Financial uncertainty, risk measures, and robust preferences

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1 The role of mathematics in finance

When one takes a look at the European financial markets as described by the fluctuation of the EuroStoxx index during the last twelve months, one sees a very irregular trajectory. When one observes it on the scale of a single day, this irregular and unpredictable character of the path does not change. Even in the very short term, there is therefore no way of making a safe extrapolation.

Given this uncertainty, the market offers many possibilities of placing a financial bet on the future behavior of the index. For instance, one can buy or sell a contract which gives the right to the value of the index six months later. In fact there is an increasing number of derivative products such as options and certificates which allow one to configure arbitrarily complex financial bets.

What is the role of mathematics, and in particular of probability theory, in this financial context? As with games of chance, mathematics cannot help to win a financial bet. In particular, it does not offer any means to compute in advance the outcome. On the other hand, mathematics may help to understand the nature of a given bet by providing methods to decide whether the bet is advantageous, fair, or acceptable, and to quantify its risk. Moreover, nowadays mathematics intervenes more and more in the construction of such bets, that is, in the design of new financial products.

A choice among the great variety of financial instruments depends upon the investor's preferences, and in particular on his or her tolerance towards the downside risk. We shall see how mathematics may help to specify preferences in a quantitative manner and, then, to determine the optimal structure of the bet according to these preferences.

2 Probabilistic models

The mathematical analysis of risks and preferences is carried out in the context of a probabilistic model. Such a model specifies first the set Ω of all scenarios which we are ready to consider. Here we focus on the behavior of a unique uncertain asset such as the EuroStoxx index between today and some time T in the future, and we exclude the occurence of sudden jumps. Thus a scenario can be described by a continuous function ω on the interval $[0, T]$, and we shall denote by $S_t(\omega) = \omega(t)$ the value of the asset at time t in this scenario.

In order to arrive at a probabilistic model, one should then choose a probability measure P on the set Ω of all scenarios. Once this choice has been made, one can now make predictions, not in the sense of guessing the future scenario, but by assigning probabilities to sets of scenarios and expectations to monetary quantities whose value may vary from scenario to scenario. If the value of a financial position at some future time t is described by a function $V_t(\omega)$ depending on the scenario up to time t, one can thus consider V_t as a random variable V_t on the probability space (Ω, P) and compute its expectation $E[V_t]$. More generally, we denote by $E_s[V_t]$ the conditional expectation of the future value V_t, given that we have observed the scenario up to some time s prior to time t.

Which probability measure should we choose? In 1900, Louis Bachelier introduced Brownian motion as a model of price fluctuations on the Paris stockmarket. A rigorous construction of the corresponding measure P on the set Ω of all continuous functions on the interval $[0, T]$ was given by Norbert Wiener in 1923. It is characterized by the property that the increments $S_t - S_s$, considered as random variables on the probability space (Ω, P), are Gaussian with means and variances proportional to the length $t - s$ of the time interval, and that they are independent for disjoint intervals. If this construction is carried out on a logarithmic scale, one obtains *geometric Brownian motion*, by now a standard model for the price fluctuation of a liquid financial asset, which was proposed by P.A. Samuelson in the 60's. Moreover, by allowing a change of clock which may depend on the scenario, one opens the door to a large variety of probabilistic models, including the solutions of stochastic differential equations. Here we are restricting the discussion to a single index, but of course we could also model the simultaneous fluctuation of several assets. Furthermore, one may want to analyze the microstructure of the market by taking into account the interactive behavior of heterogeneous agents, and then models become extremely complex.

Thus there is a vast number of modelling choices, and an entire industry is aiming to calibrate certain model classes with the help of econometric and statistical methods. In this exposition, however, we do not discuss such modelling issues in detail. We shall rather focus on some theoretical principles related to the idea of *market efficiency* and to the notion of *financial risk*.

