

Probabilistic Aspects of Options

by Hans Föllmer

0. Introduction

In recent years, derivative securities such as options have generated a lot of interest, both on a practical and on a theoretical level. Starting with the creation of the Chicago Board Options Exchange in 1973, there has been a remarkable expansion of financial markets dealing with these instruments; for example, the Swiss Options and Financial Futures Exchange opened in 1988, the Deutsche Terminbörse in 1990.

But what is really new from a theoretical, and in particular from a *probabilistic* point of view? Already in Bachelier's thesis "Théorie de la Spéculation" (1900), Brownian motion is introduced as a model for price fluctuations on a speculative market, and option prices are computed as expected values in that model. In the sixties, the pricing of American options was recognized as a problem of optimal stopping, and this motivated some of the probabilistic work in that area; cf. Samuelson (1964) and McKean (1964). As instruments of portfolio insurance, options have the form of a stop loss contract, and such contracts have been studied in the actuarial literature since the fifties; cf. Beard, Pesonen and Pentikäinen (1984).

In the seventies, due to the path-breaking work of Black and Scholes (1973) and Merton (1973), the theory of options underwent a major change. It was realized that, in a typical model for price fluctuations of a risky asset, a derivative security can be replicated by a dynamical portfolio strategy. Thus, an option can be hedged in a perfect manner: There is no intrinsic risk. This idea of dynamical hedging soon began to have a practical impact on actual trading. On the theoretical level, it became apparent that martingale theory provides a natural framework for the study of options. This has changed the way financial mathematics is being taught, and it also suggests new study programs for "Actuaries of the Third Kind"; cf. H. Bühlmann (1988).

This survey gives an introduction to those aspects of the theory of options which seem particularly interesting from a probabilistic point of view. In section 1 we describe the mathematical model of a *complete* financial market where contingent claims can be represented as stochastic integrals of the underlying price fluctuation. A fundamental representation theorem of K. Itô implies that the stan-

standard diffusion model for risky assets is indeed complete. In section 2 we consider situations which are *incomplete*: A typical claim now carries an intrinsic risk, and hedging strategies can only reduce the actual risk to that intrinsic component. Section 3 comments on some relations between option pricing, actuarial premium principles and economic equilibrium analysis. In section 4 we take a closer look at the structure of the probability measure P which models the price fluctuations of the underlying asset. We review some of the arguments in favour of geometric Brownian motion and conclude with a few tentative remarks on possible modifications.

This is the written version of two lectures given at a joint meeting of the Actuarial Society of Finland and the Finnish Mathematical Society in January 1990, and it is a pleasure to express my thanks to Heikki Bonsdorff and Hannu Niemi for their kind invitation to Helsinki. As the lectures, the paper is meant to be a first introduction to the probabilistic theory of options. For further orientation, some excellent surveys are available in the literature; cf., e.g., Harrison and Pliska (1981), Karatzas (1989), Duffie (1991), and the books of Cox and Rubinstein (1986), Duffie (1988), Huang and Litzenberger (1988), Ingersoll (1987) and Merton (1991).

Special thanks are due to Martin Schweizer: Sections 1 and 2 are an extension of Föllmer and Schweizer (1989), and he contributed to the present paper with a number of comments and corrections.

1. Pricing and Hedging of Options

Consider a risky asset (a stock, a portfolio, an exchange rate, ...) whose price fluctuation is described by a stochastic process

$$X_t(\omega) \quad (0 \leq t \leq T)$$

on some probability space (Ω, \mathcal{F}, P) . At the initial time $t = 0$, the value $X_T(\omega)$ of the asset at the terminal time T is unknown and thus constitutes a *risk*. In order to reduce this risk, we may want to consider financial instruments such as *options*.

1.1 Options and Portfolio Insurance

Consider, for example, a *call option* with striking price c . This option gives the right to buy the stock at time T at the fixed price c . The resulting pay-off is given by

$$(1.1) \quad H(\omega) = (X_T(\omega) - c)^+.$$

In the same manner, a *put option* with striking price c would yield

$$H(\omega) = (c - X_T(\omega))^+.$$

Thus, an option corresponds to a *contingent claim* $H(\omega)$ depending on $\omega \in \Omega$, i.e., to a random variable H on the underlying probability space.

Such options can be used for purposes of *portfolio insurance*. For example, we could replace the pay-off X_T by the contingent claim

$$(1.2) \quad H = \max(c, X_T),$$

either by holding the stock and a put option, or by holding the cash amount c and a call option, since

$$H = X_T + (c - X_T)^+ = c + (X_T - c)^+.$$

More generally, if we want to replace X_T by $H = f(X_T)$ with some *convex* function f , this can be achieved by a mixture of such instruments:

$$(1.3) \quad H = f(0) + f'_+(0)X_T + \int_0^\infty (X_T - c)^+ \mu(dc) \quad \text{with } f'_+(0) = f'(0) \text{ if } f \text{ is differentiable at } 0$$

where μ is the distributional derivative of the convex function f , viewed as a non-negative measure on R^+ ; cf. Leland (1980).

1.2 The Fair Price of an Option

How should we compute the *fair price* of such a contingent claim H ? In other words: What is the value V_0 of the option at the initial time 0 when the final outcome $H(\omega)$ is still uncertain? From a classical actuarial point of view, there seems to be a natural answer which goes back to Chr. Huygens and J. Bernoulli: The fair price should be equal to the *expected value* of the random variable H with respect to the probability measure P , i.e.,

$$(1.4) \quad V_0 = E[H].$$

One could include an interest rate and replace (1.1) by

$$(1.5) \quad V_0 = E\left[\frac{H}{1+r}\right].$$

To account for risk aversion, one might also want to add a safety loading, e.g., of the form $\alpha \cdot \text{var}(H)$, or defined in terms of some utility function. But (1.4) would seem to be a natural starting point.

If one accepts (1.4), or one of its modifications, then the problem is reduced to the choice of the underlying stochastic model. Here is the standard reference model. Suppose that (X_t) satisfies the stochastic differential equation

$$(1.6) \quad dX_t = \sigma X_t dW_t + m X_t dt$$

where (W_t) is a Brownian motion under P , σ is a parameter for the volatility and m is a parameter for the trend of the process. The pathwise solution of equation (1.6) is given by

$$(1.7) \quad X_t = X_0 \exp\left[\sigma W_t + \left(m - \frac{1}{2}\sigma^2\right)t\right] \quad (0 \leq t \leq T);$$

in particular, each random variable X_t has a log-normal distribution.

In the context of this standard model, we could now apply the Huygens-Bernoulli prescription (1.4), or one of the modifications above, in order to compute the fair price of an option H . But Black and Scholes (1973) gave an answer to our question which is quite different. Their prescription is as follows:

Replace P by the new measure P^ corresponding to $m^* = 0$ so that (X_t) becomes a martingale under P^* . Compute the price of H as*

$$(1.8) \quad V_0 = E^*[H].$$

Do not add any safety loading because there is no real risk.

If an interest rate r is to be included then take P^ such that the discounted process becomes a martingale and compute V_0 as in (1.5), but in terms of P^* rather than P .*

At first sight, this prescription may seem counter-intuitive. In particular, it may not be clear why the safety loading should be dropped. Before we present the general argument, let us look at an elementary two-period model; cf. Cox, Ross and Rubinstein (1979).

