Consistency Properties of Systemic Risk Measures

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Abstract

A systemic risk measure for a network of financial positions, as proposed by Chen, Iyengar, and Moallemi [6], involves an aggregation procedure and a convex risk measure that is applied to the aggregate position. We regard this structural decomposition as a consistency property with respect to a suitable σ-field. From this point of view, the dual representation of a systemic risk measure reduces to a criterium for consistency that is well known in the context of time-consistency. We also discuss conditions for spatial consistency and connect them with the analysis of spatial risk measures in [13] and [14].

1 Introduction

Consider a collection of financial positions, one for each node of a financial network, where the net gain of the position at node i is described by a real-valued function X_i on some set of possible scenarios. In view of an asymptotic analysis of large finite networks, we include the case where the set I of nodes is countably infinite.

To quantify the collective risk for such a collection, Chen, Iyengar, and Moallemi [6] introduced the notion of a systemic risk measure, defined as a functional $\bar{\rho}$ on the linear space $\bar{\mathcal{X}}$ of all collections $\bar{X} = (X_i)_{i \in I}$ that satisfies certain axioms. These axioms imply that the functional $\bar{\rho}$ is convex on $\bar{\mathcal{X}}$, and they are shown to be equivalent to a structural decomposition

$$\bar{\rho}(\bar{X}) = \rho(\Lambda(\bar{X})), \tag{1}$$

where Λ is an aggregation function on R^I that associates to each collection $\bar{X}$ an aggregate position $X = \Lambda(\bar{X})$, and where ρ is standard risk measure for real-valued positions. This is illustrated by the systemic risk measures discussed by Brunnenmaier and Cheridito [5], where ρ is taken to be a utility-based shortfall risk measure as introduced in [17]. In [6] the underlying set of scenarios is finite; for extensions to a general setting see [22], [2], and [21].

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In this paper we use the structural decomposition (1) as a starting point. Moreover, we focus on the case where the systemic risk measure $\bar{\rho}$ is a convex risk measure in the sense of [17] and [19] and can thus be interpreted as a global capital requirement. This means that, in addition to the properties required in [6], $\bar{\rho}$ has the monetary property of cash invariance. Since cash invariance is inherited by the aggregation function Λ in (1), this excludes some of the examples discussed in the literature; see, however, Remark 14.

Collections $\bar{X} \in \bar{\mathcal{X}}$ can be identified with real-valued functions on a product space. Since the systemic risk measure $\bar{\rho}$ on $\bar{\mathcal{X}}$ is assumed to be cash-invariant, the dual representation of $\bar{\rho}$ discussed in [6] and [22] now follows from standard representation results for convex risk measures in terms of suitably penalized probability measures. This is explained in Section 4. In order to determine the systemic penalty function appearing in the dual representation, we observe that the structural decomposition (1) can be regarded as a consistency condition. More precisely, $\bar{\rho}$ is consistent with a conditional convex risk measure defined in terms of the aggregation function Λ. Thus we can apply a well known criterium for consistency that has been discussed in the literature on time-consistent risk measures; cf., for example, [1] or [18, Chapter 11]. This yields a description of the systemic penalty function for $\bar{\rho}$ in terms of the penalty functions associated to ρ and Λ; see Theorem 17, and also [6] and [22] for closely related results.

There are other consistency conditions that may be relevant. In Section 5 we use spatial consistency conditions to introduce the local specification of a systemic risk measures, in analogy to the local specification of a Gibbs measure in terms of local conditional probability distributions; see [10] or [20]. For each finite set of nodes $V \subset I$ we fix a conditional risk measure $\bar{\rho}_V$ on the local collections $(X_i)_{i \in V}$ that depends on the situation outside of V and admits a structural decomposition as in (1); an axiomatic characterization of such conditional systemic risk measures is given in [21]. We assume that the family $(\bar{\rho}_V)$ is spatially consistent, that is, $\bar{\rho}_W(-\bar{\rho}_V) = \bar{\rho}_V$ for $V \subseteq W$. In analogy with the theory of Gibbs measures, our aim is to clarify the structure of the set of all global systemic risk measures $\bar{\rho}$ that are consistent with the local specification $(\bar{\rho}_V)$; in particular, this involves criteria for existence and uniqueness. As a first step in this direction, we show how these questions are connected to the analysis of spatial risk measures in [13] and [14]. Further results will be discussed in [15].

2 Preliminaries

In this section we recall some basic notions and facts about risk measures that will be used later on. For further details see, for example, [18, Chapter 4 and 11].

Let $\mathcal{X}$ denote the space of all bounded measurable functions on some underlying measurable space $(\Omega, \mathcal{F})$ of possible scenarios. A function $X \in \mathcal{X}$ is interpreted as the P&L of a financial position: for any scenario $\omega \in \Omega$, the value $X(\omega)$ denotes the resulting discounted net worth of the position at the end of a given trading period.

Definition 1. *A real-valued functional ρ on $\mathcal{X}$ is called a* monetary risk measure *if it is*

- *cash invariant, i..e., $\rho(X+m) = \rho(X) - m$ for any constant m,*
- *monotone, i.e. $X \leq Y \;\Rightarrow\; \rho(X) \geq \rho(Y)$.*
- *normalized, i.e., $\rho(0) = 0$.*

A monetary risk measure ρ is called a convex *risk measure if the functional ρ is convex on $\mathcal{X}$, i.e.,*

- $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$

for any $\lambda \in [0,1]$.

Any monetary risk measure ρ is Lipschitz-continuous with respect to the supremum norm $||X|| := \sup_{\omega\in\Omega} |X(\omega)|$, that is,

$$|\rho(X) - \rho(Y)| \leq ||X - Y||. \tag{2}$$

Moreover, ρ is uniquely determined by the associated *acceptance set*

$$\mathcal{A} := \left\{ X \in \mathcal{X} \;\middle|\; \rho(X) \leq 0 \right\},$$

since

$$\rho(X) = \inf \left\{ m \in R^1 \;\middle|\; X + m \in \mathcal{A} \right\}.$$

Thus $\rho(X)$ has the financial interpretation of a *capital requirement*: it is the minimal amount which should be added to the position X to make it acceptable. The risk measure ρ is convex if and only if its acceptance set $\mathcal{A}$ is convex.