3 Market efficiency and martingale measures

In its strongest version, the hypothesis of market efficiency demands that the measure P be such that the successive prices S_t of the uncertain asset have the probabilistic structure of a fair game. In mathematical terms, this means that the predictions of future prices coincide with the present prices, i.e.

$$E_s[S_t] = S_s$$

for every time t and every time s prior to time t. Equivalently, one may write

$$E_s[S_t - S_s] = 0,$$

which means that the conditional expectation of the net gain $S_t - S_s$ knowing the scenario up to time s is always zero. In this case, the stochastic process $(S_t)_{0 \leq t \leq T}$ is said to be a *martingale* with respect to the measure P, and P is called a *martingale measure*.

For now, let us suppose that P is indeed a martingale measure. In this case, a fundamental theorem in martingale theory due to J.L. Doob implies that there are no advantageous investment strategies. More precisely, let us consider a strategy which starts with an initial capital V_0 and divides at each time the available capital between the uncertain asset and a safe asset with interest rate r. To simplify we suppose that r is equal to 0 (in fact we have already implicitely done so in our definition of the martingale measure). Let us denote by $V_T(\omega)$ the value generated by the strategy up to time T as a function of the scenario ω. Allowing for some realistic restrictions on the strategy, one obtains the equation

$$E[V_T - V_0] = 0.$$

for the random variable V_T. Thus, there is no way to design a strategy such that the expectation of the net benefit be positive. Clearly, this conclusion is not shared by the great majority of financial analysts and consultants. In fact there are good reasons to believe that the strong version of the hypothesis of market efficiency is too rigid.

Let us therefore turn to a much more flexible way of formulating the efficiency hypothesis for financial markets. Here we admit the existence of advantageous strategies, but we exclude the possibility of a positive expected gain without any downside risk. More precisely, one requires that the probability measure P does not admit any strategy which on the one hand is advantageous in the sense that

$$E[V_T] > V_0,$$

and on the other hand is safe in the sense that the probability of a loss is zero, i.e.,

$$P[V_T < V_0] = 0.$$

Under this relaxed version of the efficiency hypothesis, the measure P is not necessarily a martingale measure. But the mathematical analysis shows that there must exist a martingale measure P^* which is equivalent to P in the sense that these two measures give positive weight to the same sets of scenarios; see Schachermayer (in this volume) or Delbaen and Schachermayer (1994).

From now on, we shall therefore assume that there exists a martingale measure P^* which is equivalent to P, and we shall denote by $\mathcal{P}^*$ the class of all these measures.

4 Hedging strategies and preferences in the face of financial risk

In the case of simple diffusion models such as geometric Brownian motion, the equivalent martingale measure P^* is in fact unique. In this case there exists a canonical way of computing the prices of all the derivative products of the underlying asset. Let us denote by $H(\omega)$ the value of such a product at time T as a function of the scenario ω. The uniqueness of the martingale measure P^* implies that there exists a strategy with initial capital

$$V_0 = E^*[H]$$

such that the value $V_T(\omega)$ generated up to time T coincides with $H(\omega)$ for every scenario ω outside of some set of probability zero. Thus, this *hedging strategy* allows a perfect replication of the financial product described by the random variable H. The initial capital it needs is given by the expectation $E^*[H]$ of the random variable H with respect to the martingale measure P^*. This expectation is also the canonical price of the product. Indeed, every other price would offer the possibility of a gain without risk. If, for instance, the price were higher than $E^*[H]$ then one could sell the product at that price and use the sum $E^*[H]$ to implement the hedging strategy which allows to pay what is needed at the final time T. The difference between the price and the initial cost $E^*[H]$ would thus be a gain without risk, and such a *free lunch* is excluded by our assumption of market efficiency.