1.3 A Binary Example

Suppose that the current value of the risky asset, say the price of 100 US \$ in SFR, is given by $X_0 = 150$. Consider a call option with a strike of $c = 150$ at time T . We assume the following binary scenario: The price X_T of 100 \$ at time T will be 180 with probability p or 90 with probability $1 - p$. Then the pay-off H of the option will be 30 with probability p and 0 with probability $1 - p$. Taking into account an interest rate r , the Huygens-Bernoulli price (1.5) would be

$$(1.9) \quad V_0 = E\left[\frac{H}{1+r}\right] = \frac{1}{1+r} \cdot p \cdot 30;$$

for $p = 0.5$ and $r = 0$ we would get $V_0 = 15$. The Black-Scholes prescription, however, would be the following. First replace p by p^* so that the exchange rate, properly discounted, behaves like a fair game:

$$(1.10) \quad X_0 = E^* \left[\frac{X_T}{1+r} \right]$$

or, more explicitly,

$$150 = \frac{1}{1+r} \cdot (p^* \cdot 180 + (1-p^*) \cdot 90);$$

for $r = 0$ we would get $p^* = \frac{2}{3}$. Now compute the fair price as the expected value

$$(1.11) \quad V_0 = E^* \left[\frac{H}{1+r} \right] = \frac{1}{1+r} \cdot p^* \cdot 30$$

in this new model; for $r = 0$ we would get $V_0 = 20$.

At first sight, this change of the model seems completely arbitrary, just as in the diffusion model above. But in the present simple case we can give a direct economic justification. For simplicity assume $r = 0$. Suppose that at time 0 you sell the option. Then you can prepare for the resulting contingent claim at time T by using the following strategy:

Sell the option at the price $\pi(H)$	$+\pi(H)$
Buy \$33.33 at the present exchange rate of 1.50	-50
Take a loan of SFR 30	$+30$

Thus, the balance at time 0 is $\pi(H) - 20$. At time T we have to distinguish two cases:

i) The dollar has risen :	Option is exercised	-30
	Sell dollars at 1.80	$+60$
	Pay back loan	-30
ii) The dollar has fallen :	Option is worthless	0
	Sell dollars at 0.90	$+30$
	Pay back loan	-30

Since the balance at time T is 0 in both cases, the balance at time 0 should also be 0, and so the price $\pi(H)$ should coincide with the Black-Scholes price 20. Any

option price different from the Black-Scholes price would enable either the option seller or the option buyer to make a sure profit without any risk: There would be an arbitrage opportunity. If the loan has to be paid back with interest then there is a similar strategy which leads to the Black-Scholes price (1.11).

In section 2.5 we are going to explain how the correct hedging strategy can be found in a systematic way. It is clear that the present example is too simplistic; even if we consider a two-period model, there is no reason to restrict our attention to a binary scenario. In fact, the situation will become less pleasant as soon as we admit a third possibility for the value X_T . In that case, a perfect hedge of the claim is no longer possible. This simple observation will motivate our general approach in section 2.

From the point of view of the continuous-time model (1.7), the binary example is relevant because it can be regarded as an infinitesimal building stone of the diffusion model. Similarly, our strategy should be viewed as an infinitesimal step in the construction of a dynamical hedging strategy in continuous time. We are now going to describe this construction. From now on, we leave completely aside the question of interest rates in order to simplify the exposition.

1.3 The general argument

Let us explain the probabilistic structure of the argument which is behind the Black-Scholes formula (1.8) and also behind the preceding example. As shown by Harrison and Kreps (1979) and Harrison and Pliska (1981), the natural mathematical framework is provided by the theory of *martingales*.

We assume that the underlying measure P admits an equivalent measure $P^* \approx P$ such that

$$(1.12) \quad (X_t) \text{ is a square-integrable martingale under } P^*.$$

This means that the increments have zero conditional expectation, i.e.,

$$(1.13) \quad E^*[X_t - X_s | \mathcal{F}_s] = 0$$

for $0 \leq s < t \leq T$, where $E^*[\cdot | \mathcal{F}_s]$ denotes conditional expectation under P^* with respect to the σ -algebra \mathcal{F}_s which specifies the information available at time s . Such a measure P^* will be called an *equivalent martingale measure*. We will comment on the economic meaning of this assumption in section 4.2. From a mathematical point of view, it guarantees that the following stochastic integrals make sense. We omit a number of technical points. For example, integrability

assumptions will be tacitly assumed whenever they are needed, but they will no longer be mentioned explicitly (up to a notable exception in 4.2).

Let us now consider a *dynamical portfolio strategy* (ξ, η) of the following form. At time t we hold the amount ξ_t in the risky asset, and the amount η_t in the riskless asset given by the constant 1. The process $\xi = (\xi_t)$ is assumed to be *predictable* and $\eta = (\eta_t)$ is assumed to be *adapted* with respect to the σ -fields (\mathcal{F}_t) . The motivation for this distinction is explained in Föllmer and Sondermann (1986); it also becomes apparent in section 2.5. For such a strategy, the *value* of the resulting portfolio at time t is given by

$$(1.14) \quad V_t = \xi_t X_t + \eta_t, \quad \# \text{ The argument is that } V_t \text{ is the value of the portfolio at time } t.$$

and the *cumulative cost* up to time t is given by

$$(1.15) \quad C_t = V_t - \int_0^t \xi_s dX_s,$$

since the stochastic integral $\int_0^t \xi_s dX_s$ measures the *gain from trade* resulting from the price fluctuations of the risky asset. Now consider a contingent claim H , and assume that H admits an *Itô representation*

$$(1.16) \quad H = H_0 + \int_0^T \xi_s^H dX_s$$

as a stochastic integral of X . Then we can *replicate* or *hedge* the contingent claim by the following dynamic portfolio strategy: Take

$$(1.17) \quad \xi = \xi^H$$

and η such that

$$(1.18) \quad V_t = \xi_t^H X_t + \eta_t = H_0 + \int_0^t \xi_s^H dX_s.$$

This implies

$$(1.19) \quad V_T(\omega) = H(\omega),$$

i.e., our strategy leads to a *perfect replication* of the claim H for any scenario $\omega \in \Omega$ which may unfold. Moreover,

$$(1.20) \quad C_t(\omega) = H_0 \quad (0 \leq t \leq T),$$

i.e., the strategy is *self-financing*: Starting with the initial amount $H_0 = C_0$ it produces, without any further risk, the final amount $V_T(\omega) = H(\omega)$. Thus, the correct price of the claim should coincide with this initial amount $V_0 = H_0$. There is no theoretical reason to add a safety loading.

A convenient way of computing this price in a direct manner, without first finding the representation (1.16), is provided by the equivalent martingale measure P^* : Since

$$E^* \left[\int_0^T \xi^H dX \right] = 0$$

due to the martingale property of X under P^* (and our implicit integrability assumptions), we obtain

$$(1.21) \quad V_0 = H_0 = E^*[H].$$

The model (Ω, \mathcal{F}, P) of our financial market is called *complete* if *every* contingent claim H admits an Itô representation of the form (1.16). For the purposes of this paper, *completeness* is equivalent to *uniqueness* of the equivalent martingale measure P^* ; cf. Jacod (1979). A fundamental theorem of K. Itô (1951) states that the canonical model for Brownian motion, given by Wiener measure P on $\Omega = C[0, T]$ with $X_t(\omega) = \omega(t)$, is indeed complete. Many diffusion processes can be constructed as pathwise functionals of Brownian motion and thereby inherit completeness. In particular, it follows from Itô's theorem that the standard reference model (1.7) is complete. Thus, the general argument leading to (1.21) also explains the Black-Scholes formula (1.8).