Typically, a convex risk measure ρ on $\mathcal{X}$ admits a *dual representation* of the form

$$\rho(X) = \sup_Q \left(E_Q\left[-X\right] - \alpha(Q) \right), \tag{3}$$

where the supremum is taken over probability measures Q on $(\Omega, \mathcal{F})$, and where the penalty function α is defined in terms of the acceptance set $\mathcal{A}$ by

$$\alpha(Q) := \sup_{X\in\mathcal{A}} E_Q\left[-X\right] \in [0,\infty]. \tag{4}$$

Such a representation requires additional regularity properties. A necessary condition is that ρ should be *continuous from above*, i.e.,

$$X_n \searrow X \;\;\Longrightarrow \rho(X_n) \nearrow \rho(X)$$

for any uniformly bounded sequence (X_n) in $\mathcal{X}$. *Continuity from below*, defined in an analogous manner, is a stronger condition, which is equivalent to the following *Lebesgue property*:

$$X_n \longrightarrow X \;\;\Longrightarrow \rho(X_n) \longrightarrow \rho(X)$$

for any uniformly bounded sequence (X_n) in $\mathcal{X}$.

Let P be a probability measure on $(\Omega, \mathcal{F})$.

Definition 2. *We say that ρ is* absolutely continuous *with respect to P, and we write $\rho \ll P$, if*

$$X = Y \ P\text{-}a.s. \implies \rho(X) = \rho(Y).$$

ρ is called sensitive *with respect to P if $P[A] > 0$ implies $\rho(-\lambda I_A) > 0$ for large enough $\lambda > 0$.*

If $\rho \ll P$ then ρ can be regarded as a convex risk measure on the Banach space $\ell^\infty(\Omega, \mathcal{F}, P)$. As shown in [17] or [18, Theorem 4.33], continuity from above is now sufficient for the dual representation of ρ. Moreover, $\alpha(Q) < \infty$ implies $Q \ll P$, and so the supremum in (3) can be taken over all probability measures $Q \ll P$. The supremum is attained if ρ is continuous from below; see [18, Corollary 4.35]. Moreover, ρ is sensitive with respect to P if and only if the dual representation holds in terms of equivalent probability measures $Q \approx P$; see [18, Theorem 4.43]. This is summarized in the following theorem.

Theorem 3. *Let ρ be a convex risk measure on $\mathcal{X}$ such that $\rho \ll P$.*

1. *If ρ is continuous from above then ρ has the dual representation* (3)*, where the supremum is taken over all probability measures $Q \ll P$.*

2. *The Lebesgue property implies that the supremum is actually attained.*

3. *Sensitivity with respect to P is equivalent to the condition that the supremum in* (3) *can be taken over all probability measures $Q \approx P$.*

We are also going to use the notion of a risk kernel introduced in [13]. Let $\mathcal{F}_0$ be a sub-σ-field of $\mathcal{F}_1 := \mathcal{F}$, and denote by $\mathcal{X}_i$ the space of all bounded measurable functions on $(\Omega, \mathcal{F}_i)$ for $i = 0, 1$.

Definition 4. *A map $\rho_{0,1} : \Omega \times \mathcal{X}_1 \to R^1$ is called a* monetary risk kernel *from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F}_1)$ if*

- *for any $\omega \in \Omega$, $\rho_{0,1}(\omega, \cdot)$ is a monetary risk measure on $\mathcal{X}_1$,*
- *for any $X \in \mathcal{X}$, $\rho_{0,1}(\cdot, X)$ is an $\mathcal{F}_0$-measurable function on Ω,*
- *$\rho_{0,1}(\cdot, X_0) = -X_0$ for any $X_0 \in \mathcal{X}_0$.*

A monetary risk kernel $\rho_{0,1}$ is called a convex risk kernel *if each risk measure $\rho_{0,1}(\omega, \cdot)$ is convex.*

Note that a monetary risk kernel $\rho_{0,1}$ from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F}_1)$ can be regarded as a map from $\mathcal{X}$ to $\mathcal{X}_0$ since (2) implies $|\rho_{0,1}(\omega, X)| \leq ||X||$ for any $X \in \mathcal{X}$.

Definition 5. *The monetary risk kernel $\rho_{0,1}$ is called* absolutely continuous *with respect to P if*

$$X = Y \ P\text{-}a.s. \implies \rho_{0,1}(\cdot, X) = \rho_{0,1}(\cdot, Y) \ P\text{-}a.s.. \tag{5}$$

Condition (5) implies that $\rho_{0,1}$ can be viewed as a map from $\ell^\infty(\Omega, \mathcal{F}, P)$ to $\ell^\infty(\Omega, \mathcal{F}_0, P)$. As such, $\rho_{0,1}$ is a convex conditional risk measure in the sense of [9] or [18, Chapter 11]. In this case, we denote by

$$\mathcal{A}_{0,1} := \left\{ X \in \mathcal{X}_1 \;\middle|\; \rho_{0,1}(X) \le 0 \;\; P\text{-a.s} \right\}. \tag{6}$$

the conditional acceptance set of $\rho_{0,1}$, and by

$$\alpha_{0,1}(Q) = \operatorname{ess\,sup}\{E_Q\left[-X|\mathcal{F}_0\right] | X \in \mathcal{A}_{0,1}\} \tag{7}$$

the conditional penalty function, defined for $Q \ll P$ such that $Q \approx P$ on $\mathcal{F}_0$; the essential supremum is taken with respect to P.

Theorem 6. *Suppose that the risk kernel $\rho_{0,1}$ is convex, absolutely continuous with respect to P, and* continuous from above *in the sense that*

$$X_n \searrow X \quad P\text{-}a.s \quad \Longrightarrow \quad \rho_{0,1}(X_n) \nearrow \rho_0(X) \quad P\text{-}a.s$$

for any uniformly bounded sequence (X_n) in $\mathcal{X}$. Then $\rho_{0,1}$ admits the dual representation

$$\rho_{0,1}(X) = \operatorname*{ess\,sup}_Q (E_Q\left[-X|\mathcal{F}_0\right] - \alpha_{0,1}(Q)), \tag{8}$$

where the essential supremum is taken under P and over all probability measures $Q \ll P$ such that $Q \approx P$ on $\mathcal{F}_0$; cf. [9, 3, 7, 1] or [18, Chapter 11].

Definition 7. *The risk measure ρ on $\mathcal{X}$ is called* consistent *with the risk kernel $\rho_{0,1}$ if*

$$\rho = \rho(-\rho_{0,1}), \tag{9}$$

that is, $\rho(X) = \rho(-\rho_{0,1}(\cdot, X))$ for any $X \in \mathcal{X}$.

