Robust Projections in the Class of Martingale Measures

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Abstract

Given a convex function f and a set $\mathcal{Q}$ of probability measures, we consider the problem of minimizing the robust f-divergence $\inf_{Q\in\mathcal{Q}} f(P|Q)$ over the class $\mathcal{P}$ of martingale measures. Under mild conditions on $\mathcal{P}$ and $\mathcal{Q}$ we show that a minimizer exists within the class $\mathcal{P}$ if $\lim_{x\to\infty} f(x)/x = \infty$. If $\lim_{x\to\infty} f(x)/x = 0$ then there is a minimizer in a class $\bar{\mathcal{P}}$ of extended martingale measures defined on the predictable σ-field. We also explain how both cases are connected to recent developments in the theory of optimal portfolio choice, in particular to robust extensions of the classical expected utility criterion.

Key words: martingale measures, relative entropy, f-divergence, robust utility, utility maximization, model uncertainty, convex duality.
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1 Introduction

Over the last three decades concepts and methods of martingale theory have played a crucial role in developing the mathematical analysis of financial risk. At the same time the field of finance has become a source of new probabilistic problems which are of intrinsic mathematical

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interest. In this paper our purpose is to analyze a projection problem for martingale measures which arises in the context of optimal portfolio choice.

The notion of a martingale measure has helped to clarify the mathematical structure of the efficient markets hypothesis. In its strong form, the hypothesis states that the price fluctuation of liquid financial assets, modelled as a stochastic process on some filtered probability space, is a martingale under the given probability measure R. In this case Doob's systems theorem would imply that there are no trading strategies with positive expected gain. In a less restrictive version, the hypothesis only requires the absence of arbitrage opportunities, i.e., of strategies which generate a positive expected gain without any downside risk. In this form it is equivalent to the existence of an equivalent *martingale measure*, i.e., a probability measure $P \approx R$ such that the price process is a local martingale under P; see Delbaen and Schachermayer [7] and Yan [44]. The model is called *complete* if there is exactly one equivalent martingale measure. It was already shown by Jacod [25], in the Proceedings of an AMS Symposium on the occasion of J. L. Doob's 65th birthday, that uniqueness of the martingale measure implies a representation property: Functionals of the price process can be represented as stochastic integrals. In the financial interpretation such a functional is viewed as a financial derivative, or a contingent claim. The integrand in the representation specifies a trading strategy in the underlying assets which provides a perfect hedge of the claim, and the arbitrage-free price of the claim is identified as the expectation under the unique equivalent martingale measure. Most realistic models, however, are *incomplete* in the sense that the representation property no longer holds, and so there is a whole class $\mathcal{P}_e$ of equivalent martingale measures.

In its general form, our projection problem consists in finding a probability measure P_0 in some class $\mathcal{P}$ of probability measures $P \ll R$ which minimizes the *robust f-divergence*

$$f(P|\mathcal{Q}) = \inf_{Q \in \mathcal{Q}} f(P|Q) \tag{1}$$

for some class $\mathcal{Q}$ of probability measures $Q \ll R$, i.e.,

$$f(P_0|\mathcal{Q}) = f(\mathcal{P}|\mathcal{Q}) := \inf_{P \in \mathcal{P}} f(P|\mathcal{Q}). \tag{2}$$

Here f is a convex function, and

$$f(P|Q) := E_Q \left[f\left(\frac{dP}{dQ} \right) \right] \tag{3}$$

denotes the f-divergence between two measures P and Q. In the classical case with $\mathcal{Q} = \{Q_0\}$ the projection problem has been considered by many authors, for instance in the context of statistical inference; see Csiszár [4] for the case $f(x) = x \log x$ where the f-divergence reduces to the relative entropy $H(P|Q)$, Rüschendorf [37], or Liese and Vajda [32].

In the financial interpretation the problem of projecting a single measure Q_0 on the class $\mathcal{P}_e$ of equivalent martingale measures arises in the context of optimal portfolio choice. Suppose we want to determine an optimal affordable claim H, given some initial capital x_0 and the possibility of trading in the underlying liquid assets. Affordability translates into the constraint

$$\sup_{P \in \mathcal{P}_e} E_P[H] \leq x_0. \tag{4}$$

If preferences are specified in terms of a concave utility function u and a probabilistic model $Q_0 \approx R$, an affordable claim is optimal if it maximizes the expected utility $E_{Q_0}[u(H)]$. In the complete case $\mathcal{P}_e = \{P_0\}$ the solution is given by

$$H_0 := (u')^{-1}\left(\lambda_0 \frac{dP_0}{dQ_0}\right), \tag{5}$$

where λ_0 is such that $E_{P_0}[H_0] = x_0$. In the incomplete case, the optimal claim is of the form (5) when P_0 is chosen to be the f-projection of Q_0 on $\mathcal{P}_e$, where $f(x) := v(\lambda_0 x)$ for some $\lambda_0 > 0$ and v denotes the convex conjugate of u; see, for instance, Karatzas and Shreve [27], Frittelli [17], Bellini and Frittelli [3], Kramkov and Schachermayer [30] and [31], Goll and Rüschendorf [21], Schachermayer [38], and also Gao, Lim, and Ng [19]. Thus the utility maximization problem is reduced to the classical projection problem of minimizing the f-divergence $f(P|Q_0)$ over the set $\mathcal{P}_e$. Existence results for classical f-projections corresponding to certain utility functions can be found in Frittelli [17] and Bellini and Frittelli [3]. Hugonnier, Kramkov and Schachermayer [24] showed that for reasonably bounded claims the existence of f-projections in the class of martingale measures is equivalent to the existence of unique marginal utility based prices.

Our robust version of the projection problem is motivated by an extension of the classical expected utility approach which takes model uncertainty into account. Instead of fixing a single model Q_0, we consider a whole class $\mathcal{Q}$ of probability measures $Q \ll R$ and define our preferences using the *robust utility functional*

$$U(H) := \inf_{Q \in \mathcal{Q}} E_Q[u(H)]. \tag{6}$$

A microeconomic characterization of such utility functionals in terms of behavioral axioms for the underlying preferences was given by Gilboa and Schmeidler [20]; see also Föllmer and Schied [16] for their relation to the theory of convex risk measures. The robust version of the optimization problem consists in maximizing the functional $U(H)$ under the constraint (4). As shown in Gundel [22], its solution is of the classical form (5) if $(P_0, Q_0) \in \mathcal{P}_e \times \mathcal{Q}$ solves the robust projection problem (2) for the sets $\mathcal{P}_e$ and $\mathcal{Q}$, i.e., if

$$f(P_0|Q_0) = f(P_0|\mathcal{Q}) = f(\mathcal{P}_e|\mathcal{Q}). \tag{7}$$

In Section 2 we analyze the robust projection problem in its general form (2). Our main result is Theorem 2.6. It states that a solution exists if

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty, \tag{8}$$

the set $\mathcal{P}$ is closed in variation, and the set $\mathcal{Q}$ is weakly compact. The key step is to show that $\{f(\cdot|\cdot) \leq c\}$, viewed as a subset of $L^1(\mathbb{R}) \times L^1(\mathbb{R})$, is weakly compact. In the classical case with $\mathcal{Q} = \{Q_0\}$ this follows easily from (8) using the de la Vallée-Poussin compactness criterion. In the general robust case the proof is more delicate. Instead of applying the compactness criterion in terms of f, we have to construct an auxiliary convex function l satisfying (8) such that the compactness condition in terms of l follows via Young's inequality in an appropriate Orlicz space. In Csiszár and Tusnády [5] existence results for robust projections were obtained in two special cases: (i) for the relative entropy $f(P|Q) = H(P|Q)$ on a finite set, and (ii) for the squared L^2-distance between the densities of P and Q.

In Section 3 we explain how the existence of a robust f-projection within the class $\mathcal{P}_e$ yields the solution of the robust utility maximization problem defined by (6) and (4). This section is largely expository: We follow Gundel [22], but we do not assume that all measures in $\mathcal{Q}$ are equivalent. Moreover, our presentation is different and contains some additional results, for example in Theorem 3.11 and Lemma 3.12. In particular we argue for a fixed value x_0 instead of using the duality properties of the maximal utility $U(H)$, viewed as a function of the initial capital x, as they were developed by Bellini and Fritelli [3], Goll and Rüschendorf [21], Kramkov and Schachermayer [30], and Gundel [22].

However, the application of our general existence result for robust f-projections involves Condition (8), and this amounts to the assumption that the utility function u is finite on the whole real line. Without this condition a robust or even a classical projection within the class $\mathcal{P}_e$ of equivalent martingale measures may not exist. Kramkov and Schachermayer [30] have shown how to develop the duality between the classical problem of utility maximization and the projection problem beyond the class $\mathcal{P}_e$: A martingale measure P is identified with the martingale of its densities with respect to the reference measure R, this class of martingales is embedded in a suitable class of supermartingales, and the projection problem is solved in this larger class. Recently Quenez [36] and Schied and Wu [41] have extended this version of the duality approach from the classical case with $\mathcal{Q} = \{Q_0\}$ to the robust case.