For simplicity, we have limited our discussion to one single financial asset. But the argument is equally valid in the multidimensional case $X = (X^1, \dots, X^n)$ where the financial market consists of n risky assets. Such a market is called *complete* if every contingent claim can be represented in the form

$$H = H_0 + \sum_{k=1}^n \int_0^T \xi_t^k dX_t^k,$$

and in that case the preceding discussion applies with only notational changes.

1.4 How to Compute the Strategy

In order to compute the hedging strategy, we have to find the integrand ξ^H in the representation (1.16). The value process is given by

$$(1.22) \quad V_t = E^*[H | \mathcal{F}_t],$$

and so the remaining component η of our strategy will be determined via (1.18). Let us first consider the case

$$(1.23) \quad H = f(X_T),$$

where the contingent claim depends only on the final value X_T of the underlying asset. Let us assume that under P^* the process (X_t) is a diffusion with generator L^* . Denote by h the solution of the *boundary value problem*

$$(1.24) \quad (L^* + \frac{\partial}{\partial t})h = 0, \quad h(\cdot, T) = f.$$

By Itô's formula we obtain

$$h(X_t, t) = h(X_0, 0) + \int_0^t h_x(X_s, s) dX_s,$$

and this implies

$$(1.25) \quad \xi_s^H = h_x(X_s, s), \quad V_s = h(X_s, s).$$

Consider the special case (1.7) of *geometric Brownian motion*. The solution of (1.24) can be computed explicitly as

$$h(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(x e^{\sigma u \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)}\right) e^{-\frac{u^2}{2}} du.$$

For a *call option* with $f(x) = (x - c)^+$ we get

$$h(x, t) = x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{c} + \frac{1}{2}\sigma^2(T-t)\right)\right) - c\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{c} - \frac{1}{2}\sigma^2(T-t)\right)\right)$$

and

$$(1.26) \quad \xi_t = \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{X_t}{c} + \frac{1}{2}\sigma^2(T-t)\right)\right),$$

where Φ denotes the distribution function of the standard normal distribution. In particular, the *price* of a call option at the initial time $t = 0$ and for an initial value $X_0 = x$ of the underlying asset is given by

$$(1.27) \quad V_0 = h(x, 0) = x\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{c} + \frac{1}{2}\sigma^2(T)\right)\right) - c\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{c} - \frac{1}{2}\sigma^2(T)\right)\right).$$

This is the *Black Scholes option pricing formula* (with $r = 0$) as it appeared in Black and Scholes (1973).

Let us mention at this point a conceptual problem. Trading strategies should be built upon the single empirical trajectory $X_t(\omega)$ ($0 \leq t \leq T$) which is actually observed. On the other hand, general stochastic integrals cannot be constructed path by path. But in our present case, we are dealing with the special class of stochastic integrals arising from Itô calculus, and these can indeed be constructed in a strictly pathwise manner, “without probabilities”; cf. Föllmer (1981). Based on this observation, a pathwise approach to the Black-Scholes formula has been developed in Bick and Willinger (1990).

For contingent claims depending only on the final value $X_T(\omega)$, we have reduced the computation of the trading strategy to the solution of a partial differential equation. But we could also consider a *look-back option* H which depends on the whole trajectory $X_t(\omega)$ ($0 \leq t \leq T$). In that case, the computation of the integrand ξ in (1.16) becomes more involved. In the context of (1.7), H may be viewed as a functional of the underlying Brownian motion. More precisely, we write $H = F(W^*)$ where $W_t^* = W_t + \frac{m}{\sigma}t$ is a standard Brownian motion under the martingale measure P^* . Suppose that F , viewed as a function on $C[0, T]$, is almost surely Fréchet differentiable and denote by $DF(W^*, \cdot)$ the derivative, regarded as a measure on $[0, T]$. Under some integrability conditions, the integrand can now be identified via *Clark's formula*

$$(1.28) \quad H = E^*[H] + \int_0^T E^*[DF(W^*, (t, T)) | \mathcal{F}_t] dW_t^*;$$

cf., e.g., Rogers and Williams (1987). In fact, since

$$dX = \sigma X dW^*,$$

(1.28) implies

$$(1.29) \quad \xi_t^H = (\sigma X_t)^{-1} E^*[DF(\cdot, (t, T)) | \mathcal{F}_t].$$

As an explicit example, consider the *maximum option*

$$(1.30) \quad H(\omega) = \max_{0 \leq t \leq T} X_t(\omega) = X_0 \exp[\sigma M_T(\omega)] = F(W^*)$$

where we use the notation

$$M_{s,t} = \max_{s \leq u \leq t} [W_u^* - \frac{1}{2}\sigma u]$$

and $M_t = M_{0,t}$. The maximum of (X_t) is attained, P^* -almost surely, at a unique random time $\tau(W^*)$. At such a trajectory, F is differentiable with

$$DF(W^*, \cdot) = \sigma F(W^*) \delta_{\tau(W^*)},$$

where δ_c denotes the Dirac measure at c . Using the Markov property, we can now compute

$$\begin{aligned} E^*[DF(W^*, (t, T)) | \mathcal{F}_t](W^*) &= E^*[\sigma F(W^*) I_{\{M_{t,T} > M_t\}} | \mathcal{F}_t](W^*) \\ &= \sigma X_t E^*[\exp(\sigma M_{T-t}); M_{T-t} > M_t(W^*)] \end{aligned}$$

For a fixed value of W^* , this expectation only depends on the distribution of the maximum of a Brownian motion with constant drift. But this distribution is known explicitly; cf. Shepp (1979). This leads to the strategy

(1.31)

$$\begin{aligned} \xi_t(\omega) &= \left(-\log \frac{M_t}{X_t}(\omega) + \frac{\sigma^2(T-t)}{2} + 2 \right) \cdot \Phi \left(\frac{-\log \frac{M_t}{X_t}(\omega) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right) \\ &\quad + \sigma\sqrt{T-t} \varphi \left(\frac{-\log \frac{M_t}{X_t}(\omega) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right), \end{aligned}$$

and to the value

$$(1.32) \quad V_0 = E^*[H] = X_0 \left(\frac{\sigma^2 T}{2} + 2 \right) \Phi \left(\frac{1}{2} \sigma \sqrt{T} \right) + \sigma \sqrt{T} X_0 \varphi \left(\frac{1}{2} \sigma \sqrt{T} \right),$$

where φ denotes the density of the standard normal distribution; cf. Goldman et al. (1979). The derivation via Clark's formula is due to M. Schweizer; it parallels IV.41.13 in Rogers and Williams (1987).

As shown by J.M. Bismut, the Clark formula (1.28) can be derived as an exercise in Malliavin calculus; cf., e.g., Rogers and Williams (1987). A systematic application of Malliavin calculus techniques leads to more refined versions, where differentiability and integrability assumptions are relaxed considerably. Actually, one of the recent extensions of Clark's formula was motivated by the application to financial markets; cf. Karatzas and Ocone (1989). Thus, the mathematics of financial markets becomes involved with very advanced techniques of stochastic analysis. Here is another example: Merton (1989) introduces contingent claims of the form

$$H = \delta_c(X_T),$$

in analogy to the *pure securities* of Arrow and Debreu. But a rigorous mathematical approach to distributions on Wiener space has been developed only recently; cf. Watanabe (1984).

2. Minimizing Risk in an Incomplete Market

So far we have described the construction of hedging strategies in a *complete* financial market where every contingent claim admits a representation (1.16). In that *ideal* case, hedging allows us to eliminate all the uncertainty involved in handling an option. But *real* situations will typically be *incomplete*. In discrete time, this happens whenever the possibilities at one single step do not reduce to a binary scenario as in section 1.3. It also happens in continuous time if the price fluctuations include jumps of varying size, or if a contingent claim depends on additional sources of randomness. In particular, the standard model (1.7) becomes incomplete if the variance is not completely known; cf. section 2.4 below.