Let ρ_0 denote the restriction of ρ to $\mathcal{X}_0$, and note that the consistency condition (9) can be written as

$$\rho = \rho_0(-\rho_{0,1}). \tag{10}$$

We denote by

$$\mathcal{A}_0 := \left\{ X \in \mathcal{X}_0 \;\middle|\; \rho(X) \le 0 \right\},$$

the acceptance set corresponding to the convex risk measure ρ_0 on $\mathcal{X}_0$, and by α_0 the corresponding penalty function defined by

$$\alpha_0(Q) := \sup_{X \in \mathcal{A}_0} E_Q\left[-X\right]. \tag{11}$$

As shown in [16, 3, 7, 4, 1] or [18, Chapter 11], consistency of ρ with the risk kernel $\rho_{0,1}$ can be characterized as follows in terms of the acceptance sets $\mathcal{A}, \mathcal{A}_0, \mathcal{A}_{0,1}$ or, equivalently, in terms of the penalty functions $\alpha, \alpha_0, \alpha_{0,1}$.

Theorem 8. *Assume that ρ and $\rho_{0,1}$ are convex, absolutely contiuous with respect to P, and continuous from above. Then the following conditions are equivalent:*

1. *ρ_0 is consistent with $\rho_{0,1}$*
2. *$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{0,1}$*
3. *$\alpha(Q) = \alpha_0(Q) + E_Q[\,\alpha_{0,1}(Q)]$*
   for any $Q \ll P$ such that $Q \approx P$ on $\mathcal{F}_0$.

3 Systemic risk measures

In addition to the measurable space $(\Omega, \mathcal{F})$ of possible scenarios we fix a countable set I, viewed as the set of nodes i in some financial network. We denote by ℓ^∞ the space of bounded sequences $\bar{x} = (x_i)_{i\in I}$ and by $||\bar{x}||$ the corresponding supremum norm. If the set I is finite then ℓ^∞ reduces to the Euclidean space R^I.

Let $\bar{\mathcal{X}}$ denote the space of bounded measurable functions on the product space $\bar{\Omega} := \Omega \times I$ with canonical product σ-field $\bar{\mathcal{F}}$. Note that any $\bar{X} \in \bar{\mathcal{X}}$ can be regarded as a configuration

$$\bar{X} = (X_i)_{i\in I} \in \mathcal{X}^I$$

of financial positions, one for each node of the network, such that

$$||\bar{X}|| := \sup_{i\in I} ||X_i|| < \infty.$$

We are going to regard both $\mathcal{X}$ and ℓ^∞ as subspaces of $\bar{\mathcal{X}}$: Any $X \in \mathcal{X}$ wiil be identified with the configuration $\bar{X}$ defined by $X_i(\omega) = X(\omega)$ (constant across all nodes), and any $\bar{x} \in \ell^\infty$ with the configuration defined by $\bar{X}(\omega) = \bar{x}$ (constant across all scenarios).

Chen, Iyengar, and Moallemi [6] introduced the notion of a systemic risk measure on the space $\bar{\mathcal{X}}$ of configurations $\bar{X} = (X_i)_{i\in I}$ that consists in applying some convex risk measure ρ on $\mathcal{X}$ to a suitable real-valued aggregate $\Lambda(\bar{X})$ of the configuration $\bar{X}$. Here we focus on systemic risk measures that are given by a monetary risk measure $\bar{\rho}$ on $\bar{\mathcal{X}}$. More precisely:

Definition 9. *A monetary risk measure $\bar{\rho}$ on the space $\bar{\mathcal{X}}$ is called a* systemic risk measure *if it admits a* structural decomposition

$$\bar{\rho}(\bar{X}) = \rho(\Lambda(\bar{X})), \tag{12}$$

also written as

$$\bar{\rho} = \rho \circ \Lambda, \tag{13}$$

where ρ is a monetary risk measure on $\mathcal{X}$ and Λ is a measurable map from ℓ^∞ to R^1 such that $\Lambda(\bar{X}) \in \mathcal{X}$ for any $\bar{X} \in \bar{\mathcal{X}}$.

Note that cash-invariance of $\bar{\rho}$ takes the form

$$\bar{\rho}(\bar{X} + \bar{m}) = \bar{\rho}(\bar{X}) - m,$$

where $\bar{m} = (m_i)_{i \in I}$ is such that $m_i \equiv m$ for some $m \in R^1$. Accordingly, $\bar{\rho}(\bar{X})$ is interpreted as a *capital requirement per node*, i.e.,

$$\bar{\rho}(\bar{X}) = \inf \left\{ m \in R^1 \mid \bar{X} + \bar{m} \in \bar{\mathcal{A}} \right\},$$

where

$$\bar{\mathcal{A}} := \left\{ \bar{X} \in \bar{\mathcal{X}} \mid \bar{\rho}(\bar{X}) \leq 0 \right\} \tag{14}$$

denotes the acceptance set of $\bar{\rho}$.

Since both $\bar{\rho}$ and ρ are required to be a monetary risk measure, the structural decomposition (12) imposes further restrictions on the map Λ:

Definition 10. *A map Λ from ℓ^∞ to R^1 will be called a* monetary aggregation function *if it satisfies the following properties for any $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$:*

1. $\Lambda(\bar{0}) = 0$
2. $\Lambda(\bar{x}) \leq \Lambda(\bar{y})$ *if* $\bar{x} \leq \bar{y}$,
3. $\Lambda(\bar{x} + \bar{y}) = \Lambda(\bar{x}) + y$ *if $\bar{y}$ is constant, i.e., $y_i = y$ for all $i \in I$.*

A monetary aggregation function Λ will be called a concave aggregation function *if Λ is concave on ℓ^∞.*

Note that these three properties simply amount to the condition that $-\Lambda$ is a normalized monetary risk measure on the space ℓ^∞. It follows from (2) that Λ has the contraction property

$$|\Lambda(\bar{x})| \leq ||\bar{x}||. \tag{15}$$

In particular, any aggregation function satisfies $\Lambda(\bar{X}) \in \mathcal{X}$ for any $\bar{X} \in \bar{\mathcal{X}}$, as required in (12).

Remark 11. *We interpret $\Lambda(\bar{x})$ as an aggregate* per node. *In particular, the aggregate of a constant position $\bar{y}$ with $y_i \equiv y$ is given by y and not by $y\,|I|$ as in [6], where I is assumed to be finite. Moreover, condition (3) in our definition of a monetary aggregation function insists on cash invariance, while this is not required in [6]. Our stronger condition implies that $-\Lambda$ is a monetary risk measure. This will be convenient in the next section, because it allows us to use standard results on the dual representation of convex risk measures. Note also that [6] argues in terms of net losses $\tilde{X} := -X$ instead of net gains X. Thus, the corresponding aggregation function on ℓ^∞ given by $\tilde{\Lambda}(\bar{x}) := -\Lambda(-\bar{x})$ is convex if Λ is concave.*

Proposition 12. *Suppose that $\bar{\rho}$ is a systemic risk measure. Then the structural decomposition* (12) *is unique, and it is given by the restrictions*

$$\rho := \bar{\rho}|_{\mathcal{X}} \quad \text{and} \quad -\Lambda := \bar{\rho}|_{\ell^\infty} \tag{16}$$

of $\bar{\rho}$ to the subspaces $\mathcal{X}$ and ℓ^∞. In particular, Λ is an aggregation function. Moreover, the systemic risk measure $\bar{\rho}$ is convex if and only if the risk measure ρ is convex on $\mathcal{X}$ and the aggregation function Λ is concave on ℓ^∞.