In Section 4 we insist on the original idea of identifying the solution of the robust optimization problem in terms of a martingale measure. In Cvitanic, Schachermayer, and Wang [6] the solution of the projection problem is described as a finitely additive measure. Here we use a different idea which goes back to Doob's construction of conditional Brownian motions corresponding to a harmonic function; see [9], Chapter 2.X. As shown in Föllmer [11], [12], any supermartingale on a sufficiently rich filtered probability space can be represented as a measure on the predictable σ-field; see also Föllmer [13] in the volume in honour of J.L. Doob mentioned above. For such measures we introduce the notion of an *extended martingale measure*. Theorem 4.5 shows how the robust projection problem can be solved in the class $\bar{\mathcal{P}}$ of extended martingale measures. Corollary 4.8 describes the application to the robust optimization problem. Some of the key arguments are essentially the same as in Quenez [36] and Schied and Wu [41]. The main novelty is that here we insist on an appropriate notion of a martingale measure.

2 Robust f-Projections

Let $(\Omega, \mathcal{F})$ be a measurable space and denote by $\mathcal{M}_1(\Omega)$ the set of probability measures on $(\Omega, \mathcal{F})$. Let the function $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ be convex and continuous. In order to define the f-divergence of $P \in \mathcal{M}_1(\Omega)$ with respect to $Q \in \mathcal{M}_1(\Omega)$, we associate to $f(\cdot)$ the function $f(\cdot, \cdot)$ on $[0, \infty) \times [0, \infty)$ defined by

$$f(x, y) := \begin{cases} 0 & \text{if } x = y = 0 \\ x \lim\limits_{z \to \infty} \frac{f(z)}{z} & \text{if } y = 0, x > 0 \\ y f\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases} \tag{9}$$

For an affine function $l(x) = ax + b$ on $[0, \infty)$ the associated function $l(\cdot, \cdot)$ on $[0, \infty) \times [0, \infty)$ is given by $l(x, y) = ax + by$. Since $f(\cdot, \cdot)$ is the supremum of the affine functions $l(\cdot, \cdot)$ associated to some affine function l on $[0, \infty)$ such that $l \le f$, $f(\cdot, \cdot)$ is lower semicontinuous and convex on $[0, \infty) \times [0, \infty)$.

Definition 2.1. *Let P, $Q \in \mathcal{M}_1(\Omega)$, and let $R \in \mathcal{M}_1(\Omega)$ be some reference measure such that $P, Q \ll R$; for example, we may take $R := (P + Q)/2$. The* f-divergence *of P with* *respect to Q is defined as*

$$f(P|Q) := \int f\left(\frac{dP}{dR}, \frac{dQ}{dR}\right) dR.$$

Remark 2.2. *Let P^a and P^s denote the absolutely continuous and the singular part in the Hahn-Lebesgue decomposition of $P \in \mathcal{M}_1(\Omega)$ with respect to $Q \in \mathcal{M}_1(\Omega)$. Then*

$$f(P|Q) = \int f\left(\frac{dP^a}{dQ}\right) dQ + \lim_{x\to\infty} \frac{f(x)}{x} \cdot P^s[\Omega] \quad \in \; (-\infty, \infty]; \tag{10}$$

note that the first term on the right-hand side is bounded from below by $f(P^a[\Omega])$ due to Jensen's inequality and that $\lim_{x\to\infty} f(x)/x > -\infty$. In particular the f-divergence is well defined, and it is independent of the choice of the reference measure R. If $P \ll Q$ or if $\lim_{x\to\infty} f(x)/x = 0$, then Equation (10) reduces to

$$f(P|Q) = \int f\left(\frac{dP^a}{dQ}\right) dQ \quad \in \; [f(P^a[\Omega]), \infty].$$

Definition 2.3. *For a subset $\mathcal{P}$ of $\mathcal{M}_1(\Omega)$ and $Q \in \mathcal{M}_1(\Omega)$, $P_Q \in \mathcal{P}$ is called an* f-projection *of Q on $\mathcal{P}$ if it minimizes the f-divergence over the set $\mathcal{P}$:*

$$f(P_Q|Q) = f(\mathcal{P}|Q) := \inf_{P\in\mathcal{P}} f(P|Q).$$

For a subset $\mathcal{Q}$ of $\mathcal{M}_1(\Omega)$ and $P \in \mathcal{M}_1(\Omega)$, $Q_P \in \mathcal{Q}$ is called a reverse f-projection *of P on $\mathcal{Q}$ if it minimizes the f-divergence of P over the set $\mathcal{Q}$:*

$$f(P|Q_P) = f(P|\mathcal{Q}) := \inf_{Q\in\mathcal{Q}} f(P|Q).$$

Finally, $P_0 \in \mathcal{P}$ is called a robust f-projection *of $\mathcal{Q}$ on $\mathcal{P}$ if it minimizes the* robust f-divergence *$f(P|\mathcal{Q}) := \inf_{Q\in\mathcal{Q}} f(P|Q)$ over the set $\mathcal{P}$:*

$$f(P_0|\mathcal{Q}) = \inf_{P\in\mathcal{P}} f(P|\mathcal{Q}) =: f(\mathcal{P}|\mathcal{Q}),$$

i.e.,

$$\inf_{Q\in\mathcal{Q}} f(P_0|Q) = \inf_{P\in\mathcal{P}} \inf_{Q\in\mathcal{Q}} f(P|Q).$$

Remark 2.4. *Since $f(P|Q) = \hat{f}(Q|P)$ where $\hat{f} : [0,\infty) \to \mathbb{R} \cup \{\infty\}$ is the convex continuous function defined by $\hat{f}(x) := xf(1/x)$, a reverse f-projection of P on $\mathcal{Q}$ may be viewed as an $\hat{f}$-projection of P on $\mathcal{Q}$; see Liese and Vajda [32] and Gundel [22]. If f is strictly convex, then so is $\hat{f}$. In this case there is at most one f-projection P_Q of Q on $\mathcal{P}$ and at most one reverse f-projection Q_P of P on $\mathcal{Q}$.*

Let us now fix two convex subsets $\mathcal{P}$ and $\mathcal{Q}$ of $\mathcal{M}_1(\Omega)$. Our aim is to show that the robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$ exists under the following assumptions.

Assumption 2.5. *All measures in $\mathcal{P}$ and $\mathcal{Q}$ are absolutely continuous with respect to some reference measure R. The convex set*

$$\mathcal{K}_{\mathcal{P}} := \left\{ \frac{dP}{dR} : P \in \mathcal{P} \right\}$$

is closed in $L^1(R)$, and the convex set

$$\mathcal{K}_{\mathcal{Q}} := \left\{ \frac{dQ}{dR} : Q \in \mathcal{Q} \right\}$$

is weakly compact in $L^1(R)$.

Note that $\mathcal{K}_{\mathcal{P}}$ is closed in $L^1(R)$ iff $\mathcal{P}$ is closed in variation, and this property implies that the convex set $\mathcal{K}_{\mathcal{P}}$ is weakly closed in $L^1(R)$.

Theorem 2.6. *Let Assumption 2.5 hold and assume furthermore that*

$$\lim_{x \to \infty} \frac{f(x)}{x} = \infty. \tag{11}$$

Then there exists a robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$. Moreover, there exists a reverse f-projection Q_0 of P_0 on $\mathcal{Q}$, i.e.,

$$f(P_0|Q_0) = f(P_0|\mathcal{Q}) = f(\mathcal{P}|\mathcal{Q}).$$

The proof will consist in three steps: First we show that the f-divergence is jointly lower semicontinuous in P and Q, then we formulate a compactness criterion in terms of some auxiliary function l, and in the third step we construct such a function l which has the required properties.

Define

$$F_R(\phi, \psi) := \int f(\phi, \psi) dR$$

for $\mathcal{F}$-measurable $\phi, \psi \geq 0$. Note that $f(\phi, \psi) \geq b\psi$ for some constant b since $f(\cdot)$ is convex and finally increasing due to our assumption (11), hence bounded from below on $[0, \infty)$. Thus $F_R(\phi, \psi) \in (-\infty, \infty]$ is well defined. Note also that

$$f(P|Q) = F_R\left(\frac{dP}{dR}, \frac{dQ}{dR}\right)$$

for P, Q, $R \in \mathcal{M}_1(\Omega)$ such that P, $Q \ll R$. We will view F_R as a functional on the closed convex subset $L^1_+(R) \times L^1_+(R)$ of the Banach space $L^1(R) \times L^1(R)$.