In an *incomplete* situation, there will be claims which carry an *intrinsic risk*. In this section, our purpose is to find a dynamic portfolio strategy which reduces the actual risk to that intrinsic component. Thus, our emphasis is *not* on the pricing of claims (we will come back to that question in section 3), but on the *reduction of risk*, as it is perceived by an agent who evaluates the situation in terms of the probability measure P . In continuous time, this reduction of risk corresponds to a regression problem for semimartingales, and this is a somewhat technical matter. But the basic idea is quite simple, and in section 2.5 it is described in an elementary two-period model. If the reader is less interested in the martingale aspects of the problem, he is invited to pass immediately on to 2.5.

2.1 Hedging with a martingale

Let us first consider the case $P = P^*$ where X is already a martingale under the initial measure P . In this case, the following criterion of *risk-minimization* was introduced in Föllmer and Sondermann (1986). For a given contingent claim H , we look for a strategy which *replicates* the claim in the sense that

$$(2.1) \quad V_T = H,$$

and which *minimizes*, at each time t , the *remaining risk*

$$(2.2) \quad E[(C_T - C_t)^2 | \mathcal{F}_t];$$

the minimum is taken over all continuations of this strategy after time t which respect (2.1). H admits a representation (1.16) if and only if this remaining risk

can be reduced to 0, and this means that the risk-minimizing strategy is self-financing. For a general contingent claim, this will no longer be true. However, the risk-minimizing strategy is always *mean-self-financing* in the sense that

$$(2.3) \quad E[C_T - C_t | \mathcal{F}_t] = 0 \quad (0 \leq t \leq T),$$

i.e., the cost process C associated to a risk-minimizing strategy is a *martingale*.

Existence and uniqueness of a risk-minimizing strategy follow from the decomposition

$$(2.4) \quad H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H$$

where (L_t^H) is a martingale orthogonal to X . The strategy is then described by

$$(2.5) \quad \xi_t = \xi_t^H, \quad \eta_t = V_t - \xi_t X_t$$

with

$$(2.6) \quad V_t = H_0 + \int_0^t \xi_s^H dX_s + L_t^H.$$

The associated cost process is of the form

$$(2.7) \quad C_t = H_0 + L_t^H,$$

and so the remaining risk at time t is given by

$$(2.8) \quad E[(L_t^H - L_t^H)^2 | \mathcal{F}_t].$$

The value process V can be computed directly as the martingale

$$(2.9) \quad V_t = E[H | \mathcal{F}_t] \quad (0 \leq t \leq T).$$

Its decomposition (2.6), and in particular the decomposition (2.4) of $V_T = H$, is obtained by applying the well-known Kunita-Watanabe projection technique in the space of square-integrable martingales: We simply project the martingale V associated to H via (2.9) on the martingale X ; cf. Föllmer and Sondermann (1986).

2.2 Hedging with a semimartingale

Let us consider the general *incomplete* case $P \approx P^*$, where P itself is no longer a martingale measure. (X_t) is now a *semimartingale* under P , and we assume that it admits a Doob decomposition of the form

$$(2.10) \quad X_t = X_0 + M_t + A_t \quad (0 \leq t \leq T)$$

where (M_t) is a martingale and (A_t) is a continuous process with paths of bounded variation. Schweizer (1988) introduces a criterion of *local risk-minimization*. For a strategy with (2.1), it is essentially equivalent to the condition that the associated cost process

$$(2.11) \quad C \text{ is a martingale orthogonal to } M.$$

Such a strategy will be called *optimal*. An optimal strategy corresponds to a decomposition

$$(2.12) \quad H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H$$

where (L_t^H) is a martingale orthogonal to M . Given such a decomposition, the associated optimal strategy (ξ, η) can be computed again by (2.5) and (2.6). Conversely, an optimal strategy leads to the decomposition

$$H = C_T + \int_0^T \xi_s dX_s = C_0 + \int_0^T \xi_s dX_s + (C_T - C_0),$$

and this is of the form (2.12). Thus, the problem of minimizing risk is reduced to the task of finding the representation (2.12). This is of course analogous to (2.4). But if X is not a martingale, we can no longer use directly the Kunita-Watanabe projection technique. An obvious idea is to shift the problem to an equivalent martingale measure. But in contrast to the complete case, the martingale measure is no longer unique, and different martingale measures may lead to different strategies. However, in a number of situations it turns out that there is a *minimal* martingale measure $P^* \approx P$ such that the optimal strategy for P can be computed in terms of P^* ; cf. Schweizer (1990b) and Föllmer and Schweizer (1990).

2.3 Incomplete information

Let us now consider a situation which would be complete if we had more information. In that case, the projection problem can be solved in a direct manner.

Recall that the information accessible to us is described by the σ -fields (\mathcal{F}_t) . Now suppose that the model is complete with respect to some larger family $(\tilde{\mathcal{F}}_t)$, and that the Doob-Meyer decomposition (2.10) of X with respect to (\mathcal{F}_t) is also valid with respect to $(\tilde{\mathcal{F}}_t)$, i.e., M is also a martingale with respect to $(\tilde{\mathcal{F}}_t)$. A class of examples will be given in section 2.4.

By assumption, a given claim H has the representation

$$(2.13) \quad H = \tilde{H}_0 + \int_0^T \tilde{\xi}_s^H dX_s$$

where \tilde{H}_0 is $\tilde{\mathcal{F}}_0$ -measurable, and where the process $\tilde{\xi}^H$ is predictable with respect to the larger filtration $(\tilde{\mathcal{F}}_t)$. In Föllmer and Schweizer (1990) it is shown that (2.13) implies

$$(2.14) \quad H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H$$

with $H_0 = E[\tilde{H}_0 | \mathcal{F}_0]$, where the integrand is obtained by *predictable projection*:

$$(2.15) \quad \xi_t^H = E[\tilde{\xi}_t^H | \mathcal{F}_t].$$

Thus, there exists a unique optimal strategy given by (2.5) and (2.6).

2.4. Incompleteness due to a random variance

As a more explicit example, consider a diffusion model on $C[0, T]$ together with an additional source of randomness, described by a probability space (S, \mathcal{S}, μ) , which affects the variance of the diffusion. Let \mathcal{F} be the natural product σ -field on

$$\Omega = C[0, T] \times S,$$

define $X_t(\omega) = \omega_0(t)$ for $\omega = (\omega_0, \eta) \in \Omega$, let (\mathcal{F}_t) denote the σ -fields generated by the process (X_t) , and let $(\tilde{\mathcal{F}}_t)$ be the larger family obtained by adding at time 0 the information in \mathcal{S} . The diffusion is given by adapted processes (β_t) and $(\sigma_t(\cdot, \eta))$ ($\eta \in S$) on $C[0, T]$ such that the stochastic differential equation

$$(2.16) \quad dX_t = \sigma_t(X, \eta) dW_t + \beta_t(X) dt$$

with respect to a standard Wiener process (W_t) has a unique solution for any $\eta \in S$. Let P_η be the corresponding distribution on $C[0, T]$, and assume that the diffusion model $(C[0, T], P_\eta)$ is complete for any $\eta \in S$.

Now let P be the probability measure on (Ω, \mathcal{F}) given by $P(d\omega_0, d\eta) = \mu(d\eta)P_\eta(d\omega_0)$. Then the conditions of the previous section are satisfied. In particular, the model (Ω, \mathcal{F}, P) is *complete* with respect to $(\tilde{\mathcal{F}}_t)$. By predictable projection as in (2.15), we obtain the representation (2.14) of a given contingent claim H and the corresponding optimal hedging strategy.