Proof. For a deterministic configuration $\bar{X}$ with $\bar{X}(\cdot) \equiv \bar{x}$ for some $\bar{x} \in \ell^\infty$, the decomposition (12) implies

$$\bar{\rho}(\bar{X}) = \rho(\Lambda(\bar{x})) = -\Lambda(\bar{x}),$$

since the normalized monetary risk measure ρ satisfies $\rho(m) = -m$ for any constant $m \in R^1$. This shows that $-\Lambda$ is given by the restriction of $\bar{\rho}$ to ℓ^∞, viewed as a subspace of $\bar{X}$. In particular, $-\Lambda$ is a monetary risk measure on ℓ^∞, and this means that Λ is an aggregation function. The first property of an aggregation function implies $|\Lambda(\bar{X}(\omega)| \leq ||\bar{X}||$, and so we obtain $\Lambda(\bar{X}) \in \mathcal{X}$ for any $\bar{X} \in \bar{\mathcal{X}}$.

Now consider a configuration $\bar{X} = (X_i)_{i \in I}$ that is constant across all nodes, that is, $X_i \equiv X$ for some $X \in \mathcal{X}$. The third property of the aggregation function Λ implies $\Lambda(\bar{X}(\omega)) = X(\omega)$, hence

$$\bar{\rho}(\bar{X}) = \rho(\Lambda(\bar{X})) = \rho(X).$$

Thus ρ is given by the restriction of $\bar{\rho}$ to $\mathcal{X}$, viewed as a subspace of $\bar{\mathcal{X}}$.

Finally, convexity of $\bar{\rho}$ on $\bar{\mathcal{X}}$ implies convexity on the subspaces $\mathcal{X}$ and ℓ^∞, and so both ρ and $-\Lambda$ are convex risk measures. Conversely, convexity of both ρ and $-\Lambda$ implies, using the monotonicity of ρ, that $\bar{\rho} = \rho(\Lambda)$ is a convex risk measure on $\bar{\mathcal{X}}$. □

Remark 13. *As observed in [6], the structural decomposition* (12) *implies stronger versions of the standard properties of a convex risk measure $\bar{\rho}$ on $\bar{\mathcal{X}}$. These stronger properties of $\bar{\rho}$ are formulated in terms of the restriction $\Lambda = \bar{\rho}|_{\ell^\infty}$ of $\bar{\rho}$ to ℓ^∞.*

Let $\succeq$ denote the partial order on ℓ^∞ defined by $\bar{x} \succeq \bar{y} :\Longleftrightarrow \Lambda(\bar{x}) \leq \Lambda(\bar{y})$, and write $\bar{X} \succeq \bar{Y}$ if $\bar{X}(\omega, \cdot) \succeq \bar{X}(\omega, \cdot)$ for each $\omega \in \Omega$. Then the structural decomposition implies

$$\bar{X} \succeq \bar{Y} \implies \bar{\rho}(\bar{X}) \leq \bar{\rho}(\bar{Y}).$$

This property, called preference consistency *in [6], is stronger than monotonicity since the pointwise inequality $\bar{X} \geq \bar{Y}$ implies $\bar{X} \succeq \bar{Y}$.*

If $\bar{\rho}$ is convex, then the structural decomposition $\bar{\rho} = \rho \circ \Lambda$ implies

$$\bar{Z} \succeq \alpha\bar{X} + (1-\alpha)\bar{Y} \implies \bar{\rho}(\bar{Z}) \leq \alpha\bar{\rho}(\bar{X}) + (1-\alpha)\bar{\rho}(\bar{X})$$

for any $\alpha \in [0, 1]$, since ρ is monotone and convex. This property, calles risk convexity *in [6], is stronger than convexity because the condition on the left hand side is satisfied whenever $\bar{Z} = \alpha\bar{X} + (1-\alpha)\bar{Y}$, due to the concavity of Λ.*

Conversely, a convex risk measure $\bar{\rho}$ admits a structural decomposition (12) *when it has these stronger properties; see [6] in the case of a finite Ω, [22] and [21] in the general case, and also [12] and [2] for closely related results.*

Remark 14. *In this paper we restrict the discussion to systemic risk measures $\bar{\rho} = \rho \circ \Lambda$ that are cash-invariant. The aggregation function Λ inherits this property, and this excludes various examples of aggregation functions that are discussed in the literature. However, some of them are included if we apply our cash-invariant risk measure $\bar{\rho}$ not to the collection $\bar{X}$ but to the collection $\bar{X}'$, where the position X_i is replaced by $X_i' := \min(X_i, 0)$. In other words, we replace $\bar{\rho}$ by the* loss-based risk measure *$\bar{\rho}'$ defined by $\bar{\rho}'(\bar{X}) := \bar{\rho}(\bar{X}')$, as proposed in [8].*

In the general case, our arguments apply if we replace the standard results for convex risk measures by their extensions to general risk functionals; see, e.g., [11]. This will be discussed in [15].

4 Dual Representation

From now on we fix a probability measure P on $(\Omega, \mathcal{F})$ and a probability measure μ on I with strictly positive weights $(\mu_i)_{i\in I}$, and we denote by $\bar{P} = P \otimes \mu$ the corresponding product measure on $(\bar{\Omega}, \bar{\mathcal{F}})$.

Throughout this section we fix a systemic risk measure $\bar{\rho} = \rho \circ \Lambda$ on $\bar{\mathcal{X}}$ that is convex. We also assume $\bar{\rho} \ll \bar{P}$; this is equivalent to $\rho \ll P$, since μ is strictly positive on I. Thus, $\bar{\rho}$ and ρ can be viewed as convex risk measures on the Banach spaces $\ell^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and $\ell^\infty(\Omega, \mathcal{F}, P)$, respectively.

Lemma 15. *The systemic risk measure $\bar{\rho} = \rho \circ \Lambda$ is continuous from above if and only if both ρ and $-\Lambda$ are continuous from above. The same equivalence holds for continuity from below.*

Proof. Clearly, continuity from above for $\bar{\rho}$ implies continuity from above for the restrictions $\rho := \bar{\rho}|_{\mathcal{X}}$ and $\Lambda := -\bar{\rho}|_{\ell^\infty}$.