The situation becomes much more complicated if the equivalent martingale measure is no longer unique. One can still construct a hedging strategy which makes sure that the final value $V_T(\omega)$ is at least equal to the value $H(\omega)$ for every scenario outside of a set with zero probability. But the initial capital which would be needed is now given by

$$V_0 = \sup E^*[H]$$

where the supremum is taken over all measures P^* in the class $\mathcal{P}^*$ of equivalent martingale measures. From a practical point of view, this sum is typically too high.

A more pragmatic approach consists in abandoning the aim of a perfect hedge and in accepting a certain *shortfall risk*. It is at this point that the investor's preferences come into play. In order to quantify the shortfall risk of a strategy with final outcome V_T, the investor may for instance take the probability

$$P[V_T < H]$$

of generating a value smaller than the amount H which will be needed in the end. Or one could define the shortfall risk as an expectation of the form $E_P[l(H - V_T)]$ where l is an increasing and convex loss function. Once this choice is made, one can then determine the strategy which minimizes the shortfall risk under the constraint of a given initial capital V_0; see Föllmer and Leukert (2000).

In specifying preferences related to risk, we have just started to use explicitly the probability measure P. Let us point out that this was not the case in the preceding discussion which only made use of the class $\mathcal{P}^*$ of equivalent martingale measures. In practice the choice of a probability measure, and therefore of a systematic procedure of making probabiliistic predictions, is usually not obvious. As Alan Greenspan said some time ago:

Our ability to predict is limited. We need some humility.

How can we rephrase this maxim of humility in mathematical terms? We shall describe a way of doing just that by explaining a recent approach to the quantification of financial risk which takes model uncertainty explicitely into account, and which does not rely on a specific probabilistic model to be fixed beforehand.

5 Risk measures and model uncertainty

Let us consider a financial position, and denote by $X(\omega)$ the net result at the end of the period $[0, T]$ for the scenario ω. The position is therefore described by a function X on the set Ω of all scenarios. We shall now describe a way of quantifying the risk $\rho(X)$ of such a position in monetary terms.

Suppose that in the space $\mathcal{X}$ of all financial positions we have singled out a subset $\mathcal{A}$ of positions which are judged to be "acceptable", for example from the point of view of a supervising agency. Let us write $X \geq Y$ if $X(\omega) \geq Y(\omega)$ for every scenario ω. It is then natural to assume that X is acceptable as soon as there exists an acceptable position Y such that $X \geq Y$. Let us define the risk $\rho(X)$ as the minimal amount of capital such that the position X becomes acceptable when this amount is added and invested in a risk free manner. In mathematical terms,

$$\rho(X) = \inf\{m | X + m \in \mathcal{A}\},$$

where the infimum is taken over the class of all constants m such that the new position $X+m$ is acceptable. It follows that ρ is a functional defined on the class $\mathcal{X}$ of all positions such that $\rho(X+m) = \rho(X) - m$ for every constant m. Moreover, the risk decreases when the outcome increases, i.e., $\rho(X) \leq \rho(Y)$ if $X \geq Y$. Let us call such a functional ρ on $\mathcal{X}$ a *monetary risk measure*.

Let us suppose, for now, that a probability measure P is given on the set Ω of all scenarios. A classical way of quantifying the risk of a position X, now considered as a random variable on the probability space (Ω, P), consists in computing its variance. But the definition of the variance, which is natural as a measure of measurement risk in the theory of errors, does not provide a monetary risk measure. In particular the symmetry of the variance does not capture the asymmetric perception of financial risk: what is important in the financial context is the *downside risk*. One might therefore decide to find a position X acceptable if the probability of a loss is below a given threshold, i.e., $P[X < 0] \leq \beta$ for a fixed constant β. The resulting risk measure, called *Value at Risk* at level β, takes the form

$$\mathrm{VaR}(X) = \inf\{m | P[X + m < 0] \leq \beta\} = -\sup\{x | P[X < x] \leq \beta\}.$$

Thus, it is given by a quantile of the distribution of X under the probability measure P. Value at Risk is a monetary risk measure, and it is very popular in the world of Finance. It has, however, a number of shortcomings. In particular the class $\mathcal{A}$ of acceptable positions is not convex. In financial terms this means that there are situations where *Value at Risk* may penalize the diversification of a position, even though such a diversification may be desirable.