In order to illustrate this general projection method, consider the standard Black-Scholes model, but with a random jump of the diffusion parameter at time t_0 . For $\eta \in S = \{+, -\}$ and $t_0 \in (0, T)$, put

$$\sigma_t(\eta) = \sigma^0 I_{[0, t_0)}(t) + \sigma^\eta I_{[t_0, T]}(t)$$

with fixed parameters $\sigma^0, \sigma^+, \sigma^- > 0$, and define μ by $\mu(\{+\}) = p$. Let P_η be the distribution of the solution of the stochastic differential equation

$$dX_t = \sigma_t(\eta) \cdot X_t dW_t + m \cdot X_t dt$$

with some drift parameter $m \in \mathbb{R}$. Any contingent claim can be written as a stochastic integral with respect to the larger filtration $(\tilde{\mathcal{F}}_t)$:

$$(2.17) \quad H = H_0^+ I_B + H_0^- I_{B^c} + \int_0^T (\xi_s^+ I_B + \xi_s^- I_{B^c}) dX_s$$

where H_0^\pm and ξ^\pm denote the usual Black-Scholes values and strategies for a *known* variance σ^\pm , and where $B = \{\eta = +\}$. The decomposition (2.14) with respect to the smaller filtration (\mathcal{F}_t) is given by

$$H_0 = E[\tilde{H}_0] = pH_0^+ + (1-p)H_0^-,$$

$$\xi = (p\xi^+ + (1-p)\xi^-)I_{[0, t_0]} + (\xi^+ I_B + \xi^- I_{B^c})I_{(t_0, T]}$$

and

$$L_t^H = (H_0^+ - H_0^-) \cdot (I_B - p) \cdot I_{[t_0, T]}(t).$$

This determines the optimal strategy. It depends explicitly on p but not on the drift parameter m . In fact, it can be computed in terms of the *minimal* martingale measure P^* which eliminates the drift but does not change the parameter p .

In the martingale case $P = P^*$, this example was introduced in Harrison and Pliska (1981), and the optimal strategy already appears in Müller (1985). The general projection method in Föllmer and Schweizer (1990) was developed in order to provide a systematic approach to examples of this kind.

2.5 A two-period example

In discrete time, the preceding discussion becomes much simpler: A unique risk-minimizing strategy does exist, and it can be computed by a backward iteration from the terminal time T . At each step, the computation reduces to the following elementary regression argument, except that expectations are replaced by conditional expectations with respect to the information available at that step; cf. Föllmer and Schweizer (1989). Here we only explain the structure of one single step.

Consider a simple two-period model for the risky asset, say for the exchange rate of US\$ against SFR. At time 0 the exchange rate X_0 is known and will be treated as a constant. The exchange rate X_1 at time 1 is a random variable on some probability space (Ω, \mathcal{F}, P) . Now assume we have sold an option described by a random variable H on the same probability space. At time 1 we will have to pay the random amount $H(\omega)$. In order to hedge against this pay-off, we buy ξ_1 dollars and put aside η_0 Swiss francs. This initial portfolio at time 0 has the value

$$(2.18) \quad V_0 = \xi_1 X_0 + \eta_0 .$$

At the terminal time 1, we want a portfolio whose value is exactly equal to H . The value of the dollar account will be $\xi_1 X_1$, and so we have to adjust the Swiss franc account from η_0 to $\eta_1 = H - \xi_1 X_1$ in order to obtain

$$(2.19) \quad V_1 = \xi_1 X_1 + \eta_1 = H .$$

For a given H , such a strategy will be determined by our choice of the constants ξ_1 and V_0 at the initial time 0; η_1 is a random variable which is determined at time 1 by (2.19).

Let us examine the resulting *cumulative costs*. At time 0 we have $C_0 = V_0$, and the additional cost due to our adjustment of the Swiss franc account at time 1 is given by

$$(2.20) \quad C_1 - C_0 = \eta_1 - \eta_0 = V_1 - V_0 - \xi_1(X_1 - X_0).$$

due to (2.18) and (2.19). Let us now choose our trading strategy described by ξ_1 and V_0 in such a way that the *remaining risk* at time 0, measured by the expected quadratic cost

$$E[(C_1 - C_0)^2] = E[(H - V_0 - \xi_1(X_1 - X_0))^2] ,$$

is minimized. But this simply means that we are looking for the *best linear estimate* of H based on $X_1 - X_0$. Thus, the problem is solved by *linear regression*: The optimal constants ξ_1 and V_0 are given by

$$(2.21) \quad \xi_1 = \frac{\text{Cov}(H, X_1)}{\text{Var}[X_1]}$$

and

$$(2.22) \quad V_0 = E[H] - \xi_1 E[X_1 - X_0].$$

In particular we obtain

$$(2.23) \quad C_0 = E[C_1].$$

Thus, the optimal strategy is *mean-self-financing*: once we have determined the initial value $V_0 = C_0$, the additional cost $C_1 - C_0$ is a random variable with expectation $E[C_1 - C_0] = 0$. By this optimal trading strategy, the remaining risk is reduced to the minimal mean square prediction error

$$(2.24) \quad R_{\min} = \text{Var}[H] - \xi_1^2 \text{Var}[X_1] = \text{Var}[H](1 - \varrho(H, X_1)^2)$$

where ϱ denotes the correlation coefficient. This value R_{\min} may be viewed as the *intrinsic risk* of the option H . It is this intrinsic risk, and not the *a priori risk* measured by the variance $\text{Var}[H]$ of H , on which any safety loading should be based.

Note that the optimal strategy $\xi_1 = \xi^H$ and $V_0 = H_0$ can be read off immediately from the decomposition

$$(2.25) \quad H = H_0 + \xi^H(X_1 - X_0) + L^H,$$

where L^H has expectation 0 and is orthogonal to $X_1 - X_0$. This decomposition is the elementary analogue of (2.12). In retrospect, we can now describe the construction in sections 2.1 and 2.2 as a *sequential regression in continuous time* with respect to the basic semimartingale (X_t) .

Looking back to the elementary example in section 1.3, we can now clearly see its crucial point: The binary structure trivially implies a *linear dependence* between H and $X_1 - X_0$. Thus, the regression of H on $X_1 - X_0$ becomes a *perfect* regression with $L^H = 0$, i.e., the model *complete*. In particular, the risk in (2.24) is completely eliminated. As soon as we admit a third possibility for the value of

$X_1(\omega)$, the situation becomes less pleasant: There will be an intrinsic risk, and hedging should be viewed as a linear regression which is no longer perfect.

Let us conclude this elementary section with two elementary questions:

- i) *Why do we concentrate only on risk reduction (instead of applying, e.g., some mean-variance criterion)?*
- ii) *Why do we insist on $V_T = H$?*

Both questions refer to the investment problem faced by the option seller. The answer is that this problem may be *decomposed* into a pure *investment* problem which does not take into account the option, and into the pure *hedging* problem whose solution we have just sketched. Suppose in fact that the option H has been sold for a price $\pi(H)$, and that the option seller invests the amount ζ into the risky asset. His gain is given by

$$\begin{aligned} G &= (\pi(H) - H) + \zeta(X_1 - X_0) \\ &= (\pi(H) - H_0) + (\zeta - \xi^H)(X_1 - X_0) - L^H. \end{aligned}$$

If his decision is based on a *mean-variance criterion* of the form

$$E[G] - \alpha \text{var}[G] = \max,$$

then the optimal amount is given by

$$\zeta - \xi^H = \frac{E[X_1 - X_0]}{2\alpha \text{var}[X_1 - X_0]}.$$

But the right side coincides with the amount ζ_{inv} which would be invested without taking into account the option. Thus, the optimal overall strategy *splits* into the pure investment strategy ζ_{inv} , and into the pure hedging strategy which was determined above:

$$(2.26) \quad \zeta = \zeta_{inv} + \xi^H.$$

This splitting argument came out of a discussion with F. Dybvig. Its extension to the general setting of the previous sections is discussed in Schweizer (1990b).

