611 and Grant No. DMS-0707602.

local residuals only have to be taken into account for elements and edges within the discrete noncoincidence set.

The paper is organized as follows: In section 2, we introduce the elliptic obstacle problem as a variational inequality involving a closed, convex subset $K \subset H_0^1(\Omega)$ and address its unconstrained formulation in terms of a Lagrange multiplier in $H^{-1}(\Omega)$. We further consider a finite element approximation by means of P1 conforming finite elements with respect to a simplicial triangulation of the computational domain. The unconstrained formulation of the discrete approximation gives rise to discrete multipliers which are Radon measures, namely a linear combination of point functionals associated with nodal points within the discrete coincidence set. The evaluation of the discrete multipliers for the nodal basis functions of the underlying finite element space and the specification of a consistency error due to the extension of the discrete multipliers to $H^{-1}(\Omega)$ and the mismatch between the continuous and discrete coincidence and noncoincidence sets are the essential keys for the subsequent a posteriori error analysis. In section 3, we present the error estimator, data oscillations, a bulk criterion taking care of the selection of elements and edges for refinement, and the refinement strategy. Furthermore, the main convergence result is stated in terms of a reduction of the discretization error in the energy norm up to the consistency error. The subsequent section 4 is devoted to the proof of the error reduction property which uses the reliability of the estimator, its discrete local efficiency, and a perturbed Galerkin orthogonality as basic tools. Finally, section 6 contains a detailed documentation of numerical results for some selected test examples displaying the convergence history of the AFEM and thus illustrating its numerical performance.

2. The obstacle problem and its finite element approximation. We assume $\Omega \subset \mathbb{R}^2$ to be a bounded, polygonal domain with boundary $\Gamma := \partial\Omega$. We use standard notation from Lebesgue and Sobolev space theory, refer to $H^k(\Omega)$, $k \in \mathbb{N}$, as the Sobolev spaces based on $L^2(\Omega)$, and denote their norms as $\|\cdot\|_{k,\Omega}$. We refer to $(\cdot,\cdot)_{0,\Omega}$ as the inner product of the Hilbert space $L^2(\Omega)$. For $k = 1$, $|\cdot|_{1,\Omega}$ stands for the associated seminorm on $H^1(\Omega)$ which actually is a norm on $V := H_0^1(\Omega) := \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}$. We refer to $V^* := H^{-1}(\Omega)$ as the dual of V and to $\langle\cdot,\cdot\rangle$ as the associated dual pairing. Likewise, $\langle\cdot,\cdot\rangle_\Gamma$ stands for the dual pairing between the trace space $H^{1/2}(\Gamma)$ and its dual. We denote by $V_+ := \{v \in V \mid v \geq 0 \text{ a.e. on } \Omega\}$ the positive cone of V and by V_+^* the positive cone of V^*, i.e., $\sigma \in V_+^*$ iff $\langle\sigma, v\rangle \geq 0$ for all $v \in V_+$.

We further refer to $C(\overline{\Omega})$ as the Banach space of continuous functions on $\overline{\Omega}$. Its dual $\mathcal{M}(\Omega) = C(\overline{\Omega})^*$ is the space of Radon measures on Ω with $\langle\langle\cdot,\cdot\rangle\rangle$ standing for the associated dual pairing. We refer to $C_+(\overline{\Omega})$ and $\mathcal{M}_+(\Omega)$ as the positive cones of $C(\overline{\Omega})$ and $\mathcal{M}(\Omega)$. In particular, $\sigma \in \mathcal{M}_+(\Omega)$ iff $\langle\langle\sigma, v\rangle\rangle \geq 0$ for all $v \in C_+(\overline{\Omega})$.

For given $f \in L^2(\Omega)$ and $\psi \in H^1(\Omega)$ with $\psi|_\Gamma \geq 0$, we consider the obstacle problem

$$\inf_{v \in K} J(v)\ , \quad J(v) := \frac{1}{2} a(v,v) - (f,v)_{0,\Omega}, \tag{2.1}$$

where K stands for the closed, convex set

$$K := \{v \in V \mid v \leq \psi \text{ a.e. on } \Omega\}.$$

and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bilinear form

$$a(v,w) := \int_\Omega \nabla v \cdot \nabla w \, dx\ ,\ v, w \in V.$$

It is well-known [21] that (2.1) admits a unique solution and that the necessary and sufficient optimality conditions are given by the variational inequality

$$a(u, v-u) \geq (f, v-u)_{0,\Omega} \ , \ \ v \in K. \tag{2.2}$$

We define the coincidence set (active set) $\mathcal{A}$ as the maximal open set in Ω such that $u(x) = \psi(x)$ f.a.a. $x \in \mathcal{A}$ and the noncoincidence set (inactive set) $\mathcal{I}$ according to $\mathcal{I} := \bigcup_{\varepsilon > 0} B_\varepsilon$, where B_ε is the maximal open set in Ω such that $u(x) \leq \psi(x) - \varepsilon$ for almost all $x \in B_\varepsilon$.

Introducing a Lagrange multiplier $\sigma \in V^*$ for the constraints, (2.2) can be written in unconstrained form as follows

$$a(u, v) = (f, v)_{0,\Omega} - \langle \sigma, v \rangle \ , \ v \in V, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ stands for the dual pairing of V^* and V. We note that $\sigma \in V_+^*$. Moreover, the following complementarity condition is satisfied

$$\langle \sigma, u - \psi \rangle = 0. \tag{2.4}$$

We assume $\{\mathcal{T}_\ell\}_{\ell \in \mathbb{N}_0}$ to be a shape regular family of simplicial triangulations of the computational domain Ω. Given $D \subseteq \overline{\Omega}$, we refer to $\mathcal{N}_\ell(D)$ and $\mathcal{E}_\ell(D)$ as the sets of vertices and edges of $\mathcal{T}_\ell$ in D, and we simply write $\mathcal{N}_\ell$ and $\mathcal{E}_\ell$, if $D = \overline{\Omega}$. For $D \subseteq \overline{\Omega}$ and $E \in \mathcal{E}_\ell$ we denote by $|D|$ and $|E|$ the area of D and length of E, and we refer to f_D as the integral mean of f with respect to D, i.e., $f_D := |D|^{-1} \int_D f dx$. Moreover, for $T \in \mathcal{T}_\ell(\Omega)$ and $E \in \mathcal{E}_\ell(T)$, we denote by ν_E the exterior unit normal on E. For $p \in \mathcal{N}_\ell$, $E \in \mathcal{E}_\ell$, and $T \in \mathcal{T}_\ell$ we refer to

$$\begin{aligned}
\omega_\ell^p &:= \bigcup \{T \in \mathcal{T}_\ell \mid p \in \mathcal{N}_\ell(T)\}, \\
\omega_\ell^E &:= \bigcup \{T \in \mathcal{T}_\ell \mid E \in \mathcal{E}_\ell(T)\}, \\
\omega_\ell^T &:= \bigcup \{T' \in \mathcal{T}_\ell \mid \mathcal{N}_\ell(T') \cap \mathcal{N}_\ell(T) \neq \emptyset\}
\end{aligned}$$

as the patches of elements associated with p, E and T, respectively. Further,

$$\mathcal{E}_\ell^p := \bigcup \{E \in \mathcal{E}_\ell \mid p \in \mathcal{N}_\ell(E)\}$$

is the set of edges sharing p as a common vertex.

We denote by S_ℓ the finite element space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_\ell$ and set

$$V_\ell := S_\ell \cap V.$$

We further define $\psi_\ell \in S_\ell$ as some approximation of $\psi \in H^1(\Omega)$. For instance, if $\psi \in C(\bar{\Omega})$, we may choose $\psi_\ell \in S_\ell$ as the nodal interpoland of ψ (cf. [17]).

The finite element approximation of (2.1) amounts to the solution of the finite dimensional constrained minimization problem

$$\min_{v_\ell \in K_\ell} J(v_\ell) \ , \quad J(v_\ell) := \frac{1}{2} a(v_\ell, v_\ell) - (f, v_\ell)_{0,\Omega} \ . \tag{2.5}$$

Here, the constrained discrete set K_ℓ is given by

$$K_\ell := \{v_\ell \in V_\ell \mid v_\ell(x) \leq \psi_\ell(x) \ , \ x \in \Omega\} \ .$$

Again, the optimality conditions give rise to the variational inequality

$$a(u_\ell, v_\ell - u_\ell) \geq (f, v_\ell - u_\ell)_{0,\Omega}\ , \quad v_\ell \in K_\ell. \tag{2.6}$$

We define the discrete coincidence set according to $\mathcal{A}_\ell := \{x \in \overline{\Omega} \mid u_\ell(x) = \psi_\ell(x)\}$ and refer to $\mathcal{I}_\ell := \overline{\Omega} \setminus \mathcal{A}_\ell$ as the discrete noncoincidence set. We note that $\mathcal{A}_\ell$ may consist of vertices and/or edges only.

The corresponding Lagrange multiplier σ_ℓ can be written as a linear combination of Dirac delta functionals δ_p associated with $p \in \mathcal{N}_\ell$ according to

$$\sigma_\ell := \sum_{p \in \mathcal{N}_\ell} \alpha_\ell(p)\delta_p\ , \quad \alpha_\ell(p) \in \mathbb{R}\ , \quad p \in \mathcal{N}_\ell. \tag{2.7}$$

As in the continuous setting, (2.6) can be written in unconstrained form as

$$a(u_\ell, v_\ell) = (f, v_\ell)_{0,\Omega} - \langle\langle \sigma_\ell, v_\ell \rangle\rangle\ , \quad v_\ell \in V_\ell\,. \tag{2.8}$$

In particular, $\sigma_\ell \in \mathcal{M}_+(\bar{\Omega})$ and the complementarity condition

$$\langle\langle \sigma_\ell, \psi_\ell - u_\ell \rangle\rangle = 0 \tag{2.9}$$

is satisfied.

Residual-type a posteriori error estimators for obstacle problems that contain the standard edge residuals $\eta_E := h_E^{1/2}\|\nu_E \cdot [\nabla u_\ell]_E\|_{0,E}$, where $[\nabla u_\ell]_E$ denotes the jump of ∇u_ℓ across E, for edges within the discrete coincidence set cannot be efficient: Assume ψ_ℓ to have a kink that aligns with some edge E in the discrete coincidence set. Then, the edge residual $\eta_E = h_E^{1/2}\|\nu_E \cdot [\nabla \psi_\ell]_E\|_{0,E}$ will be large, although the discretization error $|u - u_\ell|_{1,\Omega}$ can be arbitrarily small. The same applies to the discrete local efficiency. As will be shown in the subsequent a posteriori error analysis, the standard element and edge residuals within the discrete coincidence set do not contribute to the error estimator. They will be eliminated in essence by the discrete multiplier. However, the a posteriori error analysis requires an extension of the discrete multiplier to $V^* = H^{-1}(\Omega)$. This extension is motivated by the following explicit representation of σ_ℓ.