The following result appears also in Liese and Vajda [32], Theorem 1.47, but with a different proof.

Lemma 2.7. *Under Assumption* (11) *the functional F_R is convex and weakly lower semicontinuous on $L^1_+(R) \times L^1_+(R)$.*

Proof. Convexity of F_R follows from the convexity of $f(\cdot,\cdot)$ on $[0,\infty)^2$. In order to verify weak lower semicontinuity, we have to show that the sets

$$A_c := \left\{(\phi,\psi) \in L^1_+(R) \times L^1_+(R) : F_R(\phi,\psi) \le c\right\}$$

are closed with respect to the weak product topology. But since A_c is convex, it is enough to check that A_c is strongly closed; cf. Dunford, Schwartz [10], Theorem V.3.13. To this end, take $(\phi_n,\psi_n) \in A_c$ $(n \ge 1)$ such that $\phi_n \to \phi$ and $\psi_n \to \psi$ in $L^1(R)$ as n tends to infinity. Passing to subsequences if necessary, we may assume that both sequences converge R-almost surely. Since $f(\phi_n,\psi_n) \ge b\psi_n$ and $(\psi_n)_{n=1,2,\ldots}$ is uniformly integrable we can use the lower semicontinuity of f on $[0,\infty)^2$ and Fatou's lemma to conclude

$$\begin{aligned} F_R(\phi,\psi) &= \int f(\lim_{n\to\infty}(\phi_n,\psi_n))dR \\ &\le \int \liminf_{n\to\infty} f(\phi_n,\psi_n)dR \\ &\le \liminf_{n\to\infty} F_R(\phi_n,\psi_n) \le c. \end{aligned}$$

Since ϕ, $\psi \in L^1_+(R)$ we see that $(\phi,\psi) \in A_c$. □

Remark 2.8. *In particular the functional $F_R(dP/dR,\cdot)$ is weakly lower semicontinuous on the weakly compact set $\mathcal{K}_\mathcal{Q}$. This shows that a reverse f-projection Q_P of P on $\mathcal{Q}$ exists for any $P \in \mathcal{M}_1(\Omega)$. Thus the existence of a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$ amounts to the existence of some $P_0 \in \mathcal{P}$ which minimizes the f-divergence $f(P|Q_P)$ over $\mathcal{P}$.*

Since $F_R(\cdot,\cdot)$ is weakly lower semicontinuous on $\mathcal{K}_\mathcal{P} \times \mathcal{K}_\mathcal{Q}$, the existence of a robust f-projection will now follow if we can show that the set $\{(P,Q) : f(P|Q) \le c\}$ is compact in the weak product topology. To this end we prove the following criterion.

Lemma 2.9. *Let $l : [0,\infty) \to \mathbb{R}$ be a positive increasing function such that $\lim_{x\to\infty} l(x)/x = \infty$. Let Assumption 2.5 hold and assume that for any constant $c > 0$ there is a constant $c_0 > 0$ such that for any $P \in \mathcal{P}$*

$$f(P|\mathcal{Q}) \le c \qquad \Longrightarrow \qquad E_R\left[l\left(\frac{dP}{dR}\right)\right] \le c_0. \tag{12}$$

Then there exist a robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$and a reverse f-projection Q_0 of P_0 on $\mathcal{Q}$.

Proof. We may assume $f(\mathcal{P}|\mathcal{Q}) < \infty$ because otherwise every $P \in \mathcal{P}$ would be a robust f-projection. Take $c > f(\mathcal{P}|\mathcal{Q})$. Since $f(P|Q) = F_R\,(dP/dR, dQ/dR)$ and since F_R is weakly lower semicontinuous by Lemma 2.7, it is enough to show that $\{(P,Q) \in \mathcal{P}\times\mathcal{Q} : f(P|Q) \le c\}$, viewed as the subset

$$\mathcal{C}_c := \{(\phi,\psi) : F_R(\phi,\psi) \le c\} \cap (\mathcal{K}_\mathcal{P} \times \mathcal{K}_\mathcal{Q})$$

of $L^1(R)\times L^1(R)$, is weakly compact. Then F_R attains its minimum in some $(P_0, Q_0) \in \mathcal{P}\times\mathcal{Q}$, which implies

$$f(P_0|\mathcal{Q}) = f(P_0|Q_0) = \inf_{P\in\mathcal{P}} f(P|\mathcal{Q}),$$

and so P_0 is a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$, and Q_0 is its reverse f-projection.

Under Condition (12)

$$\mathcal{C}_c \subseteq \mathcal{K}_{\mathcal{P},c_0} \times \mathcal{K}_\mathcal{Q},$$

where

$$\mathcal{K}_{\mathcal{P},c_0} := \{\phi \in \mathcal{K}_\mathcal{P} : E_R[l(\phi)] \le c_0\}$$

is uniformly integrable by the de la Vallée-Poussin criterion, hence relatively compact in the weak topology $\sigma(L^1(R), L^\infty(R))$; see Dellacherie and Meyer [8], Theorems II.22 and II.25. Since $\mathcal{K}_\mathcal{Q}$ is weakly compact by Assumption 2.5, Tychonov's theorem implies that $\mathcal{K}_{\mathcal{P},c_0}\times\mathcal{K}_\mathcal{Q}$ is relatively compact in the weak product topology, and so is $\mathcal{C}_c$. But $\mathcal{C}_c$ is also weakly closed due to the lower semicontinuity of F_R and Assumption 2.5, and so $\mathcal{C}_c$ is in fact weakly compact. □

Remark 2.10. *Consider the classical case $\mathcal{Q} = \{Q_0\}$. Then Condition* (12) *is trivially satisfied for $l = f$ and $R = Q$, and the preceding proof reduces to the standard argument for the existence of a classical f-projection; see, e.g., Liese and Vajda [32], Proposition 8.5, and in the relative entropy case $f(x) = x\log x$ Csiszár [4], Theorem 2.1.*

Since $\mathcal{K}_\mathcal{Q}$ is assumed to be weakly compact, we can choose a function $g : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} g(x)/x = \infty$ such that

$$\sup_{Q\in\mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right] < \infty, \tag{13}$$

cf. Dellacherie and Meyer [8], Theorem II.22. Given the functions f and g, we are now going to construct a suitable function l and at the same time a convex function h such that an appropriate Young inequality with respect to h will allow us to obtain the estimate in terms of l which is required in Lemma 2.9.

For a convex function h on $[0,\infty)$ we denote by h^* its Fenchel-Legendre transform on $[0,\infty)$ defined by

$$h^*(x) := \sup_{y\geq 0}\{xy - h(y)\}. \tag{14}$$

Lemma 2.11. *There exist strictly increasing functions h and l_i $(i=1,2)$ on $[0,\infty)$ with initial value $h(0)=l_i(0)=0$ such that the following properties hold:*

(i) h is continuous, convex, strictly increasing, and $\lim_{x\to\infty} h(x)/x = \infty$.

(ii) l_i is concave and $\lim_{x\to\infty} l_i(x)=\infty$ $(i=1,2)$.

(iii) $h(xl_1(x)) \leq f(x)$ for large enough x.

(iv) $xh^*(l_2(x)) \leq g(x)$ for large enough x.

(v) $l(x) := xl_1(l_2(x)) \leq g(x)$ for large enough x.

Proof. We are going to use repeatedly the following simple fact: If $\tilde{u}$ is a function on $[0,\infty)$ such that $\lim_{x\to\infty}\tilde{u}(x)=\infty$, then there is a strictly increasing concave function u on $[0,\infty)$ such that $\lim_{x\to\infty} u(x)=\infty$, $u(0)=0$, and $u(x)\leq\tilde{u}(x)$ on $[x_1,\infty)$ for some $x_1\geq 0$. Indeed, take a sequence $0=x_0\leq x_1<x_2<\ldots$ converging to infinity such that for $n\geq 1$, $\tilde{u}(x)\geq n+1$ for all $x\geq x_n$, and the sequence $x_{n+1}-x_n$ increases in $n\geq 0$. Define $u(x_n):=n$ and u linear between x_n and x_{n+1} for $n\geq 0$. Then we have $u(x)\leq n+1\leq\tilde{u}(x)$ on $[x_n,x_{n+1})$ for $n\geq 1$, hence u is dominated by $\tilde{u}$ on $[x_1,\infty)$. Furthermore, $u'(x)=(u(x_{n+1})-u(x_n))/(x_{n+1}-x_n)=1/(x_{n+1}-x_n)$ for $x\in(x_n,x_{n+1})$ for $n\geq 0$. Since this fraction is non-increasing, u is concave.