3. Pricing via Risk Exchange

In the *complete* situation of section 1, the *fair price* of a contingent claim H was determined as the expected value

$$(3.1) \quad E^*[H]$$

with respect to the unique martingale measure P^* equivalent to P . From a modern actuarial point of view, the computation of the price in terms of a new measure may actually look familiar. For example, a premium principle in terms of exponential utility leads to the price

$$(3.2) \quad \tilde{E}[H] = \frac{1}{Z(\alpha)} E[e^{\alpha H} H],$$

computed as the expected value of H with respect to a Esscher transform \tilde{P} of P . Note, however, that any premium principle involving risk aversion would, as in (3.2), result in a price which is greater than the expected value $E[H]$. On the other hand, we could easily have $E^*[H] < E[H]$. The crucial difference is this: A premium principle like (3.2) is defined solely in terms of the distribution of H , while (3.1) involves the underlying risky asset (X_t) . This conforms with the economic point of view that the pricing of H should form part of a broader equilibrium analysis.

3.1 Consistent pricing

In a model of economic equilibrium, a consistent price system at time 0 should be given by a continuous positive linear functional on the commodity space $\mathcal{L}^2 = \mathcal{L}^2(\Omega, \mathcal{F}, P)$ of contingent claims, i.e., by a positive price density $\tilde{\varphi} \in \mathcal{L}^2$. We may assume that the corresponding measure \tilde{P} is normalized to be a probability measure. Thus, the price of $H \in \mathcal{L}^2$ at time $t = 0$ would be its expected value

$$(3.3) \quad \tilde{E}[H]$$

with respect to \tilde{P} . We are leaving aside interest rates, and so prices at times $t > 0$ should be computed as conditional expectations

$$(3.4) \quad \tilde{E}[H|\mathcal{F}_t]$$

with respect to the σ -algebras \mathcal{F}_t describing the information available at time t . But X_t is supposed to denote the price at time t of the underlying financial asset with value $H = X_T$ at time T , and so we should have

$$(3.5) \quad X_t = \tilde{E}[X_T|\mathcal{F}_t].$$

This means that the process (X_t) is a *martingale* under \tilde{P} . In the complete case, a martingale measure equivalent to P is uniquely determined, and so we get $\tilde{P} = P^*$.

Thus, a consistent price system on a *complete* financial market necessarily leads to the valuation (3.1) of contingent claims.

In a typical model of *risk theory*, where the stochastic process (X_t) involves jumps of varying size, the situation will be *incomplete*. Thus, there will be several equivalent martingale measures. Delbaen and Haezendonck (1989) develop a systematic approach to *principles of premium calculation* based on different choices of the martingale measure. For a discussion of option pricing with a view to premium rating in Insurance see also Taylor (1989) and Sondermann (1988).

In the context of general economic equilibrium theory, the question of pricing in incomplete financial markets is an area of very active research. Geanakoplos (1990) and Duffie (1991) give a survey of its present scope. Here we shall mention just a few of its probabilistic aspects.

3.2 Pricing and parametric models

We have argued that a consistent price mechanism should be of the form (3.1) for some martingale measure P^* equivalent to P . In the *incomplete* case, such a martingale measure is no longer unique, and so we are faced with a choice. The following recipe may look tempting. Suppose that we have reasons to choose the underlying model P from a parametric family P_λ ($\lambda \in \Lambda$) of equivalent probability measures on (Ω, \mathcal{F}) . For example, in the complete case of section 1 we could take the class P_λ ($\lambda \in \Lambda$) of all geometric Brownian motions with fixed variance σ^2 and varying drift parameter $\lambda = m$. Suppose that there is exactly one such measure P_{λ^*} which turns (X_t) into a martingale. Then it may seem natural to choose

$$(3.6) \quad P^* = P_{\lambda^*}$$

as our price system; cf. Brennan (1979). In the complete case, this leads necessarily to the correct answer. In the incomplete case, we have to check whether this choice is consistent with an economic equilibrium approach. Let us now recall the basic equilibrium argument for *risk exchanges* between individual agents maximizing expected utility. It goes back to Arrow (1953); cf. Borch (1960), Pesonen (1984) and Bühlmann (1984) for its relevance in actuarial literature, and Duffie (1988, 1991) for a systematic exposition from the point of view of mathematical economics.

3.3 Risk exchanges

We go back to a two-period model. Consider a finite set I of economic agents with utility functions u_i ($i \in I$). Each agent i holds risky assets whose pay-off at time $t = 1$ is given by a random variable W_i on (Ω, \mathcal{F}) . These assets are traded at time $t = 0$. The result will be a new allocation Y_i ($i \in I$) of assets such that

$$(3.7) \quad \sum_i Y_i = \sum_i W_i = W,$$

where W denotes the resulting total wealth. Agent i evaluates the possible outcomes at time $t = 1$ in terms of a probability measure P_i on (Ω, \mathcal{F}) . We assume that P_i is given by a density of the form $\psi_i = g_i(W)$ with respect to a common reference measure P . A *price system*, given by a probability measure Q on (Ω, \mathcal{F}) , and an *allocation* Y_i ($i \in I$) define an *equilibrium* if, for each $i \in I$, the random variable Y_i maximizes

$$(3.8) \quad E_i[u_i(Y)] = E[u_i(Y)\psi_i]$$

under the constraint

$$(3.9) \quad \int Y dQ = \int W_i dQ.$$

Granting some regularity, the first-order conditions for such an equilibrium are

$$(3.10) \quad E_i[u'_i(Y_i)Z] = E_i[u'_i(Y_i) \int Z dQ] = c_i \int Z dQ$$

for any bounded measurable Z . This implies

$$(3.11) \quad Q \approx P_i \quad (i \in I)$$

and the identification

$$(3.12) \quad u'_i(Y_i)\psi_i = c_i\varphi \quad (i \in I)$$

of the density φ of Q with respect to P . Note that in such an equilibrium the presence of *one* risk neutral agent j with constant u'_j would imply $Q = P_j$, i.e., the price system would be given by the valuation P_j of this risk neutral agent.

In general, Q involves an intricate *aggregation of individual preferences and expectations*. In fact, (3.12) and (3.7) imply $\varphi = f(W)$ with

$$(3.13) \quad (\log f)' = -\alpha + \sum \frac{\alpha}{\alpha_i} (\log g'_i)',$$

where $\alpha_i = -(\log u_i)'(Y_i)$ denotes the risk aversion of agent i at the level Y_i and α is defined by $\alpha^{-1} = \sum \alpha_i^{-1}$. As an example, consider the case where individual utilities are exponential with constant risk aversion α_i ($i \in I$). Then the aggregation of individual utilities and expectations takes the form

$$(3.14) \quad \varphi = \frac{1}{Z(\alpha)} e^{-\alpha W} \tilde{\psi}, \quad \tilde{\psi} = \prod \psi_i^{\frac{\alpha}{\alpha_i}},$$

i.e., the price system Q belongs to the *exponential family* with respect to W and the aggregate measure $d\tilde{P} = \tilde{\psi}dP$.