LEMMA 2.1. *The discrete Lagrange multiplier σ_ℓ has the representation*

$$\alpha_\ell(p) = \begin{cases} \sum\limits_{T \in \omega_\ell^p} (f, \varphi_\ell^p)_{0,T} - \sum\limits_{E \in \mathcal{E}_\ell^p} (\nu_E \cdot [\nabla u_\ell]_E, \varphi_\ell^p)_{0,E}, & p \in \mathcal{N}_\ell(\mathcal{A}_\ell), \\ 0, & p \in \mathcal{N}_\ell(\mathcal{I}_\ell), \end{cases} \tag{2.10}$$

where φ_ℓ^p is the nodal basis function associated with the nodal point p.

Proof. It is an immediate consequence of (2.9) that $\alpha_\ell(p) = 0$ for $p \in \mathcal{I}_\ell$. On the other hand, if $p \in \mathcal{A}_\ell$, we choose $v_\ell = \varphi_\ell^p$. It follows from (2.8) that

$$\alpha_\ell(p) = \langle\langle \sigma_\ell, \varphi_\ell^p \rangle\rangle = (f, \varphi_\ell^p)_{0,\omega_\ell^p}\ -\ (\nabla u_\ell, \nabla \varphi_\ell^p)_{0,\omega_\ell^p}\ . \tag{2.11}$$

An elementwise application of Green's formula to the second term on the right-hand side in (2.11) yields

$$(\nabla u_\ell, \nabla \varphi_\ell^p)_{0,\omega_\ell^p} = \sum_{T \in \omega_\ell^p} (\nabla u_\ell, \nabla \varphi_\ell^p)_{0,T} = \sum_{E \in \mathcal{E}_\ell^p} (\nu_E \cdot [\nabla u_\ell], \varphi_\ell^p)_{0,E}. \tag{2.12}$$

Inserting (2.12) in (2.11) we obtain the assertion. $\square$

In the a posteriori error analysis of obstacle problems, the Lagrange multiplier σ_ℓ is considered as a functional on V_ℓ and extended to V; see, e.g., [8]. Usually this is done via a representation as an L_2 function. Here, for the reasons mentioned above, the construction refers to Lemma 2.1 and edge terms are included. We set

$$\langle \tilde{\sigma}_\ell, v\rangle \;:=\; \sum_{p\in\mathcal{N}_\ell(\mathcal{A}_\ell)} \Big(\frac{1}{3}\sum_{T\in\,\Omega_\ell^p}(f,v)_{0,T} - \frac{1}{2}\sum_{E\in\,\mathcal{E}_\ell^p}(\nu_E\cdot[\nabla u_\ell],v)_{0,E}\Big)\,. \tag{2.13}$$

REMARK 2.1. *The sum in the definition of $\tilde{\sigma}_\ell$, i.e., in* (2.13) *is restricted to points in the active set. If the summation runs over all nodal points of the grid and the factors are adjusted at the boundary, then we obtain an extension $\hat{\sigma}_\ell$ with $\langle\hat{\sigma}_\ell, v\rangle = a(u_h, v) - (f, v)$ for all $v \in V$; see [10].*

We denote by $\mathcal{E}_\ell(\mathcal{A}_\ell)$ and $\mathcal{T}_\ell(\mathcal{A}_\ell)$ the sets of edges and elements having all vertices within the discrete coincidence set $\mathcal{A}_\ell$, i.e.,

$$\mathcal{E}_\ell(\mathcal{A}_\ell) \;:=\; \bigcup\{E\in\mathcal{E}_\ell(\Omega)\mid \mathcal{N}_\ell(E)\subset\mathcal{A}_\ell\}\;, \tag{2.14a}$$

$$\mathcal{T}_\ell(\mathcal{A}_\ell) \;:=\; \bigcup\{T\in\mathcal{T}_\ell(\Omega)\mid \mathcal{N}_\ell(T)\subset\mathcal{A}_\ell\}\;, \tag{2.14b}$$

and we refer to $\mathcal{E}_\ell(\mathcal{I}_\ell)$ and $\mathcal{T}_\ell(\mathcal{I}_\ell)$ as the complements

$$\mathcal{E}_\ell(\mathcal{I}_\ell) \;:=\; \mathcal{E}_\ell\setminus\mathcal{E}_\ell(\mathcal{A}_\ell)\;,\qquad \mathcal{T}_\ell(\mathcal{I}_\ell) \;:=\; \mathcal{T}_\ell\setminus\mathcal{T}_\ell(\mathcal{A}_\ell)\;. \tag{2.15}$$

We further introduce $\mathcal{E}_{\mathcal{A}_\ell}^{(i)}\subset\mathcal{E}_\ell$ and $\mathcal{T}_{\mathcal{A}_\ell}^{(i)}\subset\mathcal{T}_\ell$ as the subsets of edges and elements having i vertices in the discrete coincidence set $\mathcal{A}_\ell$, i.e.,

$$\mathcal{E}_{\mathcal{A}_\ell}^{(i)} \;:=\; \bigcup\{E\in\mathcal{E}_\ell\mid \text{card}(\mathcal{N}_\ell(E)\cap\mathcal{A}_\ell)=i\}\,,\quad i\in\{0,1,2\}\,, \tag{2.16a}$$

$$\mathcal{T}_{\mathcal{A}_\ell}^{(i)} \;:=\; \bigcup\{T\in\mathcal{T}_\ell\mid \text{card}(\mathcal{N}_\ell(T)\cap\mathcal{A}_\ell)=i\}\,,\quad i\in\{0,1,2,3\}\,, \tag{2.16b}$$

and we define $\mathcal{E}_{\mathcal{I}_\ell}^{(i)}$ and $\mathcal{T}_{\mathcal{I}_\ell}^{(i)}$ analogously. In particular, $\mathcal{E}_\ell(\mathcal{A}_\ell)=\mathcal{E}_{\mathcal{A}_\ell}^{(2)}$ and $\mathcal{T}_\ell(\mathcal{A}_\ell)=\mathcal{T}_{\mathcal{A}_\ell}^{(3)}$. Moreover, we set

$$\mathcal{T}_{\mathcal{F}_\ell} \;:=\; \mathcal{T}_\ell\setminus(\mathcal{T}_{\mathcal{A}_\ell^{(3)}}\cup\mathcal{T}_{\mathcal{I}_\ell^{(3)}})\quad,\quad \mathcal{E}_{\mathcal{F}_\ell} \;:=\; \mathcal{E}_\ell\setminus(\mathcal{E}_{\mathcal{A}_\ell^{(2)}}\cup\mathcal{T}_{\mathcal{I}_\ell^{(2)}}). \tag{2.17}$$

Now the summation in (2.13) can be reorganized such that each triangle and each edge enters only once. Taking (2.14) and (2.16) into account, from (2.13) we easily deduce that for $v\in V$ there holds

$$\langle\tilde{\sigma}_\ell, v\rangle = \sum_{i=1}^{3}\frac{i}{3}\sum_{T\in\mathcal{T}_{\mathcal{A}_\ell}^{(i)}}(f,v)_{0,T} \;-\sum_{i=1}^{2}\frac{i}{2}\sum_{E\in\mathcal{E}_{\mathcal{A}_\ell}^{(i)}}(\nu_E\cdot[\nabla u_\ell]_E, v)_{0,E}\,. \tag{2.18}$$

It follows that for $v_\ell\in V_\ell$

$$\langle\langle\sigma_\ell, v_\ell\rangle\rangle - \langle\tilde{\sigma}_\ell, v_\ell\rangle \;=\; \sum_{T\in\mathcal{T}_{\mathcal{F}_\ell}}\kappa_T(f,v_\ell)_{0,T} - \sum_{E\in\mathcal{E}_{\mathcal{F}_\ell}}\kappa_E(\nu_E\cdot[\nabla u_\ell]_E, v_\ell)_{0,E}, \tag{2.19}$$

where

$$\kappa_T := 1-\frac{i}{3},\;\; T\in\mathcal{T}_\ell^{(i)}\quad,\quad \kappa_E := 1-\frac{i}{2},\;\; E\in\mathcal{E}_\ell^{(i)}. \tag{2.20}$$

We note that $\tilde{\sigma}_\ell$ does not inherit the complementarity properties from σ_ℓ, in particular, $\tilde{\sigma}_\ell\notin V_+^*$. Obviously, the contribution of $\tilde{\sigma}_\ell$ reminds of the well-known residual estimators for linear problems. Section 4 will highlight its role in the a posteriori error analysis.

3. The a posteriori error estimator and the error reduction property. We consider the residual-type a posteriori error estimator

$$\eta_\ell := \Big(\sum_{T \in \mathcal{T}_\ell(\mathcal{I}_\ell)} \eta_T^2 + \sum_{E \in \mathcal{E}_\ell(\mathcal{I}_\ell)} \eta_E^2 \Big)^{1/2}, \tag{3.1}$$

where $\mathcal{T}_\ell(\mathcal{I}_\ell)$ and $\mathcal{E}_\ell(\mathcal{I}_\ell)$ are given by (2.15). The element residuals η_T are weighted elementwise L^2-residuals and the edge residuals η_E are weighted L^2-norms of the jumps $\nu_E \cdot [\nabla u_\ell]$ of the normal derivatives across the interior edges according to

$$\eta_T := h_T \|f_T\|_{0,T} \quad , \quad \eta_E := h_E^{1/2} \|\nu_E \cdot [\nabla u_\ell]_E\|_{0,E}. \tag{3.2}$$

They are defined as in the linear regime (see, e.g., [33]), but in contrast to that case they only have to be considered for elements T and edges E within the discrete non-coincidence set $\mathcal{I}_\ell$.

The refinement of a triangulation $\mathcal{T}_\ell$ is based on a bulk criterion that has been previously used in the convergence analysis of adaptive finite elements for nodal finite element methods [15, 24]. For the obstacle problem under consideration, the bulk criterion is as follows: Given a universal constant $\Theta \in (0,1)$, we create a set of elements $\mathcal{M}_\ell^{(1)} \subset \mathcal{T}_\ell(\mathcal{I}_\ell)$ and a set of edges $\mathcal{M}_\ell^{(2)} \subset \mathcal{E}_\ell(\mathcal{I}_\ell)$ such that

$$\Theta \sum_{T \in \mathcal{T}_\ell(\mathcal{I}_\ell)} \eta_T^2 \;\le\; \sum_{T \in \mathcal{M}_\ell^{(1)}} \eta_T^2 \,, \tag{3.3a}$$

$$\Theta \sum_{E \in \mathcal{E}_\ell(\mathcal{I}_\ell)} \eta_E^2 \;\le\; \sum_{E \in \mathcal{M}_\ell^{(2)}} \eta_E^2 \,. \tag{3.3b}$$

The bulk criterion is realized by a greedy algorithm [12, 13]. Based on the bulk criterion, we generate a fine mesh $\mathcal{T}_{\ell+1}$ as follows: If $T \in \mathcal{M}_\ell^{(1)}$ or $E = T_+ \cap T_- \in \mathcal{M}_\ell^{(2)}$, we refine T or $T_\pm$ by repeated bisection such that an interior nodal point p_T in T or interior nodal points $p_+ \in T_+$ and $p_- \in T_-$ are created [24]. In order to guarantee a geometrically conforming triangulation, new nodal points are generated, if necessary.