In a first step we construct the convex function h. Since f is convex and $\lim_{x\to\infty} f(x)/x=\infty$, its left-hand derivative f'_- is non-decreasing and tends to infinity. In particular $f'_->0$ on $[x_0,\infty)$ for some $x_0\geq 0$. Take a non-decreasing function $\zeta:[0,\infty)\to[0,\infty)$ that tends to infinity, but satisfies $\lim_{x\to\infty}\zeta(x)/x=0$. Define

$$h'(x) := \gamma(x) f'_-(\zeta(x)) \tag{15}$$

on $[x_0,\infty)$, where $\gamma:[0,\infty)\to[0,\infty)$ is decreasing, tending to 0, and such that $h'>0$ is non-decreasing and tends to infinity. For example, we may choose $\zeta(x):=\sqrt{x}$ and $\gamma(x):=(f'_-(\zeta(x)))^{-1/2}$.

Now define h such that (15) is satisfied on $[x_0,\infty)$, and h is linear on $[0,x_0)$ with $h(0)=0$ and $h(x_0)=x_0h'(x_0)$. Then h is a convex function which has the required properties. Moreover,

$$\lim_{x\to\infty}\frac{h(cx)}{f(x)} = 0 \qquad \text{for all } c>0. \tag{16}$$

Indeed, for $c \in (0, \infty)$ take $\alpha \geq x_0$ such that $\zeta(y) \leq y/c$ for $y \geq \alpha$. Then we have for $cx \geq \alpha$,

$$\begin{aligned} h(cx) &= h(\alpha) + \int_\alpha^{cx} \gamma(y) f'_-(\zeta(y))dy \\ &\leq h(\alpha) + \gamma(\alpha) \int_\alpha^{cx} f'_- \left(\frac{y}{c}\right) dy \\ &= h(\alpha) + \gamma(\alpha) c \left(f(x) - f\left(\frac{\alpha}{c}\right)\right). \end{aligned}$$

Therefore,

$$\limsup_{x \to \infty} \frac{h(cx)}{f(x)} \leq c\gamma(\alpha),$$

and this implies (16) since $\lim_{\alpha \to \infty} \gamma(\alpha) = 0$.

In order to construct the concave function l_1, consider first the function $\tilde{l}_1$ defined by $h(x\tilde{l}_1(x)) = f(x)$, i.e., $\tilde{l}_1(x) := h^{-1}(f(x))/x$. Then $\lim_{x\to\infty} \tilde{l}_1(x) = \infty$, because otherwise there would be a $c \in (0, \infty)$ and a sequence (x_n) tending to infinity such that

$$h(x_n c) \geq h(x_n \tilde{l}_1(x_n)) = f(x_n),$$

in contradiction to (16). As explained above, we can now choose a strictly increasing concave function l_1 such that $l_1(0) = 0$, $\lim_{x\to\infty} l_1(x) = \infty$, and $l_1(x) \leq \tilde{l}_1(x)$, hence $h(xl_1(x)) \leq f(x)$ for large enough x.

Finally we construct the concave function l_2. Let h^* be the Fenchel-Legendre transform of h defined in (14). Then h^* has the same properties as h specified in (i); see Neveu [34], pages 193 and 194. First we define $\tilde{l}_2(x)$ on $[0, \infty)$ such that

$$h^*(\tilde{l}_2(x)) = \frac{g(x)}{x} \quad \text{on } (0, \infty).$$

This implies $\lim_{x\to\infty} \tilde{l}_2(x) = \infty$. We can now choose a strictly increasing concave function l_2 such that $l_2(0) = 0$, $\lim_{x\to\infty} l_2(x) = \infty$ and $l_2(x) \leq \tilde{l}_2(x) \wedge l_1^{-1}(g(x)/x)$, hence $xh^*(l_2(x)) \leq g(x)$ and $xl_1(l_2(x)) \leq g(x)$, for large enough x. □

In order to conclude the proof of Theorem 2.6, we now show that the function l appearing in part (v) of Lemma 2.11 allows us to apply the criterion in Lemma 2.9.

Lemma 2.12. *The function l defined in Lemma 2.11 satisfies the conditions of Lemma 2.9.*

Proof. Observe first that $\lim_{x\to\infty} l(x)/x = \infty$. Now let us fix $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ such that $f(P|Q) \leq c$ for some $c > 0$. Then $P \ll Q$, and $\phi := dP/dQ$ and $\psi := dQ/dR$ are well defined.

Let $x_0 > 1$ be such that Conditions (iii)-(v) in Lemma 2.11 are satisfied for $x \geq x_0$. In order to verify Condition (12) we decompose the expectation on the right-hand side as follows:

$$\begin{aligned} E_R\left[l\left(\frac{dP}{dR}\right)\right] &= E_R[l(\phi\psi)] \\ &= E_R[l(\phi\psi); \phi \leq x_0] + E_R[l(\phi\psi); \phi > x_0, l_2(\psi) > 1] \\ &\quad + E_R[l(\phi\psi); \phi > x_0, l_2(\psi) \leq 1]. \end{aligned} \tag{17}$$

We are going to show that each of these three terms is bounded by some constant which only depends on c but not on the specific choice of P and Q. Since l_i is concave with $l_i(0) = 0$ for $i = 1, 2$, we have $l_i(\alpha x) \leq \alpha l_i(x)$ for any $\alpha \geq 1$, and this estimate will be used repeatedly.

On $\{\phi \leq x_0\}$ we have

$$\begin{aligned} l(\phi\psi) &\leq l(x_0\psi) \\ &= x_0\psi l_1(l_2(x_0\psi)) \\ &\leq x_0^2\psi l_1(l_2(\psi)) \\ &= x_0^2 l(\psi) \leq x_0^2(c_1 + g(\psi)), \end{aligned}$$

where $c_1 := \sup\{l(x) : x \leq x_0\}$, since $l(x) \leq g(x)$ for $x \geq x_0$, and so the first term above satisfies

$$E_R[l(\phi\psi); \phi \leq x_0] \leq x_0^2\,(c_1 + E_R[g(\psi)]) \leq x_0^2 \left(c_1 + \sup_{Q\in\mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right]\right),$$

which is finite by (13).

On $\{\phi > x_0,\ l_2(\psi) > 1\}$ we have

$$l_1(l_2(\phi\psi)) \leq l_1(\phi l_2(\psi)) \leq l_1(\phi) l_2(\psi),$$

and this implies

$$E_R[l(\phi\psi); \phi > x_0, l_2(\psi) > 1] \leq E_Q[\phi l_1(\phi) l_2(\psi)].$$

Now we use Young's inequality to conclude that

$$E_Q[\phi l_1(\phi) l_2(\psi)] \leq 2 \cdot ||\phi l_1(\phi)||_h \cdot ||l_2(\psi)||_{h^*};$$

see Neveu [34], Proposition IX.2.2. Here

$$||X||_h := \inf\left\{a > 0 : E_Q\left[h\left(\frac{|X|}{a}\right)\right] \leq 1\right\}$$

denotes the Orlicz norm with respect to h and Q, and $||X||_{h^*}$ is defined in the same manner in terms of h^* and Q. But

$$||\phi l_1(\phi)||_h \leq \max\{1, E_Q[h(\phi l_1(\phi))]\}$$

(see Neveu [34], proof of Proposition IX.2.2), and

$$\begin{aligned} E_Q[h(\phi l_1(\phi))] &\leq c_2 + E_Q[f(\phi)] \\ &= c_2 + f(P|Q) \\ &\leq c_2 + c, \end{aligned}$$

where $c_2 := \sup\{h(xl_1(x)) : x \leq x_0\}$, since $h(xl_1(x)) \leq f(x)$ for $x \geq x_0$. In the same way,

$$||l_2(\psi)||_{h^*} \leq \max\{1, E_Q[h^*(l_2(\psi))]\},$$

and

$$\begin{aligned} E_Q[h^*(l_2(\psi))] &= E_R[\psi h^*(l_2(\psi))] \\ &\leq c_3 + E_R[g(\psi)] \\ &\leq c_3 + \sup_{Q \in \mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right], \end{aligned}$$

where $c_3 := \sup\{xh^*(l_2(x)) : x \leq x_0\}$, since $xh^*(l_2(x)) \leq g(x)$ for $x \geq x_0$. This yields the desired bound for the second term on the right-hand side of Equation (17).

