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A NOTE ON MATRIX METHODS FOR LOCATION OF THE ZEROS OF POLYNOMIALS

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Abstract. It is shown that the optimal Gerschgorin disc for a companion matrix gives in fact the classical location of the zeros of a polynomial due to Cauchy. Some refinements and modifications of Eneström-Kakeya type theorems and a concrete application are discussed.

1. Introduction

Let $p(z)$ be a complex monic polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_0 \neq 0,$$

of degree $n \geq 1$; the restriction to $a_n = 1$ and $a_0 \neq 0$ throughout this note is not essential but simplifies notation. In [3, Satz 1], [2, Lemma 2] a matrix method was applied to prove that all zeros of $p(z)$ lie in the disc centered at the origin,

$$|z| \leq \max\left\{r, \sum_{j=0}^{n-1} \frac{|a_j|}{r^{n-j-1}}\right\}, \tag{1.1}$$

where r is an arbitrary positive number. Since (1.1) is used in [2] to obtain several generalizations of the famous Eneström-Kakeya theorem (see below), we will have a closer look at this matrix method in both optimality and comparison with Cauchy's theorem for the localization of polynomial zeros [7].

It is known (cf. [7, p. 144]) that Cauchy's bound can be obtained from (1.1): The minimal bound for the modulus of all the zeros of $p(z)$ is the (unique) positive root r^* of the equation

$$r^{*n} = \sum_{j=0}^{n-1} |a_j| r^{*j}. \tag{1.2}$$

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The bound r^* for the modulus of all the zeros of $p(z)$ is due to Cauchy in 1829; see, e.g., [7, Theorem(27,1)]. Therefore, r^* is sometimes called the *Cauchy radius* of p.

In this note it will be proved that the optimal bound which can be obtained by matrix methods, namely the "optimal Gerschgorin radius," is equal to the Cauchy radius. Therefore, matrix method used in the literature give better estimates than simple calculations merely using triangle inequalities.

The situation becomes different if the variable is shifted. This is discussed in the example of the Eneström-Kakeya theorem.

2. Examples and Refinement

Many Eneström-Kakeya type results given in the literature, e.g., [1, Theorem 1], [2], [5, Theorem 2], [6, Theorems 1,2,3], [9, Theorem 1] and a lot of corollaries about bounds r for the modulus of the zeros of $p(z)$ can be proved as follows:

Let $r > 0$ be explicitly given as a function of the coefficients $a_0, \ldots, a_{n-1}$ (which may have some particular properties). Then (using the particular properties) verify

$$h(r) := \sum_{j=0}^{n-1} \frac{|a_j|}{r^{n-1}} \le 1, \tag{2.1}$$

which shows $r^* \le r$. Hence, the mentioned results are all corollaries to Cauchy's estimate.

Up to particular cases in which r^* is reached (they can be attained in many cases by a rigorous discussion of the equality in the triangle inequality), there holds $h(r) < 1$. Hence $r^* < r$ and the known bound r can be improved.

Since h is strict convex and monotonically decreasing, one can compute the Cauchy radius r^* solving (1.2) or $h(r^*) = 1$, h given in (2.1), by Newton-Raphson's method or regula falsi for instance. Given r_0 with $h(r_0) < 1$ we have, e.g.,

$$r_2 := h(r_0) \cdot r_0 < r_1 := \sqrt[n]{h(r_0)} \cdot r_0,$$

which follows from $h(r_2) > 1 > h(r_1) > h(r_0)$.

3. Optimal Gerschgorin discs

Let ζ be a fixed complex number. Then the characteristic polynomial of the companion matrix (shifted by ζ)

$$B = \begin{bmatrix} -\zeta & 1 & & \\ & \ddots & \ddots & \\ & & -\zeta & 1 \\ -a_0 & \ldots & -a_{n-2} & -\zeta - a_{n-1} \end{bmatrix}$$

is equal to $(-1)^n p(\zeta+z)$. Therefore, Gerschgorin's theorem can be applied to $T^{-1}\cdot B\cdot T$, where, for instance, the regular matrix T may be diagonal with positive entries $t_1,\dots,t_n$. It follows that all zeros of $p(\zeta+z)$ lie in the disc $|z|\le R$ where

$$R := \max\left\{|\zeta| + \frac{t_2}{t_1}, |\zeta| + \frac{t_3}{t_2}, \dots, |\zeta| + \frac{t_n}{t_{n-1}}, |\zeta + a_{n-1}| + \sum_{j=0}^{n-2} |a_j| \frac{t_{j+1}}{t_n}\right\}. \tag{3.1}$$

Hence, all zeros of $p(z)$ lie in the disc $|\zeta - z| \le R$.