We further have to take into account data oscillations and a data term with respect to the right-hand side f and the obstacle ψ. The data oscillations osc_ℓ are given by

$$osc_\ell^2 \;:=\; osc_\ell^2(f) \;+\; osc_\ell^2(\psi) \,, \tag{3.4}$$

where $osc_\ell(f)$ and $osc_\ell(\psi)$ are defined by means of

$$osc_\ell^2(f) := \sum_{T \in \mathcal{T}_\ell(\Omega)} osc_T^2(f) + \sum_{E \in \mathcal{E}_\ell(\Omega)} osc_{\omega_\ell^E}^2(f) \,, \tag{3.5a}$$

$$osc_\ell^2(\psi) := \sum_{T \in \mathcal{T}_\ell(\Omega)} osc_T^2(\psi) + \sum_{E \in \mathcal{E}_\ell(\Omega)} osc_{\omega_\ell^E}^2(\psi) \,, \tag{3.5b}$$

$$osc_D(f) := \mathrm{diam}(D)\, \|f - f_D\|_{0,D} \,,$$

$$osc_D(\psi) := |\psi - \psi_\ell|_{1,D} \,, \qquad D \in \{T, \omega_\ell^E\} \,.$$

On the other hand, the data term μ_ℓ is of the form

$$\mu_\ell^2 := \sum_{E \in \hat{\mathcal{M}}_\ell^{(2)}} \mu_E^2(\psi) \quad , \quad \mu_E(\psi) := h_E \, \|\nu_E \cdot [\nabla \psi]_E\|_{0,E}, \tag{3.6}$$

where

$$\hat{\mathcal{M}}_\ell^{(2)} \ := \ \{E \in \mathcal{M}_\ell^{(2)} \mid m_E \in \mathcal{A}_{\ell+1} \text{ and } p_+ \in \mathcal{I}_{\ell+1} \text{ or } p_- \in \mathcal{I}_{\ell+1}\}$$

with $p_\pm$ denoting the interior nodal points in $T_\pm$ $(E = T_+ \cap T_-)$ (cf. case $(ii)_{2,1}$ in the proof of Lemma 5.3 in section 5 below which is the only situation where μ_ℓ^2 occurs in the a posteriori error analysis).
The refinement and the new mesh $\mathcal{T}_{\ell+1}$ shall also take care of a reduction of the data oscillations (cf., e.g., [24]). In particular, we require that

$$osc_{\ell+1}^2 \ \leq \ \rho_2 \, osc_\ell^2 \tag{3.7}$$

for some $0 < \rho_2 < 1$. This can be achieved by additional refinements if necessary. Likewise, we require that

$$\mu_{\ell+1}^2 \ \leq \ \rho_3 \, \mu_\ell^2, \tag{3.8}$$

where $0 < \rho_3 < 1$. Since these terms can be expected to arise only in the discrete noncoincidence set close to the discrete free boundary, (3.8) can be achieved by including edges in the vicinity of the discrete free boundary in the refinement process.

The convergence analysis is based on the reliability and the discrete efficiency of the estimator η_ℓ as well as on a perturbed Galerkin orthogonality which will be addressed in detail in the subsequent section. These properties involve consistency errors due to the extension $\tilde{\sigma}_\ell$ of the discrete multiplier σ_ℓ and the mismatch between the continuous and discrete coincidence and noncoincidence sets. In particular, we define

$$con_\ell \ := \ con_\ell^{rel} \ + \ con_\ell^{ort} \, . \tag{3.9}$$

Here, con_ℓ^{rel} and con_ℓ^{ort} refer to the consistency errors associated with the reliability of η_ℓ and the perturbed Galerkin orthogonality:

$$con_\ell^{rel} \ := \ |\langle \tilde{\sigma}_\ell, \psi - u \rangle| \quad , \quad con_\ell^{ort} \ := \ 2 \, \langle \sigma, \psi_\ell - u_\ell \rangle. \tag{3.10}$$

Due to the construction of $\tilde{\sigma}_\ell$, the consistency error con_ℓ^{rel} is nonzero only in the small patch $\mathcal{T}_{\mathcal{F}_\ell} \cup \mathcal{E}_{\mathcal{F}_\ell}$ in the vicinity of the discrete free boundary (cf. (2.17)) and in $\mathcal{C}_1 := \mathcal{A}_\ell \cap \mathcal{I}$. On the other hand, the consistency error con_ℓ^{ort} is nonzero only in $\mathcal{C}_2 := \mathcal{A} \cap \mathcal{I}_\ell$. The sets $\mathcal{C}_1$ and $\mathcal{C}_2$ represent the mismatch between the continuous and discrete coincidence and noncoincidence sets. Usually, the sets $\mathcal{T}_{\mathcal{F}_\ell} \cup \mathcal{E}_{\mathcal{F}_\ell}$ and $\mathcal{C}_\nu, 1 \leq \nu \leq 2$, are small and the consistency errors con_ℓ^{rel} and con_ℓ^{ort} turn out to be at least one order of magnitude smaller than the other error terms as it is the case, for instance, in the numerical examples presented in section 6. However, if necessary, the marking strategy can be extended by marking the elements and edges in $\mathcal{T}_{\mathcal{F}_\ell} \cup \mathcal{E}_{\mathcal{F}_\ell}$ and $\mathcal{C}_\nu, 1 \leq \nu \leq 2$, for refinement. To do so, we need to provide approximations of the mismatch sets $\mathcal{C}_1$ and $\mathcal{C}_2$. We denote by $\chi(D), D \subset \overline{\Omega}$, the characteristic function of D and, following [18] and [22], define

$$\chi_\ell^{\mathcal{A}} \ := \ I - \frac{\psi_\ell - u_\ell}{\gamma h_\ell^r + \psi_\ell - u_\ell}$$

with appropriately chosen $\gamma, r > 0$ as an approximation of $\chi(\mathcal{A})$. Indeed, it can be shown that $\|\chi_\ell^{\mathcal{A}} - \chi(\mathcal{A})\|_{0,T} \to 0$ as $h_\ell \to 0$ for each $T \in \mathcal{T}_\ell(\Omega)$ (cf. [18, 22]). Then, $\chi_\ell^{\mathcal{I}} := I - \chi_\ell^{\mathcal{A}}$ is an approximation of $\chi(\mathcal{I})$ and hence, $\chi_\ell^{\mathcal{C}_1} := \chi(\mathcal{A}_\ell)\chi_\ell^{\mathcal{I}}$ and $\chi_\ell^{\mathcal{C}_2} := \chi(\mathcal{I}_\ell)\chi_\ell^{\mathcal{A}}$ provide approximations of the characteristic functions $\chi(\mathcal{C}_1)$ and $\chi(\mathcal{C}_2)$.

The main result of this paper states an error reduction in the $|\cdot|_{1,\Omega}$-norm up to the consistency error con_ℓ.

THEOREM 3.1. *Let $u \in V$ and $u_\ell \in V_\ell$, $u_{\ell+1} \in V_{\ell+1}$, respectively, be the solutions of* (2.2) *and* (2.8)*, and let osc_ℓ, μ_ℓ, and con_ℓ be the data oscillations, data terms, and the consistency error as given by* (3.4), (3.6), *and* (3.9)*, respectively. Assume that* (3.7),(3.8) *are satisfied. Then, there exist constants $0 < \rho_1 < 1$ and $C_i > 0, 1 \le i \le 3$, depending only on Θ and the local geometry of the triangulations, such that*

$$\begin{pmatrix} |u-u_{\ell+1}|^2_{1,\Omega} \\ osc^2_{\ell+1} \\ \mu^2_{\ell+1} \end{pmatrix} \le \begin{pmatrix} \rho_1 & C_1 & C_2 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix} \begin{pmatrix} |u-u_\ell|^2_{1,\Omega} \\ osc^2_\ell \\ \mu^2_\ell \end{pmatrix} + \begin{pmatrix} C_3\ con_\ell \\ 0 \\ 0 \end{pmatrix}. \quad (3.11)$$

REMARK 3.1. *If the consistency error con_ℓ is negligible, the error reduction property* (3.11) *implies R-linear convergence of the finite element approximations $u_\ell \in V_\ell$ to the solution $u \in V$ of* (2.2).

The proof of Theorem 3.1 will be presented in the next section.

4. Reliability. We will show that the residual-type error estimator from (3.1) provides an upper bound for the energy norm error up to the data oscillations and the consistency error con_ℓ^{ref}.

Throughout this section, we denote by $C > 0$ a constant depending only on the geometry of the triangulation, not necessarily the same at each occurrence. Moreover, for $A, B \in \mathbb{R}$ we use the notation $A \lesssim B$, if $A \le CB$. Likewise, $A \approx B$ iff $A \lesssim B$ and $B \lesssim A$.

THEOREM 4.1. *Let $u \in V$ and $u_\ell \in V_\ell$ be the solutions of* (2.3) *and* (2.8)*, respectively, and let η_ℓ, osc_ℓ, and con_ℓ^{rel} be the error estimator* (3.1)*, the data oscillations* (3.4) *and the consistency error* (3.10)*, respectively. Then, there holds*

$$|u-u_\ell|^2_{1,\Omega} \ \lesssim\ \eta^2_\ell \ +\ osc^2_\ell \ +\ con_\ell^{rel}\ . \quad (4.1)$$

Proof. Setting $e_u := u - u_\ell$ and denoting by $P_{V_\ell} : V \to V_\ell$ Clément's quasi-interpolation operator (see, e.g., [33]), we find by straightforward computation

$$|e_u|^2_{1,\Omega} = a(e_u, e_u) = r(e_u - P_{V_\ell} e_u) + \ell_1(P_{V_\ell} e_u) + \ell_2(e_u), \quad (4.2)$$

where

$$\begin{aligned} r(v) &:= (f, v)_{0,\Omega} - a(u_\ell, v) - \langle \tilde{\sigma}, v \rangle \quad , \quad v \in V\ , \\ \ell_1(v_\ell) &:= \langle\langle \sigma_\ell, v_\ell \rangle\rangle - \langle \tilde{\sigma}_\ell, v_\ell \rangle \quad , \quad v_\ell \in V_\ell, \\ \ell_2(v) &:= \langle \tilde{\sigma}_\ell - \sigma, v \rangle \quad , \quad v \in V. \end{aligned}$$

Elementwise integration by parts and the representation (2.18) leads to

$$\begin{aligned} r(v) = &\sum_{T \in \mathcal{T}_\ell} (f, v)_{0,T} - \sum_{E \in \mathcal{E}_\ell(\Omega)} (\nu_E \cdot [\nabla u_\ell]_E, v)_{0,E} - \langle \tilde{\sigma}_\ell, v \rangle \quad (4.3) \\ = &\sum_{T \in \mathcal{T}_\ell(\mathcal{I}_\ell)} \kappa_T (f_T, v)_{0,T} - \sum_{E \in \mathcal{E}_\ell(\mathcal{I}_\ell)} \kappa_E (\nu_E \cdot [\nabla u_\ell]_E, v)_{0,E} + \\ &\sum_{T \in \mathcal{T}_\ell(\mathcal{I}_\ell)} \kappa_T (f - f_T, v)_{0,T}, \end{aligned}$$

where κ_T and κ_E are given by (2.20). Standard estimation of the terms on the right-hand side in (4.3) with $v := e_u - P_{V_\ell} e_u$ yields

$$
\begin{aligned}
|r(e_u - P_{V_\ell} e_u)| &\lesssim \sum_{T \in \mathcal{T}_\ell(\mathcal{I}_\ell)} \Big(\eta_T + osc_T(f)\Big) |e_u|_{1,\omega_\ell^T} + \sum_{E \in \mathcal{E}_\ell(\mathcal{I}_\ell)} \eta_E |e_u|_{1,\omega_\ell^E} \\
&\leq \frac{1}{10} \, |e_u|_{1,\Omega}^2 \; + \; C\Big(\eta_\ell^2 \; + \; osc_\ell^2(f)\Big). \qquad (4.4)
\end{aligned}
$$