Conversely, take a sequence $(\bar{X}_n)_{n=1,2,\ldots}$ in $\bar{\mathcal{X}}$ that decreases $\bar{P}$-a.s. to $\bar{X} \in \bar{\mathcal{X}}$. Then we have $X_{n,i}(\omega) \searrow X_i(\omega)$ $(i \in I)$ P-a.s., hence $\Lambda(\bar{X}_n(\omega)) \searrow \Lambda(\bar{X}(\omega))$ P-a.s. if Λ is continuous from above on ℓ^∞. But this implies

$$\bar{\rho}(\bar{X}_n) = \rho(\Lambda(\bar{X}_n)) \nearrow \rho(\Lambda(\bar{X})) = \bar{\rho}(\bar{X})$$

if, in addition, ρ is continuous from above on $\mathcal{X}$.

The same argument shows that continuity from below, in other words the Lebesgue property, holds for $\bar{\rho}$ if and only if it holds for ρ and $-\Lambda$. $\square$

Lemma 16. *The systemic risk measure $\bar{\rho} = \rho \circ \Lambda$ is sensitive with respect to $\bar{P}$ if and only if ρ is sensitive with respect to P and $-\Lambda$ is sensitive with respect to μ.*

Proof. Sensitivity of $\bar{\rho}$ clearly implies sensitivity for the restrictions $\rho := \bar{\rho}|_{\mathcal{X}}$ and $\Lambda := -\bar{\rho}|_{\ell^\infty}$.

Conversely, take $\bar{A} \in \mathcal{F}$ such that $P[\bar{A}] > 0$, and choose $A \in \mathcal{F}$ and $i \in I$ such that $P[A] > 0$ and $A \times \{i\} \subseteq \bar{A}$. Sensitivity of $-\Lambda$ implies $\gamma(\lambda) := -\Lambda(-\lambda I_{\{i\}}) > 0$ for some $\lambda > 0$. In fact we obtain $\lim_{\lambda\to\infty} \gamma(\lambda) = \infty$, since $\gamma(0) = 0$ and since the function $\gamma(\cdot)$ is convex, due to the concavity of Λ. Since

$$-\Lambda(-\lambda I_{A\times\{i\}}(\omega, \cdot)) = \gamma(\lambda) I_A(\omega),$$

monotonicity of $\bar{\rho}$ and sensitivity of ρ yield

$$\bar{\rho}(-\lambda I_{\bar{A}}) \geq \bar{\rho}(-\lambda I_{A\times\{i\}}) = \rho(-\gamma(\lambda) I_A) > 0$$

for large enough $\gamma(\lambda)$, that is, for large enough λ. □

From now on we also assume that the convex systemic risk measure $\bar{\rho} \ll \bar{P}$ is continuous from above. Theorem 3, applied to $\bar{\rho}$ on $\bar{\mathcal{X}}$, shows that $\bar{\rho}$ admits the dual representation

$$\bar{\rho}(\bar{X}) = \sup_{\bar{Q}\ll\bar{P}} \left(E_{\bar{Q}}\left[-\bar{X}\right] - \alpha(\bar{Q})\right), \tag{17}$$

where the penalty function $\bar{\alpha}$ is defined by

$$\bar{\alpha}(\bar{Q}) := \sup_{X\in\bar{\mathcal{A}}} E_{\bar{Q}}\left[-X\right] \tag{18}$$

in terms of the acceptance set $\bar{\mathcal{A}}$ in (14). If $\bar{\rho}$ is sensitive with respect to $\bar{P}$ then it is enough to take the supremum over equivalent probability measures $\bar{Q} \approx \bar{P}$.

Our aim is to clarify the structure of the systemic penalty function $\bar{\alpha}$. To this end, we are going to show that the structural decomposition $\bar{\rho} = \rho \circ \Lambda$ in (12) can be viewed as a consistency property of $\bar{\rho}$ with respect to a risk kernel $\bar{\rho}_{0,1}$ defined in terms of Λ. This will allow us to apply Theorem 8.

Due to Proposition 12 and Lemma 15, our assumptions on $\bar{\rho}$ imply that the monetary risk measures ρ and $-\Lambda$ are convex and continuous from above, with $\rho \ll P$ and $-\Lambda \ll \mu$. Thus, ρ admits the dual representation

$$\rho(X) = \sup_{Q\ll P} \left(E_Q\left[-X\right] - \alpha(Q)\right), \tag{19}$$

where the penalty function α is given by (4). In the same way, $-\Lambda$ admits a dual representation on ℓ^∞ with penalty function α_I. For the aggregation function Λ, this takes the form

$$\Lambda(\bar{x}) = \inf_{\pi} \left(\langle \bar{x}, \pi\rangle + \alpha_I(\pi)\right) \tag{20}$$

for any $\bar{x} \in \ell^\infty$, where the infimum is taken over all probability measures $\pi = (\pi_i)_{i\in I}$ on I. Here we use the notation $\langle \bar{x}, \pi\rangle = \sum_{i\in I} x_i \pi_i$, and the penalty function α_I is defined by

$$\alpha_I(\pi) := \sup_{\bar{x}\in\mathcal{A}_I} \langle -\bar{x}, \pi\rangle \tag{21}$$

in terms of the acceptance set

$$\mathcal{A}_I := \{\bar{x} \in \ell^\infty \mid \Lambda(\bar{x}) \geq 0\}.$$

Note that any probability measure $\bar{Q}$ on the product space $(\bar{\Omega}, \bar{\mathcal{F}})$ can be written as the product $Q \otimes \nu$ of a probability measure Q on $(\Omega, \mathcal{F})$ and a stochastic kernel $\nu(\cdot)$ from $(\Omega, \mathcal{F})$ to I, that is,

- for any $\omega \in \Omega$, $\nu(\omega) = (\nu_i(\omega))_{i \in I}$ is a probability measure on I
- for any $i \in I$, $\nu_i(\cdot)$ is a measurable function on $(\Omega, \mathcal{F})$.

More precisely, Q is the marginal distribution of the first coordinate under $\bar{Q}$, the stochastic kernel $\nu(\cdot)$ is taken as a regular version of the conditional distribution of the second coordinate under $\bar{Q}$ with respect to the first coordinate, and

$$E_{\bar{Q}}[\bar{X}] = E_{Q \otimes \nu}[\bar{X}] := \int \langle \bar{X}(\omega, \cdot), \nu(\omega) \rangle Q(d\omega) \tag{22}$$

for any $\bar{X} \in \bar{\mathcal{X}}$. Since μ has strictly positive weights, we have $\bar{Q} \ll \bar{P}$ if and only if $Q \ll P$.