Now assume that at time $t = 0$ each agent $i \in I$ holds ξ_i shares of our risky asset so that $W_i = \xi_i X_T$ and $W = X_T$. Suppose also that each agent believes in *geometric Brownian motion* with drift parameter m_i and fixed volatility σ . We denote by P_i the corresponding distribution and take as reference measure the martingale measure P^* with $m^* = 0$. Let us also assume *constant proportional risk aversion*, i.e., $u'_i(x) = x^{-\beta_i}$ with some $\beta_i \geq 0$. Then (3.12) takes the form

$$(3.15) \quad Y_i^{-\beta_i} W^{\frac{m_i}{\sigma^2}} = c_i \varphi.$$

In the special case $\beta_i = \beta$ and $m_i = \beta\sigma^2$ we get $\varphi = 1$ and $Y_i = W_i$, i.e., there is no incentive to trade. In particular, the price system Q is given by the martingale measure P^* . Thus, the recipe (3.6) is confirmed under these special assumptions. In discrete time, this is sometimes viewed as a justification for applying the *Black Scholes formula* even though the situation is now incomplete; cf. Rubinstein (1976) and Brennan (1979). In continuous time, the argument may be used to conclude that our basic model (1.7) of geometric Brownian motion is *sustained* in an economy where all agents have the same constant proportional risk aversion, and all believe in this model. This very special choice of the microeconomic model thus leads to a *rational expectations equilibrium*, i.e., to an equilibrium of plans, prices and price expectations; cf. Kreps (1982) and Bick (1987).

So we see that recipe (3.6) may work, but that its validity should be viewed as a rather lucky coincidence. Independent of this special question, the equilibrium analysis of risk exchanges is important in many other ways. For example, it can be used to explain the demand for portfolio insurance in the sense of section 1.1, i.e., for claims Y_i which are defined by *convex* functions of the underlying asset. Thus, it helps to answer the question “*Who should buy portfolio insurance?*”; cf. Leland (1980) and Müller (1989).

The analysis of sequential risk exchanges should also play an important part in reaching a deeper understanding of the standard reference model (1.7) for the underlying price fluctuations. In fact, a rigorous derivation of this model from microeconomic assumptions is one of the fundamental problems in this area. We have just seen one possible approach to this problem in terms of rational expectations equilibria. But this involves a very delicate balance of individual preferences and expectations, and so it does not really explain why geometric Brownian motion should be viewed as a robust fundamental model. Let us now look at this question from some other points of view.

4. The choice of P

So far, the stochastic model (Ω, \mathcal{F}, P) for price fluctuations of our risky asset has been fixed in advance, and our discussion of hedging strategies has been based on this model. We have seen that these strategies are to some extent independent of the specific choice of P ; this is indeed one of the attractive features of the theory. But this independence does not go too far. For example, in the context of the standard model (1.7) the strategy does not depend on the drift but it does depend on the volatility.

Let us now have a closer look at the underlying randomness.

4.1 Options as a source of Brownian motion

In his thesis “Théorie de la Spéculation”, L. Bachelier (1900) introduced *Brownian motion* as a model for the price fluctuation of a risky asset in a speculative market. A rigorous construction of the corresponding measure P on the path space $\Omega = C[0, 1]$ was given by N. Wiener (1923). Thus, we could say that Bachelier chose

$$(4.1) \quad P = \text{Wiener measure on } C[0, T]$$

with some variance parameter σ as his stochastic model; the price fluctuation is given by the coordinate process $X_t(\omega) = \omega(t)$. This choice was based on a loose equilibrium argument which concluded that (X_t) should be a *martingale* under P ; cf. also Samuelson (1965). From this qualitative assumption, one can derive the specific model (4.1) if one also requires homogeneous increments. An exact argument is provided by *Lévy's theorem*: If (X_t) is a martingale under P such that the increments have stationary conditional variance, then P is a Wiener measure.

If the martingale property is assumed for the discounted process $(X_t e^{-mt})$, and if homogeneity is required for relative rather than absolute increments, then the preceding argument leads to *geometric Brownian motion* described by the stochastic differential equation

$$(4.2) \quad dX = \sigma X dW + mX dt$$

with respect to a standard Brownian motion (W_t) . Its explicit solution is given by

$$(4.3) \quad X_t = X_0 e^{\sigma W_t + (m - \frac{1}{2}\sigma^2)t}.$$

This model, advocated by Samuelson (1964) and others since the fifties, has served as the basic reference model for fluctuations on a speculative market in continuous

time. We have seen how Black and Scholes formulated their option pricing formula in the context of this model.

It is interesting to note that Bachelier's main concern was the computation of a premium for certain financial instruments including options; Brownian motion is introduced as a model with a view toward this end. In this sense, we might say that options motivated the appearance of Brownian motion. A formula for the price of an option appears on p.51 in Bachelier (1900): It can be regarded as the *normal* analogue of the *log-normal* Black-Scholes formula. But this is just a coincidence, due to the fact that Bachelier has already chosen the martingale measure. His pricing argument is based on the classical prescription (1.4) and does not involve the deeper idea of dynamical hedging.

4.2 Absence of arbitrage

From the point of view of mathematical economics, one would like to see a more rigorous derivation of the underlying stochastic model. There is one basic result in that direction. Roughly speaking, it asserts that the absence of arbitrage opportunities implies the existence of an equivalent martingale measure $P^* \approx P$. More precisely, consider *elementary* trading strategies with pay-off

$$\int_0^T \xi dX = \sum_{i=0}^{n-1} \xi_i (X_{t_{i+1}} - X_{t_i})$$

where $0 \leq t_0 < \dots < t_n \leq 1$, and where ξ_i is bounded and \mathcal{F}_{t_i} -measurable. Such a strategy ξ provides an *arbitrage opportunity* if it has a pay-off

$$(4.4) \quad \int_0^T \xi dX \geq 0 \quad P - a.s.$$

which does *not* reduce to

$$(4.5) \quad \int_0^T \xi dX = 0 \quad P - a.s.$$

Now suppose that P admits an equivalent martingale measure P^* . Then the martingale property of X under P^* implies $E^*[\int_0^T \xi dX] = 0$; this is a special case of Doob's fundamental *systems theorem* for martingales. Since $P^* \approx P$, (4.4) holds

also $P^* - a.s.$, and so we can conclude that $\int_0^T \xi dX = 0 \quad P^* - a.s.$, hence (4.5).

Thus, the existence of an equivalent martingale measure $P^* \approx P$ implies the *absence of arbitrage*, i.e., the implication (4.4) \Rightarrow (4.5) for any elementary trading strategy. Conversely, absence of arbitrage essentially implies the existence of an equivalent martingale measure. This was observed by Harrison and Kreps (1979), and it should be viewed as a converse to Doob's theorem. In discrete time, a rigorous proof can be given in a rather straightforward manner; cf. Dalang, Morton and Willinger (1990). In continuous time, it involves deep results from the theory of semimartingales. Here is a precise statement due to Stricker (1990). For $p \in [1, \infty)$ and $q^{-1} + p^{-1} = 1$, the following two statements are equivalent:

$$(4.6) \quad \overline{\left\{ \int \xi dX \mid \xi \text{ elementary} \right\}} \cap \mathcal{L}_+^p = \{0\}$$

$$(4.7) \quad \exists \text{ martingale measure } P^* \approx P \text{ with density } \frac{dP^*}{dP} \in \mathcal{L}^q.$$

Some open questions remain. In particular, it would be nice to have a construction of P^* as a limit of suitably chosen martingale measures in discrete time, where existence is straightforward.