On $\{\phi > x_0,\ l_2(\psi) \leq 1\}$ we have

$$l_1(l_2(\phi\psi)) \leq l_1(\phi l_2(\psi)) \leq l_1(\phi),$$

and so the remaining term satisfies

$$E_R[l(\phi\psi); \phi > x_0, l_2(\psi) < 1] \leq E_R[\phi\psi l_1(\phi)] = E_Q[\phi l_1(\phi)].$$

Young's inequality yields

$$E_Q[\phi l_1(\phi)] \leq 2 \cdot ||\phi l_1(\phi)||_h \cdot \inf\left\{a > 0 : h^*\left(\frac{1}{a}\right) \leq 1\right\},$$

and we have already seen above that $||\phi l_1(\phi)||_h$ is suitably bounded. □

Remark 2.13. *For special choices of functions f and g the construction of our auxiliary function l may of course be simpler. Take for example $f(x) = x^\alpha$ and $g(x) = x^\beta$ with α,*

$\beta > 1$. Choose $\gamma > 1$ such that $\gamma < \alpha$ and $(\alpha-1)\gamma \le \beta(\alpha-\gamma)$ and define $l(x) = x^\gamma$. Condition (12) now follows by applying Hölder's inequality with exponents $p = \alpha/\gamma$ and $q = \alpha/(\alpha-\gamma)$: For $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, and $\phi = dP/dQ$, $\psi = dQ/dR$,

$$
\begin{aligned}
E_R\left[l\left(\frac{dP}{dR}\right)\right] &= E_R\left[\phi^\gamma \psi^\gamma\right] = E_Q\left[\phi^\gamma \psi^{\gamma-1}\right] \\
&\le E_Q\left[\phi^{\gamma p}\right]^{1/p} E_Q\left[\psi^{(\gamma-1)q}\right]^{1/q} \\
&\le f(P|Q)^{1/p}\left(1 + E_R\left[g\left(\frac{dQ}{dR}\right)\right]^{1/q}\right);
\end{aligned}
$$

see also Gundel [22], Lemma 4.

Proof of Theorem 2.6. Due to Lemma 2.12 we can apply Lemma 2.9 to conclude that a robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ and a reverse f-projection Q_0 of P_0 on $\mathcal{Q}$ exist. □

We conclude this section with a uniqueness result for robust f-projections.

Proposition 2.14. *If f is strictly convex and $f(\mathcal{P}|\mathcal{Q}) < \infty$, then the density of the robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ with respect to its reverse f-projection Q_0 is R-almost surely unique.*

Proof. Assume that P_1 and $P_2 \in \mathcal{P}$ are two robust f-projections of $\mathcal{Q}$ on $\mathcal{P}$ with reverse f-projections Q_1 and Q_2. Then $P_i \ll Q_i$ due to Remark 2.2. Take $\gamma \in (0,1)$ and define $P_\gamma := \gamma P_1 + (1-\gamma)P_2$, $Q_\gamma := \gamma Q_1 + (1-\gamma)Q_2$,

$$
\phi_i := \frac{dP_i}{dQ_i} \cdot 1_{\{dQ_i/dR>0\}} + \infty \cdot 1_{\{dQ_i/dR=0, dP_i/dR>0\}},
$$

and $\psi_i := dQ_i/dQ_\gamma$ for $i = 1, 2$. Note that $\gamma\psi_1 + (1-\gamma)\psi_2 = 1$ and $\gamma\psi_1\phi_1 + (1-\gamma)\psi_2\phi_2 = dP_\gamma/dQ_\gamma$. By convexity of f and minimality of P_1 and P_2,

$$
\begin{aligned}
f(P_\gamma|\mathcal{Q}) &\ge \gamma f(P_1|\mathcal{Q}) + (1-\gamma) f(P_2|\mathcal{Q}) \\
&= E_{Q_\gamma}\left[\gamma\psi_1 f(\phi_1) + (1-\gamma)\psi_2 f(\phi_2)\right] \\
&\ge E_{Q_\gamma}\left[f\left(\gamma\psi_1\phi_1 + (1-\gamma)\psi_2\phi_2\right)\right] \\
&= f(P_\gamma|Q_\gamma) \\
&\ge f(P_\gamma|\mathcal{Q}),
\end{aligned}
$$

and so we have equality everywhere. But since f is strictly convex, the second inequality can only reduce to an equality if $\phi_1 = \phi_2$ Q_γ-almost surely. This means that $\phi_1 = \phi_2$ R-almost surely on the set $\{dQ_\gamma/dR > 0\}$. On the set $\{dQ_\gamma/dR = 0\}$ we have $dP_i/dR = 0$ for $i = 1, 2$ R-almost surely since $f(P_i|Q_i) < \infty$, hence $\phi_1 = \phi_2 = 0$ R-almost surely. □

3 Robust Preferences and Least Favorable Martingale Measures

In this section we explain the connection between (robust) f-projections and one of the key problems in Mathematical Finance, namely the choice of a portfolio which is optimal with respect to certain (robust) preferences.

In its general form, the problem of optimal portfolio choice consists in finding a maximal element with respect to a given preference order $\succeq$ over some convex class of "affordable" financial positions or contingent claims, described as random variables H on a given probability space $(\Omega, \mathcal{F}, R)$. Typically such a preference order admits a numerical representation

$$H \succeq \tilde{H} \Longleftrightarrow U(H) \geq U(\tilde{H})$$

in terms of some utility functional U. In order to specify the functional U we fix an increasing concave *utility function* $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. We assume that u is strictly increasing, strictly concave, and continuously differentiable on the interior $(a, \infty) := \operatorname{int}\{x : u(x) > -\infty\}$ of its domain and satisfies the *Inada condition*

$$\lim_{x \searrow a} u'(x) = \infty, \quad \lim_{x \to \infty} u'(x) = 0. \tag{18}$$

Moreover we assume that u has *regular asymptotic elasticity* in the sense of Kramkov and Schachermayer [30], Schachermayer [38], Frittelli and Rosazza [18], i.e.,

$$\limsup_{x \to \infty} \frac{xu'(x)}{u(x)} < 1 \quad \text{and, if } a = -\infty, \quad \liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1. \tag{19}$$

In the classical framework of "expected utility", whose axiomatic foundations were clarified by von-Neumann-Morgenstern and by Savage, the utility functional is of the form

$$U(H) = E_Q[u(H)],$$

where Q is some probability measure on $(\Omega, \mathcal{F})$. In this paper we use a "robust" extension of the expected utility approach which was introduced by Gilboa and Schmeidler [20]. Instead of a single probabilistic model $Q \ll R$ we take a whole class $\mathcal{Q}$ of such models and define the preference order $\succeq$ via the utility functional

$$U(H) := \inf_{Q \in \mathcal{Q}} E_Q[u(H)].$$

Thus, model uncertainty is taken into account explicitly. As shown by Gilboa and Schmeidler [20], such robust preferences can be characterized by certain behavioral axioms, and they resolve several well-known “paradoxa” which arise in the classical framework; see, for instance, Karni and Schmeidler [28] or Föllmer and Schied [16], Chapter 2.5.

Assumption 3.1. *The measures $Q \in \mathcal{Q}$ are absolutely continuous with respect to R, and the class $\mathcal{Q}$ is equivalent to R in the sense that*

$$R[A] = 0 \quad \Longleftrightarrow \quad Q[A] = 0 \textit{ for all } Q \in \mathcal{Q}. \tag{20}$$

Due to (20) the contingent claim H satisfies $U(H) > -\infty$ only if

$$H \geq a \qquad R-a.s. \tag{21}$$

since $a = \inf\{x : u(x) > -\infty\}$. From now on we will only consider contingent claims with this property.

The class of affordable contingent claims will be specified in terms of a financial market model with d liquid financial assets and finite time horizon T. The price fluctuation of these assets, properly discounted, is described by a d-dimensional positive semimartingale $(X_t)_{0\leq t\leq T}$ on the probability space $(\Omega, \mathcal{F}, R)$, equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ is trivial for R. We assume that $(X_t)_{0\leq t\leq T}$ is *locally bounded*, i.e., there exists a sequence of stopping times $(\tau_n)_{n=1,2,\ldots}$ such that $(X_{\tau_n \wedge t})_{0\leq t\leq T}$ is bounded for each n and $\tau_n \nearrow T$ R-almost surely.

Definition 3.2. *A probability measure $P \ll R$ is called an* absolutely continuous martingale measure *if $(X_t)_{0\leq t\leq T}$ is a local martingale under P. If in addition $P \approx R$, then P is called an* equivalent martingale measure*. The class of absolutely continuous martingale measures will be denoted by $\mathcal{P}$, the class of equivalent martingale measures by $\mathcal{P}_e$.*

¿From now on we assume the existence of an equivalent martingale measure, i.e.,

$$\mathcal{P}_e \neq \emptyset.$$

This assumption is equivalent to the absence of arbitrage opportunities; see Delbaen and Schachermayer [7] and also Yan [43] and [44] for precise versions of this equivalence and for different choices of the numéraire which is used to define the discounted price process $(X_t)_{0\leq t\leq T}$.