Let us introduce the conditional acceptance set

$$\bar{\mathcal{A}}_I := \{\bar{X} \in \bar{\mathcal{X}} \mid \bar{X}(\omega, \cdot) \in \mathcal{A}_I \ P\text{-a.s}\}$$

and the conditional penalty function $\bar{\alpha}_I$, defined for any stochastic kernel $\nu(\cdot)$ from $(\Omega, \mathcal{F})$ to I by

$$\bar{\alpha}_I(\nu(\cdot)) := \operatorname{ess\,sup} \{\langle \bar{X}(\cdot, \cdot), \nu(\cdot) \rangle \mid \bar{X} \in \bar{\mathcal{A}}_I\}; \tag{23}$$

the essential supremum of the $\mathcal{F}$-measurable functions $\omega \to \langle \bar{X}(\omega, \cdot), \nu(\omega) \rangle$ is taken under P. The systemic penalty function $\bar{\alpha}$ can now be described as follows in terms of the penalty function α and the conditional penalty function $\bar{\alpha}_I(\cdot)$.

Theorem 17. *For any $\bar{Q} = Q \otimes \nu \ll \bar{P}$ such that $Q \approx P$, the penalty $\bar{\alpha}(\bar{Q})$ defined by* (18) *is given by*

$$\bar{\alpha}(\bar{Q}) = \alpha(Q) + E_Q\left[\bar{\alpha}_I(\nu(\cdot))\right]. \tag{24}$$

Proof. Let $\bar{\mathcal{F}}_0$ denote the sub-σ-field $\{A \times I | A \in \mathcal{F}\}$ of $\bar{\mathcal{F}}_1 := \bar{\mathcal{F}}$. We denote by $\bar{\mathcal{X}}_0$ the space of bounded measurable functions on $(\bar{\Omega}, \bar{\mathcal{F}}_0)$ and by $\bar{\rho}_0$ the restriction of $\bar{\rho}$ to $\bar{\mathcal{X}}_0$. Consider the convex risk kernel $\bar{\rho}_{0,1}$ from $(\bar{\Omega}, \bar{\mathcal{F}}_0)$ to $(\bar{\Omega}, \bar{\mathcal{F}}_1)$ defined by

$$\bar{\rho}_{0,1}(\bar{\omega}, \bar{X}) = -\Lambda(\bar{X}(\omega, \cdot))$$

for $\bar{X} \in \bar{\mathcal{X}}$ and $\bar{\omega} = (\omega, i) \in \bar{\Omega}$. Note that $\bar{\rho}_{0,1}$ is continuous from above and absolutely continuous with respect to $\bar{P}$. The structural decomposition $\bar{\rho} = \rho \circ \Lambda$ can now be read as the consistency relation

$$\bar{\rho} = \bar{\rho}(-\bar{\rho}_{0,1}) = \bar{\rho}_0(-\bar{\rho}_{0,1})$$

as it appears in (9), (10), and Theorem 8. Thus, the decomposition (24) will follow from the description of the penalty function in Theorem 8, applied to the risk measure $\bar{\rho}$ and the risk kernel $\bar{\rho}_{0,1}$.

It remains to identify the penalty functions of $\bar{\rho}_0$ and $\bar{\rho}_{0,1}$. Since $\bar{X} \in \bar{\mathcal{X}}_0$ iff $\bar{X}(\omega, \cdot) \equiv X(\omega)$ for some $X \in \mathcal{X}$, we have $\Lambda(\bar{X}) = X$, hence

$$\bar{\rho}_0(\bar{X}) = \bar{\rho}(\bar{X}) = \rho(\Lambda(\bar{X})) = \rho(X).$$

for $\bar{X} \in \bar{\mathcal{X}}_0$. Thus, the penalty function $\bar{\alpha}_0$ of $\bar{\rho}_0$ is given by

$$\begin{aligned} \bar{\alpha}_0(\bar{Q}) &:= \sup\{E_{\bar{Q}}[-\bar{X}] \mid \bar{X} \in \bar{\mathcal{X}}_0,\ \bar{\rho}_0(\bar{X}) \leq 0\} \\ &= \sup\{E_Q[-X] | X \in \mathcal{X}, \rho(X) \leq 0\} \\ &= \alpha(Q) \end{aligned}$$

for any $\bar{Q} = Q \otimes \nu \ll \bar{P}$.

Finally, we show that the conditional penalty function $\bar{\alpha}_{0,1}(\bar{Q})$ for the conditional risk measure $\bar{\rho}_{0,1}$ is given by

$$\bar{\alpha}_{0,1}(\bar{Q})(\bar{\omega}) = \bar{\alpha}_I(\nu(\omega))$$

for $\bar{P}$-a.a. $\bar{\omega} = (\omega, i)$. Note first that the conditional acceptance set $\bar{\mathcal{A}}_{0,1}$ of $\bar{\rho}_{0,1}$ coincides with $\bar{\mathcal{A}}_I$. Indeed, $\bar{X} \in \bar{\mathcal{X}}$ belongs to $\bar{\mathcal{A}}_{0,1}$ iff

$$\bar{\rho}_{0,1}(\bar{\omega}, \bar{X}) = -\Lambda(\bar{X}(\omega, \cdot)) \leq 0 \ \bar{P}\text{-a.s.} \iff \bar{X}(\omega, \cdot) \in \mathcal{A}_I \ P\text{-a.s.},$$

that is, iff $\bar{X} \in \bar{\mathcal{A}}_I$. Since

$$E_{\bar{Q}}[-\bar{X} \mid \bar{\mathcal{F}}_0](\bar{\omega}) = \langle -\bar{X}(\omega, \cdot), \nu(\omega) \rangle \quad \bar{P}\text{-a.s.},$$

we have, $\bar{P}$-a.s,

$$\begin{aligned} \bar{\alpha}_{0,1}(\bar{Q})(\bar{\omega}) &:= \operatorname{ess\,sup}\{E_{\bar{Q}}[-\bar{X} \mid \bar{\mathcal{F}}_0] \mid \bar{X} \in \bar{\mathcal{A}}_{0,1}\} \\ &= \operatorname{ess\,sup}\{\langle -\bar{X}(\omega, \cdot), \nu(\omega) \rangle \mid \bar{X} \in \bar{\mathcal{A}}_I\} \\ &=: \bar{\alpha}_I(\nu(\cdot))(\omega). \end{aligned}$$

□

5 Local specification and phase transition

We have seen that the structural decomposition of a systemic risk measure can be regarded as a consistency condition, and we have used this fact in our proof of the dual decomposition in Theorem 17. There are other consistency conditions that may be relevant. In this section we introduce the local specification of a systemic risk measure, defined in terms of spatial consistency conditions. This will establish a connection between systemic risk measures and the spatial risk measures discussed in [13] and [14].