For $\ell_1(P_{V_\ell} e_u)$ in (4.2) we obtain

$$
|\ell_1(P_{V_\ell} e_u)| \leq \frac{1}{10} |e_u|_{1,\Omega}^2 + C\Big(\sum_{T \in \mathcal{T}_{\mathcal{F}_\ell}} (\eta_T^2 + osc_T^2(f)) + \sum_{E \in \mathcal{E}_{\mathcal{F}_\ell}} \eta_E^2 \Big). \qquad (4.5)
$$

Moreover, for $\ell_2(e_u)$ it follows that

$$
\ell_2(e_u) \; = \; \langle \tilde{\sigma}_\ell - \sigma, u - \psi \rangle \; + \; \langle \tilde{\sigma}_\ell - \sigma, \psi - \psi_\ell \rangle \; + \; \langle \tilde{\sigma}_\ell - \sigma, \psi_\ell - u_\ell \rangle \; .
$$

From the complementarity property (2.4),(2.13) and $\sigma \in V_+^*$ we deduce

$$
\begin{aligned}
\langle \tilde{\sigma}_\ell - \sigma, e_u \rangle \; &\leq \frac{1}{10} |e_u|_{1,\Omega}^2 + C\Big(\sum_{T \in \mathcal{T}_{\mathcal{F}_\ell}} (\eta_T^2 + osc_T^2(f)) + \sum_{E \in \mathcal{E}_{\mathcal{F}_\ell}} \eta_E^2 \Big) + \qquad (4.6) \\
&+ \; osc_\ell^2(\psi) + con_\ell^{rel} + \langle \tilde{\sigma}_\ell - \sigma, \psi - \psi_\ell \rangle.
\end{aligned}
$$

It remains to estimate $\langle \tilde{\sigma}_\ell - \sigma, \psi - \psi_\ell \rangle$. Zero boundary conditions are not required for $\tilde{\sigma}_\ell - \sigma$. We note that $u \in V$ and $u_\ell \in V_\ell$ satisfy

$$
a(u, v) \; = \; (f, v)_{0,\Omega} \; + \; \langle \nu_\Gamma \cdot \nabla u, v \rangle_\Gamma \; - \; \langle \sigma, v \rangle \; , \quad v \in H^1(\Omega), \qquad (4.7)
$$

$$
a(u_\ell, v_\ell) \; = \; (f, v_\ell)_{0,\Omega} \; + \; \langle \nu_\Gamma \cdot \nabla u_\ell, v_\ell \rangle_\Gamma \; - \; \langle \tilde{\sigma}_\ell, v_\ell \rangle \; , \quad v_\ell \in S_\ell, \qquad (4.8)
$$

Setting $\delta_\psi := \psi - \psi_\ell \in H^1(\Omega)$ and denoting by $P_{S_\ell} : H^1(\Omega) \to S_\ell$ Clément's quasi-interpolation operator, we obtain

$$
\langle \tilde{\sigma}_\ell - \sigma, \delta_\psi \rangle \; = \; \langle \tilde{\sigma}_\ell - \sigma, P_{S_\ell} \delta_\psi \rangle \; + \; \langle \tilde{\sigma}_\ell - \sigma, \delta_\psi - P_{S_\ell} \delta_\psi \rangle. \qquad (4.9)
$$

We have

$$
\langle \tilde{\sigma}_\ell - \sigma, P_{S_\ell} \delta_\psi \rangle = (\langle \tilde{\sigma}_\ell, P_{S_\ell} \delta_\psi \rangle - \langle\langle \sigma_\ell, P_{S_\ell} \delta_\psi \rangle\rangle) + (\langle\langle \sigma_\ell, P_{S_\ell} \delta_\psi \rangle\rangle - \langle \sigma, P_{S_\ell} \delta_\psi \rangle). \qquad (4.10)
$$

For the first term on the right-hand side in (4.10) we get

$$
|\langle \tilde{\sigma}_\ell, P_{S_\ell} \delta_\psi \rangle - \langle\langle \sigma_\ell, P_{S_\ell} \delta_\psi \rangle\rangle| \lesssim osc_\ell^2(\psi) + \sum_{T \in \mathcal{T}_{\mathcal{F}_\ell}} (\eta_T^2 + osc_T^2(f)) + \sum_{E \in \mathcal{E}_{\mathcal{F}_\ell}} \eta_E^2. \qquad (4.11)
$$

Since $P_{S_\ell} \delta_\psi$ is an admissible test function in (4.7) and (4.8), the trace inequality

$$
\|\nu_\Gamma \cdot \nabla (u - u_\ell)\|_{-1/2,\Gamma} \; \lesssim \; |u - u_\ell|_{1,\Omega}, \qquad (4.12)
$$

and Young's inequality imply that the second term on the right-hand side in (4.10) can be bounded from above as follows

$$
\begin{aligned}
&|\langle\langle \sigma_\ell, P_{S_\ell} \delta_\psi \rangle\rangle - \langle \sigma, P_{S_\ell} \delta_\psi \rangle| \leq |a(u - u_\ell, P_{S_\ell} \delta_\psi)| \; + \qquad (4.13) \\
&|\langle \nu_\Gamma \cdot \nabla (u - u_\ell), P_{S_\ell} \delta_\psi \rangle_\Gamma| \leq \frac{1}{10} \, |u - u_\ell|_{1,\Omega}^2 \; + \; C \; osc_\ell^2(\psi).
\end{aligned}
$$

Next, using (2.13) for dealing with $\tilde{\sigma}_\ell$ and (4.7) with σ we get

$$\langle \sigma - \tilde{\sigma}_\ell, \delta_\psi - P_{S_\ell}\delta_\psi \rangle \;=\; I_1 \;+\; I_2, \tag{4.14}$$

where

$$\begin{aligned} I_1 \;&:=\; (f, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,\Omega} \;-\; \sum_{p\in\mathcal{N}_\ell(\mathcal{A}_\ell)} \frac{1}{3} \sum_{T\in\omega_\ell^p} (f, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,\Omega}, \\ I_2 \;&:=\; \langle \nu_\Gamma \cdot \nabla u, \delta_\psi - P_{S_\ell}\delta_\psi \rangle_\Gamma \;-\; a(u, \delta_\psi - P_{S_\ell}\delta_\psi) \;+ \\ &\quad + \sum_{p\in\mathcal{N}_\ell(\mathcal{A}_\ell)} \frac{1}{2} \sum_{E\in E_\ell^p} (\nu_E \cdot [\nabla u_\ell]_E, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,\Omega}. \end{aligned}$$

For the first term it follows that

$$\begin{aligned} |I_1| \;&\le\; \sum_{T\in\mathcal{T}_\ell(\mathcal{I}_\ell)} (1-\kappa_T) \Big(|(f_T, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,T}| + |(f - f_T, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,T}| \Big)| \;\lesssim \\ &\lesssim\; \sum_{T\in\mathcal{T}_\ell(\mathcal{I}_\ell)} \Big(h_T \, \|f_T\|_{0,T} + h_T \, \|f - f_T\|_{0,T} \Big) \, |\delta_\psi|_{1,\omega_T} \;\lesssim \\ &\lesssim\; \sum_{T\in\mathcal{T}_\ell(\mathcal{I}_\ell)} \Big(\eta_T^2 \;+\; osc_T^2(f) \Big) \;+\; osc_\ell^2(\psi). \end{aligned}$$

Moreover, using (4.12) and Young's inequality again, the second term I_2 is estimated from above

$$\begin{aligned} |I_2| &\le |a(e_u, \delta_\psi - P_{S_\ell}\delta_\psi)| \;+ \sum_{E\in\mathcal{E}_\ell(\mathcal{I}_\ell)} (1-\kappa_E) \, |(\nu_E \cdot [\nabla u_\ell]_E, \delta_\psi - P_{S_\ell}\delta_\psi)_{0,E}| \\ &\quad + \sum_{E\in\mathcal{E}_\ell(\Gamma)} |\langle \nu_E \cdot \nabla(u - u_\ell), \delta_\psi - P_{S_\ell}\delta_\psi \rangle_E| \\ &\le \frac{1}{10} \, |e_u|_{1,\Omega}^2 + C\Big(\sum_{E\in\mathcal{E}_\ell(\mathcal{I}_\ell)} \eta_E^2 \;+\; osc_\ell^2(\psi) \Big). \end{aligned}$$

The preceding two estimates give

$$|\langle \sigma - \tilde{\sigma}_\ell, \delta_\psi - P_{S_\ell}\delta_\psi \rangle| \;\le\; \frac{1}{10} \, |e_u|_{1,\Omega}^2 \;+\; C \, \Big(\eta_\ell^2 + osc_\ell^2(f) + osc_\ell^2(\psi) \Big). \tag{4.15}$$

Finally, combining (4.4)-(4.6), (4.10), (4.11) (4.13) and (4.15) we complete the proof of (4.1). □

5. Discrete local efficiency, perturbed Galerkin orthogonality, and proof of the error reduction property. We will prove discrete efficiency of the error estimator in the sense that it provides a lower bound for the energy norm of the difference $u_\ell - u_{\ell+1}$ between the coarse and fine mesh approximation up to the data oscillations and the data terms.

THEOREM 5.1. *Let $u_\ell \in V_\ell, u_{\ell+1} \in V_{\ell+1}$ be the solutions of* (2.8) *and let η_ℓ, osc_ℓ as well as μ_ℓ be the error estimator, the data oscillations, and the data terms as given by* (3.1), (3.4), *and* (3.6), *respectively. Then, there holds*

$$\eta_\ell^2 \;\lesssim\; |u_\ell - u_{\ell+1}|_{1,\Omega}^2 \;+\; osc_\ell^2 \;+\; \mu_\ell^2. \tag{5.1}$$

As usual in the convergence analysis of adaptive finite element methods, the proof of Theorem 5.1 follows from the discrete local efficiency. The guaranteed improvements that can be associated to the volume terms and the edge terms will be established by the subsequent two lemmas. We adjust the concept in [9] to general obstacles, but it would be possible also to adopt ideas from [10] or [28].

LEMMA 5.2. *Let $T \in \mathcal{M}_\ell^{(1)}$ with an interior nodal point $p \in \mathcal{N}_{\ell+1}(T)$.*
(i) If $p \in \mathcal{N}_{\ell+1}(\mathcal{I}_{\ell+1})$, we have

$$\eta_T^2 \ \lesssim \ |u_\ell - u_{\ell+1}|_{1,T}^2 \ + \ osc_T^2(f). \tag{5.2}$$

(ii) If $p \in \mathcal{N}_{\ell+1}(\mathcal{A}_{\ell+1})$, due to $T \in \mathcal{M}_\ell^{(1)}$ there exists $\hat{p} \in \mathcal{N}_\ell(T) \cap \mathcal{N}_\ell(\mathcal{I}_\ell)$, and there holds

$$\eta_T^2 \ \lesssim \ h_T^2 \ \|f - f_{\omega_\ell^{\hat{p}}}\|_{0,\omega_\ell^{\hat{p}}}^2 \ + \ \sum_{E \in E_\ell^{\hat{p}}} \eta_E^2, \tag{5.3}$$

where $f_{\omega_\ell^{\hat{p}}} := |\omega_\ell^{\hat{p}}|^{-1} \int_{\omega_\ell^{\hat{p}}} f dx$.