Remark 3.3. *Since the price process $(X_t)_{0\leq t\leq T}$ is assumed to be locally bounded, the class $\mathcal{P}$ of absolutely continuous martingale measures is closed in the sense of Assumption 2.5 since their densities ϕ can be characterized by the conditions $E_R[\phi X_\tau] = X_0$ for stopping times $\tau \leq T$ such that $X_\tau \in L^\infty(R)$; see, for instance, Frittelli [17] or Bellini and Frittelli [3].*

Let us fix an initial wealth $x_0 > a$. Consider a contingent claim H, given as an $\mathcal{F}_T$-measurable random variable at the final time T such that (21) holds.

Definition 3.4. *Let us say that H is* affordable with limited downside risk *if there exist some $P \in \mathcal{P}_e$ such that $H \in L^1(P)$ and a trading strategy in the underlying liquid assets, described by a d-dimensional predictable and suitably integrable process $(\xi_t)_{0\leq t\leq T}$, such that the corresponding value process*

$$V_t := x_0 + \int_0^t \xi_s dX_s \qquad (0 \leq t \leq T) \tag{22}$$

satisfies

$$V_t \geq E_P[H|\mathcal{F}_t] \qquad (0 \leq t \leq T) \tag{23}$$

and in particular $V_T \geq H$ R-almost surely. For $\mathcal{P}_0 \subseteq \mathcal{P}$ such that $\mathcal{P}_0 \cap \mathcal{P}_e \neq \emptyset$ we will say that the strategy has $\mathcal{P}_0$-limited downside risk *if $H \in L^1(P)$ and (23) holds for any $P \in \mathcal{P}_0$.*

Note that the value process (22) is a local martingale under any $P \in \mathcal{P}$, and that it is a supermartingale under any $P \in \mathcal{P}_0$. This implies the constraint

$$\sup_{P\in\mathcal{P}_0} E_P[H] \leq x_0 \tag{24}$$

for any contingent claim H which is affordable with $\mathcal{P}_0$-limited downside risk.

Remark 3.5. *Suppose that the contingent claim H is bounded from below by some constant c. If H is affordable with limited downside risk, then the corresponding value process is bounded from below by c, and hence (23) is in fact satisfied for all $P \in \mathcal{P}_e$. In particular the constraint (24) is satisfied for $\mathcal{P}_0 = \mathcal{P}_e$. A key result in the theory of superhedging implies that, conversely, a claim which is bounded from below and satisfies the constraint (24) for $\mathcal{P}_0 = \mathcal{P}_e$ is in fact affordable with $\mathcal{P}_e$-limited downside risk. More precisely, there exists a trading strategy whose value process (V_t) is bounded from below and satisfies $V_T \geq H$ R-almost surely, and this implies (23) for any $P \in \mathcal{P}$ since (V_t) is a P-supermartingale. See, for instance Kramkov [29], Delbaen and Schachermayer [7], or Yan [43], and also Föllmer and Kramkov [15] and Föllmer and Kabanov [14] for an extension to trading strategies with convex constraints. Moreover, if the supremum $\sup_{P\in\mathcal{P}_e} E_P[H]$ is assumed by some $P \in \mathcal{P}_e$, then H is even attainable by some trading strategy in the sense that $H = V_T$; see Ansel and Stricker [1], Theorem 3.2.*

We are going to discuss the problem of maximizing the robust utility

$$U(H) := \inf_{Q\in\mathcal{Q}} E_Q[u(H)] \tag{25}$$

under the constraint

$$\sup_{P \in \mathcal{P}_0} E_P[H] \leq x_0 \tag{26}$$

for a suitable choice of the set $\mathcal{P}_0$. It will turn out that the resulting contingent claim H_0 is in fact affordable by means of a strategy with $\mathcal{P}_0$-limited downside risk, and this may be viewed as an extension of the superhedging result recalled in Remark 3.5.

Recall that the utility function u is finite on (a, ∞) for some $a \in [-\infty, \infty)$. From now on we assume that

$$a = 0 \qquad \text{or} \qquad a = -\infty \tag{27}$$

and that

$$u(\infty) := \lim_{x \to \infty} u(x) = \infty \qquad \text{or} \qquad u(\infty) = 0; \tag{28}$$

In view of our optimization problem this is no loss of generality since we can shift the origin along the two axes if necessary.

In order to connect this robust optimization problem to our discussion of robust f-projections, let us introduce the convex conjugate function $v : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ of the concave utility function u:

$$v(y) := \sup_{x > a}\{u(x) - xy\} = u(I(y)) - yI(y), \tag{29}$$

where $I := (u')^{-1} : (0, \infty) \to (a, \infty)$ is decreasing from ∞ to a. Note that $v(0) := \lim_{x \searrow 0} v(x) = u(\infty)$, that v is finite and differentiable with $v' = -I$ on $(0, \infty)$, and that

$$\lim_{x \to \infty} \frac{v(x)}{x} = \lim_{x \to \infty} v'(x) = -a. \tag{30}$$

due to the Inada condition (18). Moreover, our Assumption (19) of regular asymptotic elasticity implies that for any $\lambda > 0$ there are constants $a(\lambda)$ and $b(\lambda)$ such that

$$v(\lambda x) \leq a(\lambda) v(x) + b(\lambda)(x + 1); \tag{31}$$

see, for instance, Schachermayer [38] or Frittelli and Rosazza [18]. We define $v_\lambda(x) := v(\lambda x)$ for $\lambda > 0$, and we denote by

$$v_\lambda(P|Q) = v_\lambda(P^a|Q) - a\lambda P^s[\Omega]$$

the v_λ-divergence of $P \in \mathcal{P}$ with respect to $Q \in \mathcal{Q}$; for $\lambda = 1$ we simply write $v(P|Q)$. Note that $v(P|Q) < \infty$ implies $Q \ll P$ whenever $v(0) = u(\infty) = \infty$ and $P \ll Q$ whenever $a = -\infty$.

For $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ with densities $\phi := dP/dR$ and $\psi := dQ/dR$ we denote by

$$\frac{dP}{dQ} := \frac{\phi}{\psi} 1_{\{\psi>0\}} + \infty \cdot 1_{\{\psi=0,\phi>0\}}$$

the generalized Radon-Nikodym density of P with respect to Q. Note that

$$I\left(\lambda \frac{dP}{dQ}\right) = I\left(\lambda \frac{dP^a}{dQ}\right) 1_{A^c} + a 1_A,$$

where $A := \{\psi = 0, \ \phi > 0\}$ is the support of the singular part of P. In particular $I(\lambda dP/dQ) = I(\lambda dP^a/dQ)$ R-almost surely if $a = 0$, or if $a = -\infty$ and $v(P|Q) < \infty$.

Lemma 3.6. *For $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ the following conditions are equivalent:*

(i) $v(P|Q) < \infty$*,*

(ii) $v_\lambda(P|Q) < \infty$ *for any* $\lambda > 0$*,*

(iii) For any $\lambda > 0$ *the contingent claim*

$$H_\lambda := I\left(\lambda \frac{dP}{dQ}\right)$$

satisfies

$$H_\lambda \in L^1(P) \text{ and } u(H_\lambda) \in L^1(Q), \tag{32}$$

(iv) $H_\lambda^- \in L^1(P)$ *and* $u(H_\lambda)^+ \in L^1(Q)$ *for any* $\lambda > 0$*.*

Proof. The equivalence of (i) and (ii) follows from (31). In order to check the equivalence of (ii) to (iv), define $\rho := dP^a/dQ$ and note that (ii) is equivalent to $aP^s[\Omega] > -\infty$ and $E_Q[v(\lambda\rho)] < \infty$ for any $\lambda > 0$. For $0 < \lambda_1 < \lambda < \lambda_2$, the two estimates

$$v(\lambda_i \rho) \geq v(\lambda\rho) + v'(\lambda\rho)(\lambda_i - \lambda)\rho \qquad \text{on} \qquad \{0 < \rho < \infty\}$$

for $i = 1, 2$ show that $v'(\lambda\rho)\rho \in L^1(Q)$ and hence $I(\lambda\rho) \in L^1(P^a)$ and $H_\lambda = I(\lambda\rho) + a \cdot 1_A \in L^1(P)$, as soon as (ii) holds. Since $u(H_\lambda) = u(I(\lambda\rho))$ Q-almost surely and

$$u(I(\lambda\rho)) = v(\lambda\rho) + \lambda\rho I(\lambda\rho) \tag{33}$$

by (29), Condition (ii) also implies $u(H_\lambda) \in L^1(Q)$. Clearly, (iii) implies (iv). Conversely, (33) allows us to verify (ii) as soon as $u^+(H_\lambda) \in L^1(Q)$ and $H_\lambda^- \in L^1(P)$. Indeed, $v^-(\lambda\rho) \in L^1(Q)$ by convexity of v and $v^+(\lambda\rho) \leq u^+(I(\lambda\rho)) + \lambda\rho H_\lambda^-$. Moreover, if $a = -\infty$, then $|a|P^s[\Omega] \leq E_P[H_\lambda^-]$, hence $P \ll Q$ and $v_\lambda(P|Q) = E_Q[v(\lambda\rho)] < \infty$. $\square$