From now on we assume that the underlying measurable space is a product space of the form

$$(\Omega, \mathcal{F}) = (S^I, \mathcal{S}^I),$$

where S is some polish state space with Borel σ–field $\mathcal{S}$. Thus, a possible scenario is given by a map $\omega : I \to S$ that specifies a state $s \in S$ for each site $i \in I$. For any subset $J \subseteq I$, we denote by $\mathcal{F}_J$ the σ-field on Ω generated by the projection maps $\omega \to \omega(i)$ for any $i \in J$, by $\mathcal{G}_J$ the σ-field on I generated by the sets $\{j\}$ with $j \in J$, and by $\bar{\mathcal{F}}_J$ the product-σ-field $\mathcal{F}_J \times \mathcal{G}_J$ on $\bar{\Omega} = \Omega \times I$.

We assume that I is countably infinite, and we denote by $\mathcal{V}$ the class of non-empty finite subsets V of I. For a given set $V \in \mathcal{V}$, the σ-field $\mathcal{F}_V$ contains the events that are observable on V, while the events in $\mathcal{F}_{V^c}$ depend on the environment of V, that is, the situation on $V^c := I \backslash V$.

For each $V \in \mathcal{V}$, we fix a systemic risk kernel $\bar{\rho}_V(\cdot, \cdot)$ that associates to any local configuration $\bar{X}_V = (X_i)_{i \in V}$ of positions at the sites in V a capital requirement $\bar{\rho}_V(\omega, \bar{X}_V)$ that depends on the environment of V. More precisely,

1. for any $\omega \in \Omega$, $\bar{\rho}_V(\omega, \cdot)$ is a systemic risk measure on $\mathcal{X}^V$ with structural decomposition
$$\bar{\rho}_V(\omega, \cdot) = \rho_V(\omega, \cdot) \circ \Lambda_V, \tag{25}$$
where $\rho_V(\omega, \cdot)$ is a convex risk measure on $\mathcal{X}$ and Λ_V is a concave aggregation function on R^V,

2. for any $\bar{X}_V \in \mathcal{X}^V$, $\bar{\rho}_V(\cdot, \bar{X}_V)$ is an $\mathcal{F}_{V^c}$-measurable function on Ω.

An axiomatic characterization of such conditional systemic risk measures is given in [21].

Note that the second condition implies that each $\rho_V(\cdot, \cdot)$ is a risk kernel from $(\Omega, \mathcal{F}_{V^c})$ to $(\Omega, \mathcal{F})$. Thus, the composition $\rho_W(-\rho_V)$ of two kernels ρ_V and ρ_W is well defined as a risk kernel from $(\Omega, \mathcal{F}_{W^c})$ to $(\Omega, \mathcal{F})$. The following definition is taken from [13] and [14].

Definition 18. *A collection* $(\rho_V)_{V \in \mathcal{V}}$ *of convex risk kernels* ρ_V *from* $(\Omega, \mathcal{F}_{V^c})$ *to* $(\Omega, \mathcal{F})$ *is called a* local specification of a convex risk measure on $\mathcal{X}$ *if it satisfies the consistency condition*

$$\rho_W(-\rho_V) = \rho_W$$

for any $V, W \in \mathcal{V}$ *such that* $V \subseteq W$.

We denote by $\mathcal{R}$ *the set of all convex risk measures* ρ *on* $\mathcal{X}$ *that are consistent with the local specification* $(\rho_V)_{V \in \mathcal{V}}$*, that is,*

$$\rho(-\rho_V) = \rho \quad \textit{for} \quad \textit{any} \quad V \in \mathcal{V}.$$

In order to introduce the analogous notion at the systemic level, we are going to regard $\bar{\rho}_V(\cdot, \cdot)$ as a risk kernel from $(\bar{\Omega}, \bar{\mathcal{F}}_{I-V})$ to $(\bar{\Omega}, \bar{\mathcal{F}})$. To this end, we identify the local aggregation function Λ_V on R^V with the conditional aggregation function on ℓ^∞ given by

$$\Lambda_V(i, \bar{x}) = \begin{cases} \Lambda_V(\bar{x}_V) & \text{if } i \in V, \\ x_i & \text{if } i \in V^c. \end{cases}$$

Thus, Λ_V associates to each profile $\bar{x}$ a new profile $\Lambda_V(\cdot, \bar{x})$, where the values on V are replaced by the aggregate value $\Lambda_V(\bar{x}_V)$ while the values outside of V remain unchanged.

Note that $-\Lambda_V(\cdot, \cdot) : I \times \ell^\infty \to R^1$ can be regarded as a convex risk kernel from $(I, \mathcal{G}_{V^c})$ to $(I, \mathcal{G}_I)$. As in the preceding definition, we can thus define the *local specification of a concave aggregation function on* ℓ^∞ as a family $(\Lambda_V)_{V \in \mathcal{V}}$ of conditional aggregation functions that satisfies the consistency condition

$$\Lambda_W(i, \Lambda_V(\cdot, \bar{x})) = \Lambda_W(i, \bar{x})$$

for $i \in I$, $\bar{x} \in \ell^\infty$, and for any $V, W \in \mathcal{V}$ such that $V \subseteq W$.

Definition 19. *We denote by $\mathcal{L}$ the set of all concave aggregation functions Λ on ℓ^∞ that are consistent with the local specification $(\Lambda_V)_{V \in \mathcal{V}}$, i.e.,*

$$\Lambda(-\Lambda_V(\cdot, \bar{x})) = \Lambda(\bar{x})$$

for any $V \in \mathcal{V}$.

With this notation, the convex risk kernel $\bar{\rho}_V$ from $(\bar{\Omega}, \bar{\mathcal{F}}_{I-V})$ to $(\bar{\Omega}, \bar{\mathcal{F}})$ corresponding to (25) can now be defined by

$$\bar{\rho}_V(\bar{\omega}, \cdot) := \rho_V(\omega, \cdot) \circ \Lambda_V(i, \cdot) \tag{26}$$

for any $\bar{\omega} = (\omega, i) \in \bar{\Omega}$, that is,

$$\bar{\rho}_V(\bar{\omega}, \bar{X}) = \rho_V(\omega, \Lambda_V(i, \bar{X})) = \begin{cases} \rho_V(\omega, X_i) & \text{if } i \in V \\ \rho_V(\omega, \Lambda_V(\bar{X}_V)) & \text{if } i \in V^c \end{cases} \tag{27}$$

for any $\bar{X} \in \bar{\mathcal{X}}$.