If one does not want to discourage diversification then one should insist on the convexity of the set $\mathcal{A}$. In this case, the functional ρ is a *convex risk measure* in the sense of Frittelli and Rosazza Gianin (2002) and Föllmer and Schied (2002, 2004). If the class $\mathcal{A}$ is also a cone then we obtain a *coherent risk measure* as defined by Artzner, Delbaen, Eber and Heath (1999) in a seminal paper which marks the beginning of the theory of risk measures. Note, however, that the cone property of the class $\mathcal{A}$ implies that a position X remains acceptable if one multiplies it by a positive factor, even if this factor becomes very large. Various problems arising in practice, such as a lack of liquidity as one tries to unwind a large position, suggest a more cautious approach. This is why we only insist on convexity.

Here is a first example. Suppose that we have chosen a probabilistic model of the financial market. Let us say that a position X is acceptable if one can find a trading strategy with initial capital 0 and final value V_T such that the value of the combined position $X + V_T$ is at least equal to 0 for all scenarios outside of a set with probability 0. If we impose no constraints on the trading strategies then the resulting risk measure will be coherent, and it takes the form

$$\rho(X) = \sup E^*[-X]$$

where the supremum is taken over the class $\mathcal{P}^*$ of all equivalent martingale measures. Convex restrictions for the trading strategies will lead to convex rather than coherent risk measures. This is another motivation to pass from coherent to convex risk measures.

Let us consider a second example. Suppose that the investor's preferences are represented by a utility functional U on the space $\mathcal{X}$ in the sense that a position X is preferred to a position Y if and only if $U(X) > U(Y)$. Then the investor may find a position X acceptable as soon as the value $U(X)$ does not fall below a given level. If the functional U takes the classical form of an *expected utility*

$$U(X) = E_P[u(X)],$$

defined with the help of a probability measure P and some concave and increasing utility function u, then one obtains a convex risk measure. For example, the utility function

$u(x) = 1 - \exp(-\gamma x)$ induces, up to an additive constant, the *entropic* risk measure

$$\rho(X) = \frac{1}{\gamma} \log E_P[\exp(-\gamma X)] = \sup(E_Q[-X] - \frac{1}{\gamma} H(Q|P)).$$

Here, the supremum is taken over the class of all probability measures Q on the set Ω of all scenarios, and $H(Q|P)$ denotes the relative entropy of Q with respect to P, defined by $H(Q|P) = E_Q[\log \varphi]$ if Q admits the density φ with respect to P and by $H(Q|P) = +\infty$ otherwise.

In both examples, the risk measure is of the form

$$\rho(X) = \sup(E_Q[-X] - \alpha(Q)) \tag{1}$$

where the supremum is taken over all probability measures Q on the set Ω of all scenarios, and where α is a penalisation function which may take the value $+\infty$. In the first example one has $\alpha(Q) = 0$ if Q is an equivalent martingale measure and $\alpha(Q) = +\infty$ otherwise. In the second example, $\alpha(Q)$ is equal to the relative entropy $H(Q|P)$ divided by the parameter γ.

In fact, the representation (1) gives the general form of a *convex risk measure*, granting some mild continuity conditions. This follows by applying a basic duality theorem for the Legendre-Fenchel transformation to the convex functional ρ; see, for example, Delbaen (2000), Frittelli and Gianin (2003) or Föllmer and Schied (2004). In order to compute the value $\rho(X)$ from the representation (1), it obviously suffices to consider the class $\mathcal{Q}$ of all probability measures Q such that $\alpha(Q)$ is finite. In the coherent case, the penalisation function only takes the two values 0 and $+\infty$. Thus, the representation (1) reduces to the form

$$\rho(X) = \sup E_Q[-X] \tag{2}$$

where the supremum is taken over the class $\mathcal{Q}$, and this is the general form of a *coherent* risk measure.