Remark 3.7. *Consider the following standard choices of a utility function u:*

(i) $u(x) = \log x$ *on* $(0, \infty)$ *(logarithmic utility),*

(ii) $u(x) = \frac{1}{\gamma} x^\gamma$ *on* $(0, \infty)$, $0 \neq \gamma \in (-\infty, 1)$ *(power utility),*

(iii) $u(x) = -\frac{1}{\alpha} e^{-\alpha x}$ *on* $\mathbb{R}^1$, $\alpha \in (0, \infty)$ *(exponential utility).*

The corresponding divergences $v_\lambda(P|Q)$ *are given by*

(i) $H(Q|P) - (1 + \log \lambda)$,

(ii) $\frac{1}{\beta} \lambda^{-\beta} E_Q \left[\left(\frac{dQ}{dP} \right)^\beta \right]$ *for* $\beta = \frac{\gamma}{1-\gamma}$,

(iii) $\frac{\lambda}{\alpha} (H(P|Q) + \log \lambda - 1)$,

where

$$H(P|Q) := \begin{cases} E_Q \left[\frac{dP}{dQ} \log \left(\frac{dP}{dQ} \right) \right] & \text{if } P \ll Q, \\ \infty & \text{otherwise} \end{cases}$$

denotes the relative entropy of P *with respect to* Q*. In particular* $v_\lambda(P|Q) < \infty$ *for all* $\lambda > 0$ *as soon as* P *and* Q *satisfy the corresponding condition* (i) $H(Q|P) < \infty$, (ii) $1/\beta E_Q \left[(dQ/dP)^\beta \right] < \infty$, *or* (iii) $H(P|Q) < \infty$.

For fixed $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ such that $P \approx Q$, it is well known how to solve the classical problem of maximizing the expected utility $E_Q[u(H)]$ under the simple constraint $E_P[H] \leq x_0$; see, for instance, Karatzas and Shreve [27]. For the convenience of the reader we summarize the solution in a slightly more general form, which will then be extended to the robust case. Note that here we only assume that $v(P|Q) < \infty$.

Theorem 3.8. *Suppose that* $P \in \mathcal{P}$ *and* $Q \in \mathcal{Q}$ *are such that* $v(P|Q) < \infty$*.*
(i) *The function* $h : (0, \infty) \to \mathbb{R}^1$ *defined by*

$$h(\lambda) := v_\lambda(P|Q) + \lambda x_0$$

is strictly convex and continuously differentiable with derivative

$$h'(\lambda) = x_0 - E_P \left[I \left(\lambda \frac{dP}{dQ} \right) \right]. \tag{34}$$

In particular h *attains its minimum in the unique value* $\lambda_{P,Q} > 0$ *such that*

$$E_P \left[I \left(\lambda_{P,Q} \frac{dP}{dQ} \right) \right] = x_0.$$

(ii) *The contingent claim*

$$H_{P,Q} := I\left(\lambda_{P,Q}\frac{dP}{dQ}\right) \in L^1(P)$$

maximizes the expected utility $E_Q[u(H)]$ under the constraint $E_P[H] \le x_0$, and the maximizer is R-almost surely unique on the set $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$. The maximal expected utility is given by $v_{\lambda_{P,Q}}(P|Q) + \lambda_{P,Q}x_0$:

$$\begin{aligned}\max_{H: E_P[H] \le x_0} E_Q[u(H)] &= E_Q[u(H_{P,Q})] \\ &= v_{\lambda_{P,Q}}(P|Q) + \lambda_{P,Q}x_0 \\ &= \min_{\lambda>0}\{v_\lambda(P|Q) + \lambda x_0\}.\end{aligned} \tag{35}$$

Proof. The function $g(\lambda) := v(\lambda) + \lambda x_0$ is strictly convex and differentiable on $(0, \infty)$ with $g(0) = v(0) = u(\infty)$, $g' = x_0 - I$, $g'(0+) = -\infty$, and $\lim_{\lambda\to\infty} g'(\lambda) = x_0 - a > 0$, hence $\lim_{\lambda\to\infty} g(\lambda) = \infty$. In particular g is bounded from below. For $\rho = dP^a/dQ$, Jensen's inequality implies

$$\begin{aligned}h(\lambda) &= E_Q\left[v\left(\lambda\rho\right) + \lambda x_0\rho\right] + x_0\lambda P^s[\Omega] \\ &\ge E_Q\left[g\left(\lambda\rho\right)\right] \ge g(\lambda)\end{aligned}$$

since $P_s[\Omega] = 0$ if $a = -\infty$ and $v(P|Q) < \infty$, and $x_0 > 0$ if $a = 0$. Note that $g'\left(\lambda\rho\right)\rho \in L^1(Q)$ for any $\lambda > 0$ by Lemma 3.6. Using the monotonicity of g' in order to get an integrable bound, we can apply Fubini's theorem to conclude

$$\begin{aligned}h(\lambda_2) &= h(\lambda_1) + E_Q\left[\int_{\lambda_1}^{\lambda_2} g'\left(\lambda\rho\right)\rho\, d\lambda\right] + x_0P^s[\Omega](\lambda_2 - \lambda_1) \\ &= h(\lambda_1) + \int_{\lambda_1}^{\lambda_2} E_{P^a}\left[g'\left(\lambda\rho\right)\right] d\lambda + x_0P^s[\Omega](\lambda_2 - \lambda_1) \\ &= h(\lambda_1) + x_0(\lambda_2 - \lambda_1) - \int_{\lambda_1}^{\lambda_2} E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] d\lambda,\end{aligned}$$

and this implies (34). Moreover, $h(\cdot)$ attains its unique minimum in some $\lambda := \lambda_{P,Q} > 0$ such that $h'(\lambda) = 0$ since h' is continuous by (32), $h(\infty) = g(\infty) = \infty$, and since (34) implies $h'(0+) = -\infty$ by monotone convergence. Since I is strictly decreasing, the minimizing value $\lambda_{P,Q}$ is uniquely determined by the condition

$$E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] = x_0.$$

Finally, any $H \in L^1(P)$ such that $H \geq a$ R-almost surely and $E_P[H] \leq x_0$ satisfies

$$\begin{aligned} E_Q[u(H)] &\leq E_Q[u(H)] + \lambda(x_0 - E_P[H]) \\ &= E_Q\left[u(H) - \lambda\rho H\right] + \lambda x_0 - \lambda E_{P^s}[H] \\ &\leq E_Q\left[v(\lambda\rho)\right] + \lambda x_0 - \lambda a P^s[\Omega] \\ &= v_\lambda(P|Q) + \lambda x_0 \qquad (36) \\ &= E_Q\left[v(\lambda\rho) + \lambda\rho I(\lambda\rho)\right] + \lambda\left(x_0 - E_{P^a}[I(\lambda\rho)] - aP^s[\Omega]\right) \\ &= E_Q\left[u\left(I\left(\lambda\frac{dP}{dQ}\right)\right)\right] + \lambda\left(x_0 - E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right]\right), \end{aligned}$$

for any $\lambda > 0$, and the two inequalities reduce to equalities iff $\lambda = \lambda_{P,Q}$ and $H = H_{P,Q}$ due to (29). The uniqueness on the set $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$ follows from the strict concavity of u. □

¿From now on we assume

$$v(\mathcal{P}|\mathcal{Q}) < \infty, \tag{37}$$

and we consider the robust divergences $v_\lambda(\mathcal{P}|\mathcal{Q})$ for $\lambda > 0$. As shown in Gundel [22], Theorem 2, the function $\lambda \mapsto v_\lambda(\mathcal{P}|\mathcal{Q}) + \lambda x$ is convex on $(0,\infty)$, and it follows as in the proof of Theorem 3.8 (i) that it attains its minimum in some positive value λ_0.

Remark 3.9. *In all three cases considered in Remark 3.7,*

$$v_\lambda(\mathcal{P}|\mathcal{Q}) = v_\lambda(P_0|Q_0) \text{ for any } \lambda > 0$$

whenever P_0 is a robust v-projection of $\mathcal{Q}$ on $\mathcal{P}$ and Q_0 is its reverse projection. In such a situation we can simply apply Theorem 3.8, and the minimizing value of λ is given by $\lambda_0 := \lambda_{P_0,Q_0}$.