Naturally, we are interested in the smallest R which can be obtained in (3.1) by an appropriate choice of $t_1,\dots,t_n > 0$. This problem was considered in a more general situation by Varga, Medley, Todd, Elzner and others, cf. [4], [8] and the reference given there. In this particular case there holds.

Theorem 3.1. *The smallest bounds $R^* > |\zeta|$ in (3.1) is given as the (unique) root of*

$$h(R^*) = 1, \quad h(R) := \sum_{j=0}^{n-2} \frac{|a_j|}{(R - |\zeta + a_{n-1}|)\cdot(R - |\zeta|)^{n-j-1}}, \quad R > |\zeta|. \tag{3.2}$$

This estimate is best possible in the sense of minimal Gerschgorin discs, that is, for $t_j = (R^* - |\zeta|)^{j-1}$, $j = 1,\dots,n$, (3.1) gives $R = R^*$ while for any other choice of $t_1,\dots,t_n > 0$ (3.1) gives $R \ge R^*$.

Proof. We only give an elementary proof for optimality: If we can find $t_1,\dots,t_n > 0$ such that (3.1) gives $R < R^*$ then

$$\frac{t_j}{t_n} = \prod_{k=j}^{n-1} \frac{t_k}{t_{k+1}} > \frac{1}{(R^* - |\zeta|)^{n-j}}, \quad j = 1,\dots,n.$$

Therefore

$$R^* - |\zeta + a_{n-1}| = \sum_{j=0}^{n-2} \frac{|a_j|}{(R^* - |\zeta|)^{n-j-1}} < \sum_{j=0}^{n-2} |a_j| \frac{t_{j+1}}{t_n} \le R - |z + a_{n-1}|,$$

which contradicts $R < R^*$. Moreover, $R = R^*$ is only possible for

$$t_j = t\cdot(R^* - |\zeta|)^{j-1},\ j = 1,\dots,n,\ t > 0. \quad \square$$

Remark 3.1. (i) Note that for $\zeta = 0$ Theorem 3.1 states that the *optimal Gerschgorin radius* R^* is equal to the *Cauchy radius* r^*.

(ii) Note also that $R^* = |\zeta| + r^*$ if and only if $|a_{n-1}| + |\zeta| = |a_{n-1} + \zeta|$. Otherwise we obtain $R^* - |\zeta| < r^*$ which gives additional information about the zeros of $p(z)$. Therefore, matrix methods are advantageous using a shift.

(iii) We remark that the location of Theorem 3.1, namely $|z-\zeta| \le R^*$ for all the zeros of $p(z)$, can be proved by obvious modifications as in Cauchy's theorem merely using triangle inequalities.

Naturally, Greschgorin's theorem can also be used to obtain exclusion discs. By some obvious modifications of the arguments given above one can prove

Theorem 3.2. *If there exists R' with $0 < R' < \min\left\{|\zeta|, |\zeta + a_{n-1}|\right\}$ such that*

$$1 = \sum_{j=0}^{n-2} \frac{|a_j|}{(|\zeta + a_{n-1}| - R')\cdot(|\zeta| - R')^{n-j-1}},$$

then the polynomial $p(z)$ does not vanish in the disc $|z-\zeta| < R'$. This estimate is best possible in the sense of minimal Gerschgorin discs and it is obtained by taking $t_j = (|\zeta| - R')^{j-1}$, $j = 1, \dots, n$.

4. Application of Eneström-Kakeya theorem

If the coefficient of the polynomial p satisfy

$$0 =: a_{-1} < a_0 \le a_1 \le \cdots \le a_{n-1} \le a_n =: 1 \tag{4.1}$$

then the famous Eneström-Kakeya theorem states that all the zeros of $p(z)$ lie in the unit disc $|z| \le 1$. As usual, we consider the polynomial

$$g(z) := (z-1)\cdot p(z) = z^{n+1} + \sum_{j=0}^{n} (a_{j-1} - a_j) z^j$$

and note that, due to (4.1), the Cauchy radius $\hat{r}$ of g is equal to 1, which proves the Eneström-Kakeya theorem. Hence, this result cannot be improved by the classical Cauchy theorem although some refinements are possible; see, e.g., [5, Theorem D.5].