Thus, the computation of the risk of a position X reduces to the following procedure. Instead of fixing one single probabilistic model, one admits an entire class $\mathcal{Q}$ of probability measures on the set of all scenarios. For every measure Q in this class, one computes the expectation $E_Q[-X]$ of the loss $-X$. But one does not deal with these measures in an equal manner: some may be taken more seriously than others, and this distinction is quantified by the subtraction of the value $\alpha(Q)$. Once this subtraction is made, one then considers the least favorable case amongst all models in the class $\mathcal{Q}$, and this defines the monetary risk $\rho(X)$ of the position X.

6 Analogies with the microeconomic theory of preferences

In the financial context, the preferences of an investor are described by a partial order on the space $\mathcal{X}$ of all financial positions. Under some mild conditions, such a partial order admits a numerical representation in terms of some utility functional U on $\mathcal{X}$. This means that the financial position X is preferred to the position Y if and only if $U(X) > U(Y)$. In the classical paradigm of *expected utility*, there is one single probability measure P on the set of all scenarios which is given a priori, and the functional U is of the form

$$U(X) = E_P[u(X)] \tag{3}$$

with some concave and increasing utility function u, as we have seen it already in the second example above. In this classical setting, the value $U(X)$ depends only on the distribution μ of the position X, considered as a random variable on the probability space (Ω, P), since it can be expressed as the integral

$$U(X) = \int u d\mu \tag{4}$$

of the utility function u with respect to μ. One may thus consider preferences as a partial order on the class of probability distributions on the real line. J. von Neumann and O. Morgenstern have characterized, via some axioms which formalize a rather strict notion of *rationality*, those partial orders on the class of distributions which admit a representation of the form (4) in terms of an implicit utility function u.

More generally, and without fixing a priori a probability measure P on the set of all scenarios, L. Savage, R. Aumann (Nobel prize 2005) and others have specified axioms for a partial order on the class $\mathcal{X}$, which allow us to reconstruct from the given preferences an implicit probability measure P and an implicit utility function u which yield a numerical representation of the form (3).

It is well known, however, that the paradigm of expected utility is not compatible with a number of empirical observations of people's behavior in situations involving uncertainty. In order to take into account such findings, and in particular certain "paradoxes" due to D. Ellsberg, M. Allais (Nobel prize 1988) and D. Kahneman (Nobel prize 2002), Gilboa and Schmeidler (1989) have proposed a more robust notion of rationality. In their axiomatic setting, the utility functional takes the form

$$U(X) = \inf E_Q[u(X)] \tag{5}$$

where the infimum is taken over a whole class $\mathcal{Q}$ of probability measures Q on the set of scenarios. Such *robust preferences* are not determined by a unique probabilistic model. Instead, they involve a whole class of probabilistic model, and thus they take explicitely into account the model uncertainty which is typical in real world situations. Up to a change of sign, there is an obvious analogy between coherent risk measures and with robust preferences characterized by the representation (5). The theory of convex risk measures suggests to go one step further and to consider a modification of the Gilboa-Schmeidler axioms which leads to a functional of the form

$$U(X) = \inf(E_Q[u(X)] + \alpha(Q)) \tag{6}$$

where α is a penalisation function, in analogy with the representation (1) of a convex risk measure. In fact, the preferences on $\mathcal{X}$ which do admit a numerical representation of the form (6) have recently been characterized by Maccheroni, Marinacci and Rustichini (2006).

7 Optimisation problems and robust projections

In view of the large variety of financial bets which are available due to the increasing number of derivative products, it is not easy to make a choice. Let us now consider the mathematical problem of computing an optimal financial position X, given an initial capital V_0 as well as the investor's preferences.