Proof. Let $p \in \mathcal{N}_{\ell+1}(T)$ be an interior node. We choose $\chi_{\ell+1}^{(p)} := \kappa \varphi_{\ell+1}^{(p)}$, $\kappa \approx f_T$, as an appropriate multiple of the level $\ell+1$ nodal basis function $\varphi_{\ell+1}^{(p)}$ associated with p such that

$$h_T^2 \|f_T\|_{0,T}^2 \ \le \ h_T^2 \ (f_T, \chi_{\ell+1}^{(p)})_{0,T}.$$

Observing $\nabla u_\ell \in P_0(T)$ we find by partial integration

$$a(u_\ell, v) = 0 \quad \text{if } \operatorname{supp} v \subset T \text{ and } v \in H_0^1(T). \tag{5.4}$$

In particular, the preceding inequality yields

$$h_T^2 \|f_T\|_{0,T}^2 \ \le \ h_T^2 \Big((f_T, \chi_{\ell+1}^{(p)})_{0,T} - a(u_\ell, \chi_{\ell+1}^{(p)}) \Big) \ . \tag{5.5}$$

Since $\chi_{\ell+1}^{(p)}$ is an admissible level $\ell+1$ test function in (2.8), we have

$$a(u_{\ell+1}, \chi_{\ell+1}^{(p)}) \ - \ (f, \chi_{\ell+1}^{(p)})_{0,T} \ + \ \langle\langle \sigma_{\ell+1}, \chi_{\ell+1}^{(p)} \rangle\rangle \ = \ 0 \ . \tag{5.6}$$

Adding (5.5) and (5.6) results in

$$\begin{aligned} h_T^2 \|f_T\|_{0,T}^2 = h_T^2 \Big(&(f_\ell - f, \chi_{\ell+1}^{(p)})_{0,T} \ + \\ &+ a(u_{\ell+1} - u_\ell, \chi_{\ell+1}^{(p)}) \ + \ \langle\langle \sigma_{\ell+1}, \chi_{\ell+1}^{(p)} \rangle\rangle \Big) \ . \end{aligned} \tag{5.7}$$

Case (i): $p \in \mathcal{N}_{\ell+1}(\mathcal{I}_\ell)$ implies that

$$\langle\langle \sigma_{\ell+1}, \chi_{\ell+1}^{(p)} \rangle\rangle \ = \ \kappa \alpha_{\ell+1}(p) \ = \ 0 \, ,$$

and we readily deduce from (5.7)

$$h_T^2 \|f_T\|_{0,T}^2 \le |u_\ell - u_{\ell+1}|_{1,T} \, h_T^2 |\chi_{\ell+1}^{(p)}|_{1,T} + osc_{\ell,T}(f) \, h_T \|\chi_{\ell+1}^{(p)}\|_{0,T} \, . \tag{5.8}$$

Observing

$$h_T^2 |\chi_{\ell+1}^{(p)}|_{1,T} \ \approx \ h_T^2 |\kappa| \ \approx \ h_T \|f_T\|_{0,T} \ , \tag{5.9a}$$

$$h_T \|\chi_{\ell+1}^{(p)}\|_{0,T} \ \approx \ h_T |T|^{1/2} |\kappa| \ \approx \ h_T \|f_T\|_{0,T} \ , \tag{5.9b}$$

we obtain (5.2).

Case (ii): We have

$$h_T^2\,\|f_T\|_{0,T}^2 \;\le\; h_T^2\,(f_T,\chi_{\ell+1}^{(p)})_{0,T} \;=\; h_T^2\,(f_T-f,\chi_{\ell+1}^{(p)})_{0,T} \;+\; h_T^2\,(f,\chi_{\ell+1}^{(p)})_{0,T}\,. \quad (5.10)$$

We set $\chi_\ell^{(\hat p)} := \kappa\varphi_\ell^{(\hat p)}$, where $\varphi_\ell^{(\hat p)}$ is the level ℓ nodal basis function associated with $\hat p$, and we choose $\alpha > 0$ such that

$$\int\limits_{\hat\omega_\ell} \Big(\varphi_{\ell+1}^{(p)} - \alpha\varphi_\ell^{(\hat p)}\Big)\,dx \;=\; 0. \quad (5.11)$$

Since $\chi_\ell^{(\hat p)}$ is an admissible level ℓ test function, there holds

$$a(u_\ell,\chi_\ell^{(\hat p)}) \;=\; (f,\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}}. \quad (5.12)$$

On the other hand, by Green's formula

$$a(u_\ell,\chi_\ell^{(\hat p)}) \;=\; \sum_{E\in E_\ell^{\hat p}} (\nu_E\cdot[\nabla u_\ell]_E,\chi_\ell^{(\hat p)})_{0,E}. \quad (5.13)$$

Using (5.11)–(5.13) yields

$$\begin{aligned} h_T^2\,(f,\chi_{\ell+1}^{(p)})_{0,T} \;&=\; h_T^2\,(f,\chi_{\ell+1}^{(p)}-\alpha\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}} \;+\; \alpha\,h_T^2\,(f,\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}} \;= \quad (5.14)\\ &=\; h_T^2\,(f-f_{\omega_\ell^{\hat p}},\chi_{\ell+1}^{(p)}-\alpha\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}} \;+\; \alpha\,h_T^2\,a(u_\ell,\chi_\ell^{(\hat p)}) \;=\\ &=\; h_T^2\,(f-f_{\omega_\ell^{\hat p}},\chi_{\ell+1}^{(p)}-\alpha\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}} \;+\; \alpha\,h_T^2 \sum_{E\in E_\ell(\hat p)} (\nu_E\cdot[\nabla u_\ell]_E,\chi_\ell^{(\hat p)})_{0,E}. \end{aligned}$$

The right-hand sides in (5.14) can be estimated as follows

$$\begin{aligned} &h_T^2\,|(f-f_{\omega_\ell^{\hat p}},\chi_{\ell+1}^{(p)}-\alpha\chi_\ell^{(\hat p)})_{0,\omega_\ell^{\hat p}}| \;\lesssim \quad (5.15)\\ &\quad\lesssim\; h_T\,\|f-f_{\omega_\ell^{\hat p}}\|_{0,\omega_\ell^{hatp}}\,\Big(h_T\,\|\chi_{\ell+1}^{(p)}\|_{0,T}+\alpha\,|\omega_\ell^{\hat p}|^{1/2}\,\|\chi_\ell^{(\hat p)})\|_{0,\omega_\ell^{\hat p}}\Big),\\ &h_T^2\,|(\nu_E\cdot[\nabla u_\ell],\chi_\ell^{(\hat p)})_{0,E} \;\lesssim\; h_E^{1/2}\,\|\nu_E\cdot[\nabla u_\ell]_E\|_{0,E}\,h_E^{3/2}\,\|\chi_\ell^{(\hat p)}\|_{0,E}. \quad (5.16) \end{aligned}$$

Using (5.9b) and

$$\begin{aligned} |\omega_\ell^{\hat p}|^{1/2}\,\|\chi_\ell^{(\hat p)})\|_{0,\omega_\ell^{\hat p}} = |\omega_\ell^{\hat p}|^{1/2}\;\;|\kappa|\,\|\varphi_\ell^{(\hat p)})\|_{0,\omega_\ell^{\hat p}} \;\lesssim\; h_T\,\|f_T\|_{0,T},\\ h_E^{3/2}\,\|\chi_\ell^{(\hat p)}\|_{0,E} = h_E^{3/2}\,|\kappa|\,\|\varphi_\ell^{(\hat p)}\|_{0,E} \;\lesssim\; h_T\,\|f_T\|_{0,T}, \end{aligned}$$

in (5.15),(5.16), we find that (5.14) results in

$$\begin{aligned} h_T^2\,|(f,\chi_{\ell+1}^{(p)})_{0,T}| \lesssim \Big(&h_T\,\|f-f_{\omega_\ell^{\hat p}}\|_{0,\hat\omega_\ell} \;+ \quad (5.17)\\ &+\; \sum_{E\in E_\ell^{\hat p}} h_E^{1/2}\,\|\nu_E\cdot[\nabla u_\ell]_E\|_{0,E}\Big)\,h_T\,\|f_T\|_{0,T}. \end{aligned}$$

Finally, using (5.9a),(5.17) in (5.10), we deduce (5.3). □

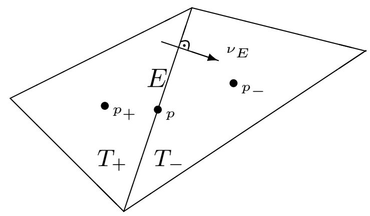

FIG. 5.1. *Notation for $E \in \mathcal{M}_\ell^{(2)}$ and the adjacent elements T_+, T_-.*

LEMMA 5.3. *Let $E \in \mathcal{M}_\ell^{(2)}$, $E = T_+ \cap T_-$, $T_\pm \in \mathcal{T}_\ell$, be a refined edge with midpoint $m_E \in \mathcal{N}_{\ell+1}(E)$ and associated patch $\omega_\ell^E := T_+ \cup T_-$. Then, there holds*

$$\eta_E^2 \;\lesssim\; |u_\ell - u_{\ell+1}|^2_{1,\omega_\ell^E} \;+\; osc^2_{\omega_\ell^E}(f) \;+\; osc^2_{\omega_\ell^E}(\psi) \;+\; \mu_E^2(\psi)\,. \tag{5.18}$$

Proof. Let $p_\pm \in \mathcal{N}_{\ell+1}(T_\pm)$ be interior nodes in $T_\pm$ and $w_{\ell+1} := u_{\ell+1} - \psi_{\ell+1}$ (cf. Fig. 5.1). We distinguish the two cases

$$\begin{aligned} &(i) \quad w_{\ell+1}(p_+) \;=\; w_{\ell+1}(p_-) \;=\; 0\,,\\ &(ii) \quad w_{\ell+1}(p_+) < 0 \quad \text{or} \quad w_{\ell+1}(p_-) < 0\,. \end{aligned}$$

Case (i): For $w_\ell := u_\ell - \psi_\ell$ we have

$$h_E\|\nu_E \cdot [\nabla u_\ell]\|^2_{0,E} \;\lesssim\; h_E\|\nu_E \cdot [\nabla w_\ell]\|^2_{0,E} \;+\; \mu_E^2(\psi). \tag{5.19}$$

Since $\nabla w_\ell|_T$, $T \in \{T_\pm\}$, is a constant vector, there exists at least one element $T' \in \mathcal{T}_{\ell+1}(T)$ such that $\nu_E \cdot \nabla w_\ell|_{T'}$ and $\nu_E \cdot \nabla w_{\ell+1}|_{T'}$ have different signs or are zero on T'. Hence,

$$|\nu_E \cdot \nabla w_\ell|_{T'}| \;\le\; |\nu_E \cdot \nabla(w_\ell - w_{\ell+1})|_{T'}| \;\le\; |\nabla(w_\ell - w_{\ell+1})|_{T'}|.$$

Since $|T'| \approx |T| \approx h_E|E|$, it follows that

$$\begin{aligned} h_E\|\nu_E \cdot [\nabla w_\ell]_E\|^2_{0,E} \;&\lesssim\; |w_\ell - w_{\ell+1}|^2_{1,T_+} \;+\; |w_\ell - w_{\ell+1}|^2_{1,T_-} \;\lesssim \\ &\lesssim\; |u_\ell - u_{\ell+1}|^2_{1,\omega_\ell^E} \;+\; osc^2_{\omega_\ell^E}(\psi). \end{aligned} \tag{5.20}$$

Combining (5.20) and (5.19) we obtain (5.18).