The equivalence of (4.6) and (4.7) extends to financial markets with a finite number of risky assets X^1, \dots, X^n where a martingale measure P^* is defined by the property that each process (X^i) becomes a martingale under P^* . As an illustration, recall the elementary example of section 1.3 with $r = 0$. The martingale measure P^* is uniquely determined by the condition $X_0 = E^*[X_T]$. Let us now introduce the option H as a second marketable asset, with value $V_T = H$ at the terminal time T and some market price $V_0 = \pi(H)$ at the initial time 0. For this extended financial market with two asset processes X and V , a martingale measure \tilde{P} must satisfy the two conditions $\tilde{E}[X_T] = X_0$ and $\tilde{E}[V_T] = V_0$. The first implies $\tilde{P} = P^*$, and so the second reduces to $\pi(H) = E^*[H]$. In the case $\pi(H) \neq E^*[H]$ there would be no martingale measure, and in section 1.3 we have seen that there would indeed be an arbitrage opportunity.

From a mathematical point of view, the equivalence of (4.6) and (4.7) is a beautiful and deep result. Note, however, that there remains a considerable gap between the requirement (4.7) and the specific choice of a model such as (4.3). In fact, almost any probability space which one might think of as a model for the price fluctuations of one single asset would be compatible with (4.7). With a view to reducing this gap, let us now sketch some more specific arguments.

4.3 Some further remarks

As in section 3 we consider a financial market with a finite set I of economic agents. Given a price p proposed at time t , these agents form their demands

$$(4.8) \quad d_i(\omega, p) \quad (i \in I)$$

based on the previous price $X_{t-1}(\omega)$, and on preferences, budget constraints and price expectations which all depend on the updated information available at time t . Thus, $d_i(\cdot, p)$ is \mathcal{F}_t -measurable. The new price $X_t(\omega)$ at time t is given by the equilibrium value \hat{p} such that

$$(4.9) \quad \frac{1}{|I|} \sum_{i \in I} d_i(\omega, \hat{p}) = 1,$$

if the available stock per capita is normalized to 1. Under standard i.i.d. assumptions on the random variables in (4.8), one can establish asymptotic normality of X_t as the number $|I|$ of agents becomes large. In fact, the equilibrium price $X_t(\omega)$ has the structure of an M -estimator in *robust statistics*, and one can simply apply well-known asymptotical results for such estimators; cf. e.g., Huber (1981). For a direct proof of asymptotic normality of equilibrium prices in large random economies see Bhattacharja and Majumdar (1973). A passage from discrete to continuous time, with suitable rescaling, would lead to a diffusion model, i.e., to a probability measure P on the basic path space $\Omega = C[0, T]$. This is one possible approach to (4.3).

Let us look at a simple special case; here we keep the number $|I|$ of agents fixed. For an agent $i \in I$, denote by $\xi_i(\omega)$ the amount of the risky asset held at time $t - 1$, and assume that the demand is given by

$$d_i(\omega, p) = \xi_i(\omega) + c_i(\omega) (p_i(\omega) - p),$$

where

$$p_i(\omega) = X_{t-1}(\omega)(1 + r_i(\omega))$$

denotes the price expected for period $t + 1$, based on the updated information available at time t . The resulting equilibrium price $X_t(\omega) = \hat{p}$ is given by

$$X_t(\omega) = X_{t-1}(\omega)(1 + R_t(\omega))$$

where $R_t(\omega)$ is a mixture of the returns $r_i(\omega)$ ($i \in I$) expected at time t for period $t + 1$. Suppose that R_t is independent of \mathcal{F}_{t-1} with stationary distribution. A

suitable exponential version of the invariance principle as in Duffie and Protter (1988) then leads to the basic reference model (4.3) of *geometric Brownian motion*.

The previous i.i.d. assumptions correspond to the idea that the speculative traders are identical in their behavior, except for a random fluctuation in their demand pattern. The structure of the model becomes more involved, but probably also more realistic, if one distinguishes different types of behavior. Black (1986) advocates a distinction between *information traders* and *noise traders*. In the words of Keynes, an information trader, or *fundamentalist*, would “purchase investments on the best long term expectations he can find”. Noise traders react to fluctuations in the price, e.g., by extrapolation, even though these may be purely random. The basic claim is that the presence of fundamentalists alone would *not* explain what we actually observe. This claim can be substantiated in a rather striking manner. In Smith et al. (1988), experiments are reported where all uncertainty about the fundamentals is eliminated: Agents are faced with an i.i.d. stream of dividends D_t ($t = 1, \dots, T$), and they are provided with all the information about the distribution. Thus, all uncertainty is eliminated from the formation of price expectations insofar these are solely based on dividend forecasts. In a purely fundamentalist model, the resulting price process should become deterministic. But in fact “fourteen out of twenty-two experiments exhibit price bubbles followed by crashes with respect to intrinsic dividend value”. This is compared “to a panic of the sort that sometimes occurs in a crowded theatre”. The authors express their doubt that these critical phenomena can be formulated in terms of a conventional model for price dynamics. But over the last 20 years, there has been great progress in understanding phenomena of this kind in the context of probabilistic models for large systems with many interacting components. This suggests a different approach, based on the idea of *stochastic interaction* in the formation of random demands of individual agents. In the context of a static equilibrium analysis as in (4.9), Markov random field models for interacting preferences were introduced in Föllmer (1974). Recently, the use of such models in dynamic stochastic games in economics was advocated in Durlauf (1989) and Kelly (1989). The application of stochastic interaction models to financial markets seems particularly appropriate. It presents a challenge for present research, both from an economic and a probabilistic point of view.

One can also try to model the effects of interaction in a simpler way, without invoking ideas from the theory of Markov random fields. Suppose, for example, that the proportion of noise traders is random and follows some Markov chain; such models are investigated in joint work with A. Kirman. Then the price fluctuation will be a *diffusion process in a random environment*. Now suppose that the

equilibrium distribution of this Markov chain is *U*-shaped. This implies that, most of the time, either the fundamentalists or the noise traders will prevail. During a fundamentalist regime, prices will follow a stable pattern dominated by a *recurrent* drift toward the perceived fundamental level. But if the proportion shifts so that noise traders take over, the drift will become *transient*, and this will lead to the spontaneous appearance of bubbles and crashes.

In the context of such models, one can begin to study the impact of *Black-Scholes traders* who simply apply the Black-Scholes recipe as if the underlying price process would follow a logarithmic Brownian motion. For such a trader, the demand for stock is a technical demand of the form

$$d_i(\omega, p) = f_i(p, t)$$

which only depends on the proposed price p and on the present mixture of call and put options in his portfolio. The above invariance principle would lead to a non-linear modification of the drift in the basic reference model of logarithmic Brownian motion. Due to the shape of the hedge ratio ξ in (1.26), this modification of the drift has a transient effect. Thus, the impact of technical traders on the underlying price process is similar to the effect of noise traders although their motivation is quite different. For a fixed proportion of technical traders, this modification of the drift would not change the structure of the optimal hedging strategies. But if this proportion is random, in analogy to the previous model, then we would get again a diffusion in a random environment. In particular, we would get a random variance. As we have seen in section 2, the model would become incomplete, and so there would be strong reasons to reconsider the strategies.

In any case, there seems to be a need for a thorough look at the probabilistic structure of basic price fluctuations. This is a challenging program in itself. But it would also lead to a reconsideration of hedging strategies. Furthermore, it would be a crucial step towards a more rigorous analysis of the impact of such strategies on the underlying price process.

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