As an application of Theorem 3.1 we discuss the influence of the shift ζ in the situation (4.1). We assume

$$1 - a_{n-1} =: \epsilon > 0 \quad \text{and} \quad \hat{R} - |\zeta| =: \hat{\rho} > 0 \tag{4.2}$$

and apply Theorem 3.1 to $g(z)$ (using the definitions $\hat{r}$ and $\hat{R}$ for $g(z)$ analogously to r^* and R^* for $p(z)$) such that (3.2) becomes $h(\hat{R}) = 1$ with

$$\begin{aligned} h(|\zeta| + \rho) &= \sum_{j=0}^{n-1} \frac{a_j - a_{j-1}}{(\rho + |\zeta| - |\zeta + a_{n-1} - a_n|)\rho^{n-j}} \\ &= \frac{\dfrac{a_{n-1}}{\rho} + (1-\rho)\cdot \displaystyle\sum_{j=0}^{n-2} \frac{a_j}{\rho^{n-j}}}{\rho + |\zeta| - |\zeta - \epsilon|}, \end{aligned} \tag{4.3}$$

for $\rho > 0$. Note that $\hat{\rho} \in (0,1]$ and $\hat{\rho} = 1$ if and only if $\epsilon + |\zeta| = |\zeta - \epsilon|$. Since h is strictly monotonously decreasing we claim $\hat{\rho} \le \rho$ whenever we can find $\rho \in (0,1]$ with $h(|\zeta| + \rho) \le 1$. For instance we have, by (4.1),

$$h(|\zeta| + \rho) \le \frac{a_{n-1}/\rho + a_{n-2} \cdot (1/\rho^n - 1/\rho)}{\rho + |\zeta| - |\zeta - \epsilon|} \le \frac{a_{n-1}/\rho^n}{\rho + |\zeta| - |\zeta - \epsilon|}. \tag{4.4}$$

If we compute ρ such that the last expression is equal to 1, then we can claim

Theorem 4.1. *Let $\tilde{\rho}$ denote the positive root of*

$$0 = \tilde{\rho}^{n+1} + (|\zeta| - |\zeta - 1 + a_{n-1}|) \cdot \tilde{\rho}^n - a_{n-1}.$$

Then all the zeros of the polynomial $p(z)$ with the coefficients (4.1) lie in the annulus

$$|\zeta - z| \le \hat{R} \le |\zeta| + \tilde{\rho} \le |\zeta| + 1.$$

Remark 4.1. (iv) Note that $\hat{R} = \zeta + \tilde{\rho}$ if and only if $\hat{\rho} = 1$ or $p(z) = z^n + a_{n-1} \cdot (z^{n-1} + \cdots + z + 1)$.

(v) Note that we obtain nothing new if $a_{n-1} = 1$ and that the last example shows that in general the assumption $\epsilon > 0$, that is, $a_{n-1} < 1$, cannot be dropped.

Theorem 4.1 has some interesting corollaries, provided $a_{n-1} < 1$. Indeed, we refine the Eneström-Kakeya theorem if and only if we choose $\zeta \notin (-\infty, 0]$. For instance, $\zeta = (1 - a_{n-1})/2 + iy$ yields $\tilde{\rho} = \sqrt[n+1]{a_{n-1}}$ and $y \to -\infty, 0, +\infty$ leads to

$$\begin{aligned} |z| < 1 \quad \wedge \quad & |z - (1 - a_{n-1})/2| < (1 - a_{n-1})/2 + \sqrt[n+1]{a_{n-1}} \\ \wedge \quad & -\sqrt[n+1]{a_{n-1}} < Im\, z < \sqrt[n+1]{a_{n-1}}, \end{aligned}$$

while $\zeta = 1 - a_{n-1}$ further implies $\tilde{\rho} \le \sqrt[n]{a_{n-1}}$ and

$$|z - 1 + a_{n-1}| < 1 - a_{n-1} + \sqrt[n]{a_{n-1}}$$

for all zeros of $p(z)$.

Finally, we mention that Theorem 4.1 can easily be refined by computing $\rho > 0$ such that the first bound in (4.4) is equal to 1. Moreover, as mentioned above, any bound $R = \rho + |\zeta|$ with $h(R) \le 1$ can be improved, e.g., by $R \cdot \sqrt[n+1]{h(R)}$.

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