Case (ii): Without loss of generality we may assume that $w_{\ell+1}(p_+) < 0$. We distinguish the subcases

$$(ii)_1 \quad w_{\ell+1}(m_E) < 0\,, \qquad (ii)_2 \quad w_{\ell+1}(m_E) \;=\; 0.$$

Case $(ii)_1$: Denoting by $\varphi_{\ell+1}^{(m_E)}$ and $\varphi_{\ell+1}^{(p_+)}$the nodal basis functions associated with m_E and p_+, we have

$$a(u_{\ell+1}, \varphi_{\ell+1}^{(m_E)}) \;=\; (f, \varphi_{\ell+1}^{(m_E)})_{0,\Omega} \quad \text{and} \quad a(u_{\ell+1}, \varphi_{\ell+1}^{(p_+)}) \;=\; (f, \varphi_{\ell+1}^{(p_+)})_{0,\Omega}. \tag{5.21}$$

The latter and (5.4) yield

$$a(u_{\ell+1} - u_\ell, \varphi_{\ell+1}^{(p_+)}) \;=\; (f, \varphi_{\ell+1}^{(p_+)})_{0,\Omega}. \tag{5.22}$$

We set $\varphi_{\ell+1}^{(E)} := \varphi_{\ell+1}^{(m_E)} - \alpha\varphi_{\ell+1}^{(p_+)}, \alpha > 0$, and choose α such that $\varphi_{\ell+1}^{(E)} \in H_0^1(\omega_\ell^E)$ and $\int_{\Omega_\ell^E} \varphi_{\ell+1}^{(E)}\, dx = 0$. It follows from (5.21) and (5.22) that

$$\begin{aligned}\frac{1}{2}\int\limits_E \nu_E \cdot [\nabla u_\ell]_E\, ds &= \int\limits_E \nu_E \cdot [\nabla u_\ell]_E\, \varphi_{\ell+1}^{(E)}\, ds \\ &= a(u_\ell - u_{\ell+1}, \varphi_{\ell+1}^{(E)}) \,+\, (f, \varphi_{\ell+1}^{(E)})_{0,\Omega_\ell^E} \\ &= a(u_\ell - u_{\ell+1}, \varphi_{\ell+1}^{(E)}) \,+\, (f - f_{\omega_\ell^E}, \varphi_{\ell+1}^{(E)})_{0,\omega_\ell^E}.\end{aligned}$$

We deduce

$$\eta_E^2 \;\lesssim\; |u_\ell - u_{\ell+1}|_{1,\Omega_\ell^E}^2 \;+\; osc_{\omega_\ell^E}^2(f),$$

which proves (5.18).
Case $(ii)_2$: We distinguish between

$$(ii)_{2,1} \quad \nu_E \cdot [\nabla u_\ell]_E \le 0 \qquad \text{and} \qquad (ii)_{2,2} \quad \nu_E \cdot [\nabla u_\ell]_E > 0.$$

Case $(ii)_{2,1}$: There exist $T'_\pm \in \mathcal{T}_{\ell+1}(T_\pm)$ such that

$$\nu_E \cdot \nabla w_{\ell+1}|_{T'_+} \;\ge\; 0 \;\ge\; \nu_E \cdot \nabla w_{\ell+1}|_{T'_-},$$

and hence,

$$\begin{aligned}0 \le -\,\nu_E \cdot [\nabla u_\ell]_E \;&=\; -\Big(\nu_E \cdot \nabla w_\ell|_{T'_+} - \nu_E \cdot \nabla w_\ell|_{T'_-}\Big) \;-\; \nu_E \cdot [\nabla \psi_\ell]_E \;\le \\ &\le\; -\Big(\nu_E \cdot \nabla(w_\ell - w_{\ell+1})|_{T'_+} - \nu_E \cdot \nabla(w_\ell - w_{\ell+1})|_{T'_-}\Big) \;-\; \nu_E \cdot [\nabla \psi_\ell]_E \;\le \\ &\le\; |\nabla(w_\ell - w_{\ell+1})|_{T'_+}| \;+\; |\nabla(w_\ell - w_{\ell+1})|_{T'_-}| \;+\; |\nu_E \cdot [\nabla \psi_\ell]_E|.\end{aligned}$$

Observing $|\omega_\ell^E| \approx |T'_\pm| \approx h_E^2$, it follows that

$$\eta_E^2 \;\lesssim\; |u_\ell - u_{\ell+1}|_{1,\omega_\ell^E}^2 \;+\; \mu_E^2(\psi_\ell),$$

which shows (5.18).
Case $(ii)_{2,2}$: We have

$$a(u_{\ell+1}, \varphi_{\ell+1}^{(m_E)}) \;\le\; (f, \varphi_{\ell+1}^{(m_E)})_{0,\Omega} \qquad \text{and} \qquad a(u_{\ell+1}, \varphi_{\ell+1}^{(p_+)}) \;=\; (f, \varphi_{\ell+1}^{(p_+)})_{0,\Omega}.$$

We construct $\varphi_{\ell+1}^{(E)}$ as in Case $(ii)_1$ and obtain

$$\begin{aligned}0 < \frac{1}{2}\int\limits_E \nu_E \cdot [\nabla u_\ell]_E\, ds \;&=\; \int\limits_E \nu_E \cdot [\nabla u_\ell]_E\, \varphi_{\ell+1}^{(E)}\, ds \\ &\le a(u_\ell - u_{\ell+1}, \varphi_{\ell+1}^{(E)}) +\, (f, \varphi_{\ell+1}^{(E)})_{0,\Omega_\ell^E} \\ &=\; a(u_\ell - u_{\ell+1}, \varphi_{\ell+1}^{(E)}) \,+\, (f - f_{\omega_\ell^E}, \varphi_{\ell+1}^{(E)})_{0,\omega_\ell^E},\end{aligned}$$

from which we deduce (5.18). □

Proof of Theorem 5.1. The upper bound (5.1) follows directly from (5.2), (5.3) in Lemma 5.2 and from (5.18) in Lemma 5.3 by summing over all $T \in \mathcal{M}_\ell^{(1)}$ and all $E \in \mathcal{M}_\ell^{(2)}$ and taking advantage of the finite overlap of the patches ω_ℓ^E. □

The final ingredient of the proof of the error reduction property is the following perturbed Galerkin orthogonality:

THEOREM 5.4. *Let $u \in V$ and $u_k \in V_k$, $k \in \{\ell, \ell+1\}$, be the solutions of* (2.2), (2.8)*, and let osc_ℓ and con_ℓ^{ort} be the data oscillations* (3.4) *and the consistency error* (3.10)*. Assume that* (3.7) *is satisfied. Then, for any $\varepsilon > 0$ there holds*

$$|u_\ell - u_{\ell+1}|_{1,\Omega}^2 \leq (1 + \frac{\varepsilon}{2})\, |u - u_\ell|_{1,\Omega}^2 \; - \; (1-\varepsilon)\, |u - u_{\ell+1}|_{1,\Omega}^2 \; + \\ + \frac{4}{\varepsilon} \rho_2 \, osc_\ell^2(f) \; + \; \frac{2}{\varepsilon}(1+\rho_3)\, osc_\ell^2(\psi) \; + \; con_\ell^{ort}. \tag{5.23}$$

Proof. By straightforward computation

$$|u_\ell - u_{\ell+1}|_{1,\Omega}^2 \; = \; |u - u_\ell|_{1,\Omega}^2 - |u - u_{\ell+1}|_{1,\Omega}^2 + 2\, a(u - u_{\ell+1}, u_\ell - u_{\ell+1})\,. \tag{5.24}$$

Now, (2.2) and (2.8) imply

$$2a(u - u_{\ell+1}, u_\ell - u_{\ell+1}) = 2(f - f_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega} + \\ + 2\Big(\langle\langle \sigma_{\ell+1}, u_\ell - u_{\ell+1} \rangle\rangle - \langle \sigma, u_\ell - u_{\ell+1} \rangle \Big). \tag{5.25}$$

Using that $f - f_{\ell+1}$ has zero integral mean on each $T \in \mathcal{T}_{\ell+1}$, applying Young's inequality and (3.5), we obtain

$$2\, |(f - f_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}| \; \leq \; \frac{\varepsilon}{2}\Big(|u - u_\ell|_{1,\Omega}^2 + |u - u_{\ell+1}|_{1,\Omega}^2 \Big) + \frac{4}{\varepsilon} \rho_2 osc_\ell^2(f). \tag{5.26}$$

On the other hand, taking advantage of $\sigma_{\ell+1} \in \mathcal{M}_+(\Omega)$, the complementarity condition (2.9), and $\sigma \in V_+^*$, we find

$$2\Big(\langle\langle \sigma_{\ell+1}, u_\ell - u_{\ell+1} \rangle\rangle - \langle \sigma, u_\ell - u_{\ell+1} \rangle \Big) \; = \\ = \; 2\underbrace{\langle\langle \sigma_{\ell+1}, u_\ell - \psi_\ell \rangle\rangle}_{\leq\, 0} \; + \; 2\Big(\langle\langle \sigma_{\ell+1} - \sigma, \psi_\ell - \psi_{\ell+1} \rangle\rangle - \langle \sigma, \psi_\ell - \psi_{\ell+1} \rangle \Big) \; + \\ + \, 2\underbrace{\langle\langle \sigma_{\ell+1}, \psi_{\ell+1} - u_{\ell+1} \rangle\rangle}_{=\, 0} \; + \; \underbrace{2\langle \sigma, \psi_\ell - u_\ell \rangle}_{=\, con_\ell^{ort}} \; - 2\, \underbrace{\langle \sigma, \psi_{\ell+1} - u_{\ell+1} \rangle}_{\leq\, 0}. \tag{5.27}$$

For the estimation of the second term on the right-hand side in (5.27) we set $\delta_{\psi_\ell} := \psi_\ell - \psi_{\ell+1}$ and recall (4.7) as well as

$$a(u_{\ell+1}, v_{\ell+1}) \; = \; (f, v_{\ell+1})_{0,\Omega} \; - \\ - \, (\nu_\Gamma \cdot \nabla u_{\ell+1}, v_{\ell+1})_{0,\Gamma} - \langle\langle \sigma_{\ell+1}, v_{\ell+1} \rangle\rangle \; , \; v_{\ell+1} \in S_{\ell+1}. \tag{5.28}$$

Since $\delta_{\psi_\ell} \in S_{\ell+1}$ is an admissible test function in (4.7) and (5.28), by the trace inequality (4.12) and by Young's inequality we find

$$|2\Big(\langle\langle \sigma_{\ell+1} - \sigma, \delta_{\psi_\ell} \rangle\rangle - \langle \sigma, \delta_{\psi_\ell} \rangle \Big)| \leq \\ \leq \; |2a(u - u_{\ell+1}, \delta_{\psi_\ell})| + |\langle \nu_\Gamma \cdot \nabla(u - u_{\ell+1}), \delta_{\psi_\ell} \rangle_\Gamma| \; \leq \\ \leq \; \frac{\varepsilon}{2}\, |u - u_{\ell+1}|_{1,\Omega}^2 \; + \; \frac{2}{\varepsilon}(1+\rho_3)\, osc_\ell^2(\psi). \tag{5.29}$$

Using (5.25)–(5.27) and (5.29) in (5.24) gives (5.23). □

We have now provided the prerequisites to prove the error reduction property (3.11) as stated in Theorem 3.1.