Assuming that preferences are represented via a functional U on the space $\mathcal{X}$ of financial positions, the problem consists in maximizing the value $U(X)$ under the condition that the position X can be financed by the available capital V_0. This means that one can find a trading strategy with initial value V_0 such that the resulting final value V_T is at least equal to X. It can be shown that this feasibility condition amounts to the constraint

$$\sup E^*[X] \leq V_0$$

where the supremum is taken over the class $\mathcal{P}^*$ of the equivalent martingale measures. We shall now assume that U is a functional with robust utility of the form (5), defined by a class $\mathcal{Q}$ of probability measures which are equivalent to P. Thus, our optimisation problem involves the two classes of measures $\mathcal{P}^*$ and $\mathcal{Q}$.

Let us first consider the case where each one of these two classes only contains one single element. In particular, the preferences are of the form (3) for one single measure Q, and there is a unique equivalent martingale measure P^*. In this classical case, the solution to the optimisation problem is given by

$$X = (u')^{-1}(\lambda\varphi), \tag{7}$$

where $(u')^{-1}$ is the inverse of the derivative of the utility function u, φ is the density of P^* with respect to Q and the parameter λ is such that $E^*[X] = V_0$.

If the preferences are robust but the martingale measure P^* is still unique, then the optimisation problem can be solved by using a basic result from robust statistics. In their robust version of the Neyman-Pearson lemma, Huber and Strassen (1973) had shown how to find a measure Q_0 which is *least favorable* in the class $\mathcal{Q}$ with respect to a given reference measure. In our financial context, Schied (2004) proved that the solution to the optimisation problem is of the form (7) if φ denotes the density of the martingale measure P^* with respect to that measure Q_0 in the class $\mathcal{Q}$ which is least favorable with respect to P^*.

If on the other hand the martingale measure P^* is no longer unique but the preferences are of the classical form (3) for a single measure Q, then our optimisation problem reduces to a dual *projection problem*. More precisely, one has to project the measure Q onto the class $\mathcal{P}^*$ of martingale measures by minimizing the functional

$$F(P^*|Q) = E_Q[f(\frac{dP^*}{dQ})],$$

where f is a convex function obtained from the utility function u by a Legendre-Fenchel transformation. In the exponential case $u(x) = 1 - \exp(-\gamma x)$, the functional F is given by the relative entropy $H(P^*|Q)$, and we are back to a classical projection problem in probability theory. For more general utility functions u, the dual problem has been systematically studied, particularly by Kramkov and Schachermayer (1999), Goll and Rüschendorf (2001) and Bellini and Frittelli (2002).

Let us now consider the general case where preferences are robust and the martingale measure is no longer unique. Now the problem consists in projecting the whole class $\mathcal{Q}$ onto the class $\mathcal{P}^*$. This amounts to finding a minimum of F, considered as a functional on the product set $\mathcal{P}^* \times \mathcal{Q}$. If one has identified a minimizing couple (P_0^*, Q_0), then the solution to the optimization problem is of the form (5) where φ is the density of Q_0 with respect to P_0^*. This robust version of a probabilistic projection problem, which is of intrinsic mathematical interest independently of the financial interpretation, has been studied by Gundel (2005) and by Föllmer and Gundel (2006).

It is, however, not always possible to find a solution to the dual problem within the class $\mathcal{P}^*$ of equivalent martingale measures. This difficulty may already appear if the preferences are of the classical form (3). In this case, Kramkov and Schachermayer (1999) have shown how one may extend the class $\mathcal{P}^*$ in order to find a solution in some larger setting. For preferences of the form (5), new robust variants of the projection problem have recently been studied by Quenez (2004), Schied and Wu (2005) and Föllmer and Gundel (2006). Moreover, Schied (2007) has just solved the optimization problem for robust preferences in the general form (6).

Thus, we find that the maxim of humility when facing financial uncertainty is in fact a very rich source of new problems in probability theory.

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