Remark 20. *The value $\rho_V(\omega, X_i)$ would reduce to $-X_i(\omega)$ if the position X_i would only depend on the situation at the node i, or only on the situation on V^c. In general, however, the position taken at the node i may also depend on the situation in V.*

Definition 21. *The collection $(\bar{\rho}_V)_{V \in \mathcal{V}}$ is called a* local specification of a convex systemic risk measure *on $\bar{\mathcal{X}}$ if it satisfies the consistency condition*

$$\bar{\rho}_W(-\bar{\rho}_V) = \bar{\rho}_W \tag{28}$$

for any $V, W \in \mathcal{V}$ such that $V \subseteq W$.

We denote by $\bar{\mathcal{R}}$ the set of all convex systemic risk measures $\bar{\rho}$ on $\bar{\mathcal{X}}$ that are consistent with the local specification $(\bar{\rho}_V)_{V \in \mathcal{V}}$, that is,

$$\bar{\rho}(-\bar{\rho}_V) = \bar{\rho} \tag{29}$$

for any $V \in \mathcal{V}$.

Our aim is to clarify the structure of the systemic risk measures in $\bar{\mathcal{R}}$, and in particular the question of existence and uniqueness, possibly under additional regularity constraints. The following theorem shows how this is related to the analysis of spatial risk measures in [13] and [14].

Let us denote by $\bar{\mathcal{R}}_L$ the class of all systemic risk measures $\bar{\rho}$ that have the Lebesgue property, that is, they are continuous from below. The notation $\mathcal{R}_L$ and Λ_L is used in the same manner.

Theorem 22. *Let $(\bar{\rho}_V)_{V\in\mathcal{V}}$ be a* local specification of a systemic risk measure *on $\bar{\mathcal{X}}$, where the kernels $\bar{\rho}_V$ are given by* (26).

1. *$(\rho_V)_{V\in\mathcal{V}}$ is a local specification of a convex risk measure on $\mathcal{X}$, and $(\Lambda_V)_{V\in\mathcal{V}}$ is a local specification of a concave aggregation function on ℓ^∞,*
2. *For any systemic risk measure $\bar{\rho} = \rho \circ \Lambda \in \bar{\mathcal{R}}$ we have $\rho \in \mathcal{R}$ and $\Lambda \in \mathcal{L}$.*
3. *For any systemic risk measure $\bar{\rho} = \rho \circ \Lambda \in \bar{\mathcal{R}}_L$ we have $\rho \in \mathcal{R}_L$ and $\Lambda \in \mathcal{L}_L$.*

Proof. To check the second statement, take $\bar{\rho} = \rho \circ \Lambda \in \bar{\mathcal{R}}$. If $\bar{X}(\cdot, i) = X(\cdot)$ for some $X \in \mathcal{X}$ (constant across nodes), then we have $\Lambda(\bar{X}) = X$ and $\Lambda_V(\cdot, \bar{X}) = X$, hence $\bar{\rho}(\bar{X}) = \rho(X)$ and $\bar{\rho}_V((\omega, i), \bar{X}) = \rho_V(\omega, X)$. Thus the consistency condition (29) implies

$$\rho(X) = \bar{\rho}(\bar{X}) = \bar{\rho}(-\bar{\rho}_V(\cdot, \bar{X})) = \bar{\rho}(-\rho_V(\cdot, X)) = \rho(-\rho_V(\cdot, X)),$$

and so we have $\rho \in \mathcal{R}$.

On the other hand, if $\bar{X}(\cdot, i) = \bar{x}_i$ for some $\bar{x} \in \ell^\infty$ (constant across scenarios), then $\bar{\rho}(\bar{X}) = -\Lambda(\bar{x})$ and $\bar{\rho}_V((\omega, i), \bar{X}) = -\Lambda_V(i, \bar{x})$. Thus we obtain

$$\Lambda(\bar{x}) = -\bar{\rho}(\bar{X}) = -\bar{\rho}(-\bar{\rho}_V(\cdot, \bar{X})) = \Lambda(\Lambda_V(\cdot, \bar{x})),$$

that is, $\Lambda \in \mathcal{L}$.

The proof of the first statement proceeds in the same way, and the third statement follows by combining the second with Lemma 15. □

Theorem 22 sets the stage for analysing the structure of the global systemic risk measures in $\bar{\mathcal{R}}$ (or $\bar{\mathcal{R}}_L$) in terms of the spatial risk measures in $\mathcal{R}$ (or $\mathcal{R}_L$) and the aggregation functions in $\mathcal{L}$ (or $\mathcal{L}_L$). In particular, it shows that the existence of a global systemic risk measure implies existence of a global spatial risk measure and of a global aggregation function, i.e.,

$$\bar{\mathcal{R}} \neq \emptyset \implies \mathcal{R} \neq \emptyset \text{ and } \mathcal{L} \neq \emptyset,$$

and the same is true if we require the Lebesgue condition:

$$\bar{\mathcal{R}}_L \neq \emptyset \implies \mathcal{R}_L \neq \emptyset \text{ and } \mathcal{L}_L \neq \emptyset.$$

In the same way, non-uniqueness at the systemic level implies that there must be non-uniqueness either for the spatial risk measures or for the aggregation functions, that is,

$$|\bar{\mathcal{R}}| > 1 \implies |\mathcal{R}| > 1 \text{ or } |\mathcal{L}| > 1,$$

and

$$|\bar{\mathcal{R}}_L| > 1 \implies |\mathcal{R}_L| > 1 \text{ or } |\mathcal{L}_\mathrm{L}| > 1.$$

This is just a first step; the converse implications are less immediate. Typically, a global aggregation function in $\mathcal{L}_L$ will be unique, and so the focus will be on the class $\mathcal{R}_L$ and its interplay with the class $\bar{\mathcal{R}}_L$. In [13] and [14], the structure of the spatial risk measures in $\mathcal{R}_L$ is analyzed under the assumption that their local specification is tied to an underlying probabilistic structure, namely to the local specification of a Gibbs measure. At the probabilistic level, non-uniqueness of the global Gibbs measure is also called a *phase transition*. It is shown in [13] and [14] how such a phase transition is related to non-uniqueness of the spatial risk measures in $\mathcal{R}_L$. The application of these results to the systemic risk measures in $\bar{\mathcal{R}}_L$ will be explored in [15].

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