Proof of Theorem 3.1. The reliability (4.1), the bulk criterion (3.3a), (3.3b), the discrete efficiency (5.1), and the assumption (3.7) imply the existence of a constant $C > 0$, depending only on Θ and on the local geometry of the triangulation, such that

$$|u - u_\ell|^2_{1,\Omega} \;\leq\; C \left(|u_\ell - u_{\ell+1}|^2_{1,\Omega} + osc^2_\ell + con^{ref}_\ell \right) .$$

Now, invoking the perturbed Galerkin orthogonality (5.23), we deduce

$$|u - u_{\ell+1}|^2_{1,\Omega} \;\leq\; \frac{C(1+\varepsilon/2) - 1}{C(1-\varepsilon)} \, |u - u_{\ell+1}|^2_{1,\Omega} \;+\; CC_\varepsilon \left(osc^2_\ell + \mu^2_\ell \right) + C \; con_\ell,$$

where $C_\varepsilon := \max((4/\varepsilon + \varepsilon/2)\rho_2, 8(1+\rho_3)/\varepsilon)$. Together with (3.5) this proves (3.11) with $\rho_1 := (C(1+\varepsilon/2) - 1)/(C(1-\varepsilon)) < 1$ for $\varepsilon < 2/(3C)$. $\square$

6. Numerical results. In this section, we provide a detailed documentation of the convergence history of the AFEM for two illustrative elliptic obstacle problems.

Example 1. We consider an obstacle problem of the form (2.1) in an L-shaped domain where the obstacle is an 'inverted' pyramid. The data are as follows

$$\Omega \;:=\; (-2,2)^2 \setminus ([0,2] \times [-2,0]) \quad , \quad \psi(x) \;:=\; 0.5(2.01 - \mathrm{dist}(x, \partial[-2,2]^2)) \;, \; x \in \overline{\Omega},$$
$$f(r,\varphi) \;:=\; -r^{2/3} \sin(2\varphi/3)(\gamma_1'(r)/r + \gamma_1''(r)) - \frac{4}{3} r^{-1/3} \gamma_1'(r) \sin(2\varphi/3) - \gamma_2(r) \;,$$
$$\gamma_1(r) \;=\; \begin{cases} 1, & \bar{r} < 0, \\ -6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1, & 0 \leq \bar{r} < 1, \\ 0, & \bar{r} \geq 1, \end{cases} \qquad \gamma_2(r) \;=\; \begin{cases} 0 \;, \; r \leq 5/4, \\ 1 \;, \; \text{elsewhere}, \end{cases}$$

where $\bar{r} = 2(r - 1/4)$ and (r, φ) stand for polar coordinates.

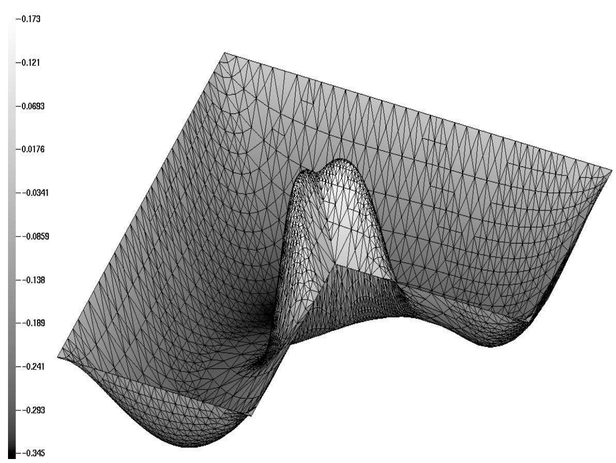

FIG. 6.1. *Visualization of the solution of the obstacle problem in Example 1*

Figure 6.1 displays a visualization of the solution, whereas Figure 6.2 shows the adaptively generated finite element meshes after 7 (left) and 10 (right) refinement steps of the adaptive loop ($\Theta = 0.6$ in the bulk criterion (3.3), (3.3a)). The coincidence set is a small

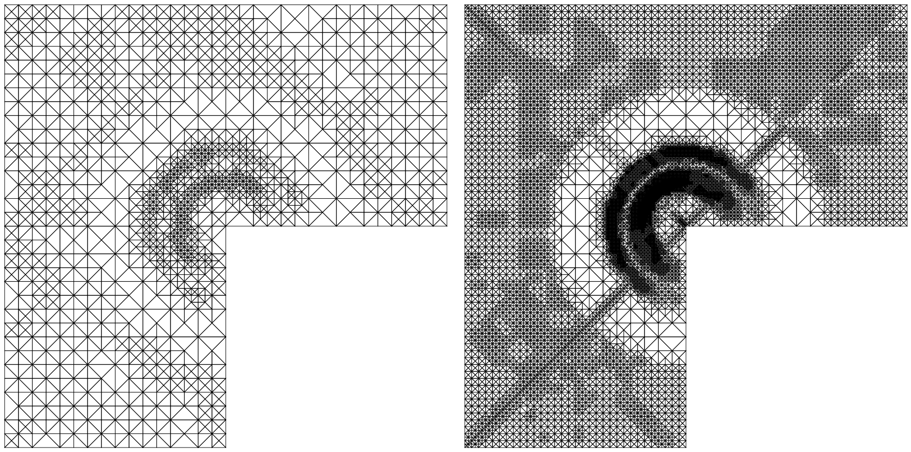

FIG. 6.2. *Adaptive refined grid after 7 (left) and 10 (right) refinement steps ($\Theta = 0.6$ in the bulk criterion)*

region at the upper fore side of the hill-like structure seen in Figure 6.1 where the solution is in contact with the inverted pyramid. We see that the refinement is dominant along the diagonal and in a circular region around the reentrant corner where the solution exhibits singular behavior.

Table 6.1 reflects the convergence history of the AFEM where ℓ stands for the refinement level and N_ℓ for the total number of degrees of freedom at level ℓ. Further, ε_ℓ, η_ℓ, $osc_\ell(f)$, and $\mu_\ell(\psi)$ denote the energy norm of the discretization error, the error estimator, and the data oscillations in f and ψ, respectively. The quantity $M_{\eta,\ell}$ refers to the percentage of elements/edges refined at level ℓ due to the bulk criterion (3.3a), (3.3b). Finally, $M_{osc,\ell}$ denotes the percentage of additional elements/edges that had to be refined in order to guarantee a reduction of the data oscillations.

TABLE 6.1
Convergence history of the adaptive refinement process in Example 1

ℓ	N_ℓ	ε_ℓ	η_ℓ	$osc_\ell(f)$	$\mu_\ell(\psi)$	$M_{\eta,\ell}$	$M_{osc,\ell}$
1	15	1.19e+00	5.61e+00	7.96e+00	2.45e+00	49.5	34.9
2	37	1.09e+00	5.57e+00	5.29e+00	1.73e+00	33.1	19.4
3	76	7.18e-01	3.90e+00	2.07e+00	1.37e+00	27.3	15.4
4	171	5.08e-01	2.70e+00	8.12e-01	1.09e+00	33.4	14.1
5	361	3.38e-01	1.82e+00	3.78e-01	8.79e-01	36.7	9.9
6	851	2.16e-01	1.20e+00	2.22e-01	7.29e-01	31.0	3.2
7	1596	1.54e-01	8.52e-01	1.46e-01	6.06e-01	34.5	3.6
8	3273	1.06e-01	5.85e-01	7.29e-02	5.04e-01	34.1	2.4
9	6356	7.54e-02	4.17e-01	4.50e-02	4.21e-01	35.2	2.0
10	12340	5.41e-02	2.98e-01	2.57e-02	3.51e-01	35.4	1.2
11	23988	3.90e-02	2.16e-01	1.60e-02	2.92e-01	34.4	0.9
12	45776	2.79e-02	1.56e-01	9.63e-03	2.44e-01	35.4	0.6
13	88439	1.99e-02	1.14e-01	5.92e-03	2.04e-01	36.0	0.4
14	166926	1.37e-02	8.36e-02	3.46e-03	1.71e-01	33.8	0.3

Figure 6.3 displays the energy norm of the discretization error ε_ℓ as a function of the degrees of freedom (DOFs) for adaptive and uniform refinement. We see that in this case the adaptive refinement is only slightly beneficial with both refinements showing the same rate of convergence.

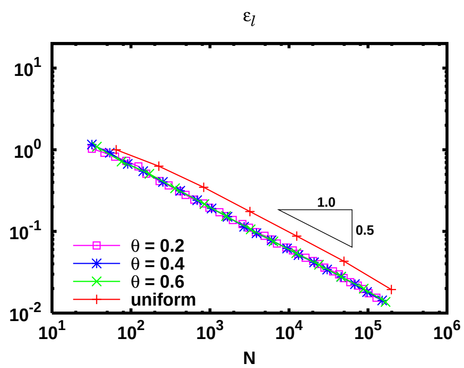

FIG. 6.3. *Energy norm of the error as a function of the DOFs for adaptive and uniform refinement in Example 1*

Example 2. We consider the torsion of an elastic, perfectly plastic cylindrical bar $Q := \Omega \times (0, L)$ of cross section $\Omega \subset \mathbb{R}^2$ and length $L > 0$. Denoting by $\partial Q_L := \Omega \times \{L\}$, $\partial Q_0 := \Omega \times \{0\}$, and $\partial Q_s := \partial\Omega \times (0, L)$ the top and the bottom of the bar as well as its lateral surface, at ∂Q_L the bar is twisted about the x_3-axis by an angle $\theta > 0$, whereas ∂Q_s is supposed to be stress free.

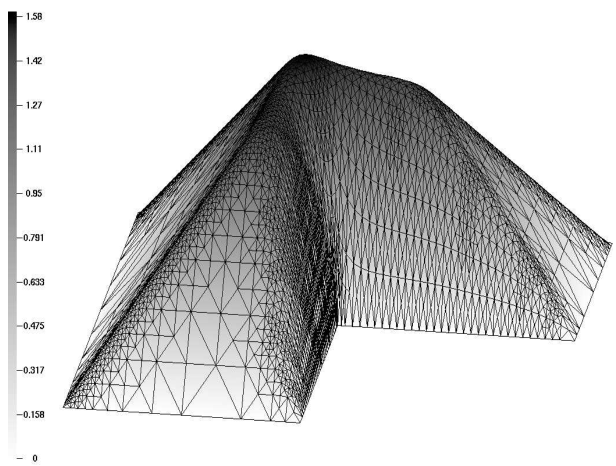

FIG. 6.4. *Visualization of the solution of the elastic-plastic problem*

Using Hencky's law for an isotropic material, modeling the plastic region by the von Mises yield criterion, and normalizing physical constants, it can be shown that the equilibrium stress tensor $\sigma = (\sigma_{ij})_{i,j=1}^3$ is given by $\sigma_{ij} = \partial u/\partial x_2$, $(i, j) \in \{(1, 3), (3, 1)\}$, $\sigma_{ij} = -\partial u/\partial x_1$, $(i, j) \in \{(2, 3), (3, 2)\}$, and $\sigma_{ij} = 0$ otherwise. Here $u \in H_0^1(\Omega)$ is the solution of the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v - u)\, dx \;\geq\; 2C \int_\Omega (v - u)\, dx\,, \qquad v \in K, \tag{6.1}$$

and K stands for the closed, convex set

$$K \;:=\; \{v \in H_0^1(\Omega) \mid v \leq \psi := \mathrm{dist}(\cdot, \partial\Omega) \text{ a.e. on } \Omega\}.$$

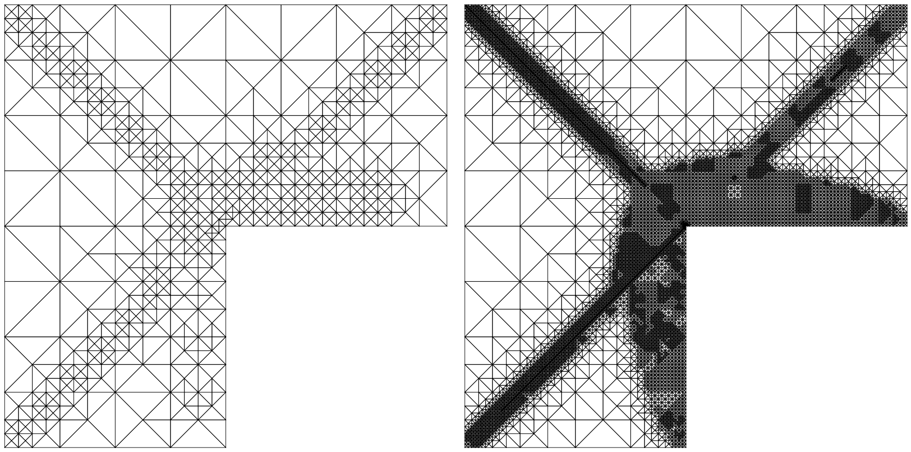

FIG. 6.5. *Adaptive refined grid after 7 (left) and 12 (right) refinement steps ($\Theta = 0.6$ in the bulk criterion)*

We have chosen Ω as the L-shaped domain $\Omega := (-2, 2)^2 \setminus ([0, 2] \times [-2, 0])$ and $C = 5$.

The computed solution and adaptively refined grids after 7 (left) and 12 (right) refinement steps ($\Theta = 0.6$ in the bulk criterion (3.3a), (3.3b)) are shown in Figure 6.4 and 6.5. The coincidence and non-coincidence sets correspond to the plastic and elastic region, respectively. The non-coincidence set consists of the union of a neighborhood of the edges forming the reentrant corner and a neighborhood around the diagonals. As can be expected from the properties of the solution, the refinement is concentrated within the non-coincidence set.

The convergence history of the AFEM is documented in Table 5.2 with the same notations as in the first example. Since the right-hand side in the variational inequality is a constant, the associated data oscillations are zero. Figure 6.6 displays the energy norm of the discretization error as a function of the degrees of freedom for adaptive and uniform refinement and demonstrates the benefits of the adaptive approach for this example.

TABLE 6.2
Convergence history of the adaptive refinement process in Example 2

ℓ	N_ℓ	ε_ℓ	η_ℓ	$\mu_\ell(\psi)$	$M_{\eta,\ell}$	$M_{\mu,\ell}$
2	65	2.49e+00	8.42e+00	3.46e+00	7.5	6.2
3	84	1.95e+00	4.99e+00	2.83e+00	10.9	4.3
4	113	1.73e+00	5.73e+00	2.29e+00	9.8	4.9
5	192	1.21e+00	5.91e+00	1.90e+00	18.3	4.1
6	336	9.26e-01	4.72e+00	1.57e+00	18.6	2.6
7	533	7.21e-01	3.67e+00	1.26e+00	20.1	3.6
8	1151	5.22e-01	2.49e+00	1.05e+00	20.0	1.3
9	1849	3.77e-01	1.77e+00	8.79e-01	25.2	2.1
10	3373	2.69e-01	1.30e+00	7.36e-01	24.2	0.9
11	5720	2.01e-01	9.50e-01	6.15e-01	26.2	1.4
12	11014	1.47e-01	6.85e-01	5.14e-01	27.1	0.5
13	19461	1.08e-01	5.06e-01	4.30e-01	26.1	0.8
14	34942	7.73e-02	3.71e-01	3.60e-01	31.8	0.4
15	67114	5.52e-02	2.75e-01	3.01e-01	26.5	0.4
16	123427	3.75e-02	2.01e-01	2.52e-01	30.8	0.2

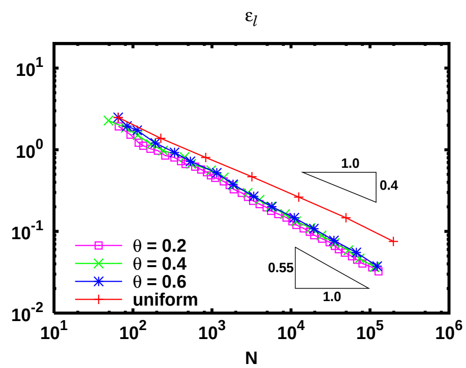

FIG. 6.6. *Energy norm as a function of the DOFs for adaptive and uniform refinement*

REFERENCES

[1] M. AINSWORTH AND J.T. ODEN, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley, Chichester 2000.

[2] M. AINSWORTH, J.T. ODEN, AND C.Y. LEE, *Local a posteriori error estimators for variational inequalities*, Numer. Methods for Partial Differential Equations, 9 (1993), pp. 23–33.

[3] I. BABUŠKA AND T. STROUBOULIS, *The Finite Element Method and its Reliability*, Clarendon Press, Oxford 2001.

[4] W. BANGERTH AND R. RANNACHER, *Adaptive Finite Element Methods for Differential Equations*, Lectures in Mathematics. ETH-Zürich. Birkhäuser, Basel 2003.

[5] S. BARTELS AND C. CARSTENSEN, *Averaging techniques yield reliable a posteriori finite element error control for obstacle problems*, Numer. Math., 99 (2004), pp. 225–249.

[6] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219–268.

[7] H. BLUM AND F.T. SUTTMEIER, *An adaptive finite element discretizaion for a simplified Signorini problem*, Calcolo, 37 (2000), pp. 65–77.

[8] D. BRAESS, *A posteriori error estimators for obstacle problems – another look*, Numer. Math., 101 (2005), pp. 415–421.

[9] D. BRAESS, C. CARSTENSEN, AND R.H.W. HOPPE, *Convergence analysis of a conforming adaptive finite element method for an obstacle problem*, Numer. Math., 107 (2007), pp. 455–471.

[10] D. BRAESS, R.H.W. HOPPE, AND J. SCHÖBERL, *A posteriori estimators for obstacle problems by the hypercircle method*, to appear in J. Comp. Visual. Sci. (2008).

[11] C. CARSTENSEN AND R.H.W. HOPPE, *Convergence analysis of an adaptive edge finite element method for the 2d eddy current equations*, J. Numer. Math., 13 (2005), pp. 19–32.

[12] C. CARSTENSEN AND R.H.W. HOPPE, *Error reduction and convergence for an adaptive mixed finite element method*, Math. Comp., 75 (2006), pp. 1033–1042.

[13] C. CARSTENSEN AND R.H.W. HOPPE, *Convergence analysis of an adaptive nonconforming finite element method*, Numer. Math., 103 (2006), pp. 251–266.

[14] Z. CHEN AND R.H. NOCHETTO, *Residual type a posteriori error estimates for elliptic obstacle problems*, Numer. Math., 84 (2000), pp. 527–548.

[15] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.

[16] K. ERIKSSON, D. ESTEP, P. HANSBO, AND C. JOHNSON, *Computational Differential Equations*, Cambridge University Press, Cambridge 1995.

[17] R. S. FALK, *Error estimates for the approximation of a class of variational inequalities*, Comp. Math., 28 (1974), pp. 963–971.

[18] M. HINTERMÜLLER AND R.H.W. HOPPE (2007). *Goal-oriented adaptivity in control constrained optimal control of partial differential equations.* to appear in SIAM J. Control Optim.

[19] R.H.W. HOPPE AND R. KORNHUBER, *Adaptive multilevel methods for obstacle problems*, SIAM J. Numer. Anal., 31 (1994), pp. 301–323.

[20] C. JOHNSON, *Adaptive finite element methods for the obstacle problem*, Math. Models Methods Appl. Sci., 2 (1992), pp. 483–487.

[21] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities*, Academic Press,

New York 1980.

[22] R. LI, W. LIU, H. MA, AND T. TANG (2002). *Adaptive finite element approximation for distributed elliptic optimal control problems.* SIAM J. Control Optim. **41**, 1321–1349.

[23] W. LIU AND N. YAN, *A posteriori error estimates for a class of variational inequalities*, J. Sci. Comput., 15 (2000), pp. 361–393.

[24] P. MORIN, R.H. NOCHETTO AND K.G. SIEBERT, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. 38 (2000), pp. 466–488.

[25] P. NEITTAANMÄKI AND S. REPIN, *Reliable Methods for Mathematical Modelling. Error Control and A Posteriori Estimates*, Elsevier, New York 2004.

[26] R.H. NOCHETTO, K.G. SIEBERT, AND A. VEESER, *Pointwise a posteriori error control for elliptic obstacle problems*, Numer. Math., 95 (2003), pp. 163–195.

[27] R.H. NOCHETTO, K.G. SIEBERT, AND A. VEESER, *Fully localized a posteriori error estimators and barrier sets for contact problems*, SIAM J. Numer. Anal., 42 (2005), pp. 2118–2135.

[28] K. SIEBERT AND A. VEESER, *A unilaterally constrained quadratic minimization with adaptive finite elements*, SIAM J. Optimization, 18 (2007), pp. 260–289.

[29] R. STEVENSON, *Optimality of a standard adaptive finite element method*, Fund. Math., 108 (2007), pp. 93–119.

[30] F.T. SUTTMEIER, *On a direct approach to adaptive FE-discretisations for elliptic variational inequalities*, J. Numer. Math., 13 (2005), pp. 73–80.

[31] A. VEESER, *Efficient and reliable a posteriori error estimators for elliptic obstacle problems*, SIAM J. Numer. Anal., 39 (2001), pp. 146–167.

[32] A. VEESER, *Convergent adaptive finite elements for the nonlinear Laplacian*, Numer. Math., 92 (2002), pp. 743–770.

[33] R. VERFÜRTH, *A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, New York – Stuttgart 1996.