Let us write $f := v_{\lambda_0}$. Note that $f(\mathcal{P}|\mathcal{Q}) < \infty$ due to Assumption (37) and Lemma 3.6. Suppose that the robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ and its reverse f-projection Q_0 exist. For $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, and $\alpha \in (0,1]$, we define $P_\alpha := \alpha P + (1-\alpha)P_0$, $Q_\alpha := \alpha Q + (1-\alpha)Q_0$,

$$\begin{aligned} \mathcal{Q}_0 :&= \{Q \in \mathcal{Q} : f(P_0|Q_\alpha) < \infty \text{ for some } \alpha \in (0,1]\} \\ &\supseteq \{Q \in \mathcal{Q} : f(P_0|Q) < \infty\}, \end{aligned} \tag{38}$$

and

$$\begin{aligned} \mathcal{P}_0 :&= \{P \in \mathcal{P} : f(P_\alpha|Q_0) < \infty \text{ for some } \alpha \in (0,1]\} \\ &\supseteq \{P \in \mathcal{P} : f(P|Q_0) < \infty\}. \end{aligned} \tag{39}$$

Let us first consider the reduced problem of maximizing

$$U_0(H) := \inf_{Q \in \mathcal{Q}_0} E_Q[u(H)] \tag{40}$$

under the constraint

$$\sup_{P \in \mathcal{P}_0} E_P[H] \leq x_0. \tag{41}$$

Remark 3.10. (i) *If $a > -\infty$ as for the logarithmic and the power utility functions, then*

$$\mathcal{P}_0 = \mathcal{P}. \tag{42}$$

Indeed, take $P \in \mathcal{P}$ and define $\rho_0 := dP_0^a/dQ_0$, $\rho := dP^a/dQ_0$, and $\rho_\alpha := dP_\alpha^a/dQ_0$ for $\alpha \in (0,1)$. Since $f = v_{\lambda_0}$ is convex with derivative $f'(x) = -\lambda_0 I(\lambda_0 x) \leq -\lambda_0 a$,

$$\begin{aligned} f(\rho_\alpha) &\leq f(\rho_0) - f'(\rho_\alpha)(\rho_0 - \rho_\alpha) \\ &\leq f(\rho_0) + \lambda_0 I(\lambda_0(1-\alpha)\rho_0)\rho_0 - \lambda_0 a \rho_\alpha \qquad \text{on } \{0 < \rho_\alpha < \infty\}. \end{aligned}$$

Since $\rho_0 I(\lambda\rho_0) \in L^1(Q_0)$ for any $\lambda > 0$ by Lemma 3.6, we obtain $f(\rho_\alpha) \in L^1(Q_0)$ for any $\alpha \in (0,1)$, hence $f(P_\alpha|Q_0) = E_{Q_0}[f(\rho_\alpha)] - a\lambda P_\alpha^s[\Omega] < \infty$ and $P \in \mathcal{P}_0$.

(ii) *If u is bounded from above as for the exponential utility function, then*

$$\mathcal{Q}_0 = \mathcal{Q}. \tag{43}$$

Indeed, take $Q \in \mathcal{Q}$ and define θ_0, θ, and θ_α as the densities of the absolutely continuous parts of Q_0, Q, and Q_α with respect to P_0. Recall from Remark 2.4 that

$$f(P_0|Q_\alpha) = \hat{f}(Q_\alpha|P_0) = \hat{f}'(\infty)Q_\alpha^s[\Omega] + \int \hat{f}(\theta_\alpha)dP_0$$

for $\hat{f}(x) := xf(1/x)$. Note that $\hat{f}'(\infty) = f(0) = v(0) = u(\infty)$ and that

$$\hat{f}'(x) = v\left(\frac{\lambda_0}{x}\right) + \frac{\lambda_0}{x} I\left(\frac{\lambda_0}{x}\right) = u\left(I\left(\frac{\lambda_0}{x}\right)\right). \tag{44}$$

As above we see that

$$\hat{f}(\theta_\alpha) \leq \hat{f}(\theta_0) - u\left(I\left(\frac{\lambda_0}{1-\alpha}\rho_0\right)\right)\theta_0 + u(\infty)\theta_\alpha \qquad \text{on } \{0 < \theta_\alpha < \infty\}.$$

Since $u\left(I\left(\lambda\rho_0\right)\right) \in L^1(Q_0)$ for any $\lambda > 0$ by Lemma 3.6, we obtain $\hat{f}(\theta_\alpha) \in L^1(P_0)$ and $\hat{f}(Q_\alpha|P_0) = E_{P_0}[\hat{f}(\theta_\alpha)] + u(\infty)Q_\alpha^s[\Omega] < \infty$ for any $\alpha \in (0,1)$, hence $Q \in \mathcal{Q}_0$.

Let us now show how the existence of a robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ yields the solution of the reduced optimization problem.

Theorem 3.11. *Assume that a robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ and its reverse f-projection Q_0 on $\mathcal{Q}$ exist. Then the robust utility maximization problem defined by* (40) *and* (41) *has the solution*

$$H_0 := I\left(\lambda_0 \frac{dP_0}{dQ_0}\right), \tag{45}$$

and the solution is R-almost surely unique on the set $\{dP_0/dR > 0\} \cup \{dQ_0/dR > 0\}$. The maximal value of the robust utility is given by

$$U_0(H_0) = f(\mathcal{P}|\mathcal{Q}) + \lambda_0 x_0.$$

Moreover, the contingent claim H_0 is affordable with $\mathcal{P}_0$-limited downside risk if $P_0 \approx Q_0 \approx R$.

Proof. For any $H \geq a$ satisfying the constraint (41), the estimate (36) applied to P_0, Q_0, and $\lambda > 0$ shows that

$$\begin{aligned} U_0(H) = \inf_{Q \in \mathcal{Q}_0} E_Q[u(H)] &\leq E_{Q_0}[u(H)] \\ &\leq \inf_{\lambda > 0}\{v_\lambda(P_0|Q_0) + \lambda x_0\} \\ &= v_{\lambda_0}(P_0|Q_0) + \lambda_0 x_0 \\ &= E_{Q_0}\left[u\left(H_0\right)\right] + \lambda_0(x_0 - E_{P_0}[H_0]), \end{aligned}$$

where we have used (29) in the last step. Note that $\lambda \mapsto v_\lambda(P_0|Q_0) + \lambda x_0$ attains its minimum in λ_0. Thus, Theorem 3.8 implies that $E_{P_0}[H_0] = x_0$, and this yields

$$\begin{aligned} U_0(H) &\leq v_{\lambda_0}(P_0|Q_0) + \lambda_0 x_0 \\ &= E_{Q_0}\left[u\left(H_0\right)\right]. \end{aligned}$$

Lemma 3.12 shows that H_0 satisfies the constraint (41) and that

$$E_{Q_0}\left[u\left(H_0\right)\right] = U_0(H_0) = \min_{Q \in \mathcal{Q}_0} E_Q\left[u\left(H_0\right)\right].$$

This concludes the proof that H_0 is optimal, with $U_0(H_0) = v_{\lambda_0}(P_0|Q_0) + \lambda_0 x_0$.

In order to show uniqueness, assume that $\tilde{H} \geq a$ solves the problem defined by (40) and (41). Then we have $E_{P_0}[\tilde{H}] \leq x_0$ and hence

$$\inf_{Q \in \mathcal{Q}} E_Q[u(\tilde{H})] \leq E_{Q_0}[u(\tilde{H})] \leq E_{Q_0}[u(H_0)].$$

The second inequality holds strictly unless $\tilde{H} = H_0$ R-almost surely on $\{dP_0/dR > 0\} \cup \{dQ_0/dR > 0\}$. This follows from the fact that H_0 maximizes $E_{Q_0}[u(H)]$ under the constraint

$E_{P_0}[H] \leq x_0$ and from the uniqueness result in Theorem 3.8. But the strict inequality is a contradiction to $E_{Q_0}[u(H_0)] = \inf_{Q\in\mathcal{Q}} E_Q[u(H_0)]$. Thus $\tilde{H} = H_0$ R-almost surely on $\{dP_0/dR > 0\} \cup \{dQ_0/dR > 0\}$.

Moreover, we obtain from Goll and Rüschendorf [21], Theorem 3.2, that

$$H_0 = x_0 + \int_0^T \xi_s dX_s \tag{46}$$

for some trading strategy $(\xi_t)_{0\leq t\leq T}$ such that the corresponding value process $V_t := \int_0^t \xi_s dX_s$ $(0 \leq t \leq T)$ is a P_0-martingale; this representation is based on results due to Yor [45] and Jacod [26]. For any $P \in \mathcal{P}_0$ the value process is a local martingale under P, and the conditional estimates (49) show that it is bounded from below by the P-martingale $E_P[H_0|\mathcal{F}_t]$, $0 \leq t \leq T$. Thus, H_0 is affordable with $\mathcal{P}_0$-limited downside risk if $P_0 \approx Q_0 \approx R$. Uniqueness follows from the strict concavity of u, and this is consistent with the uniqueness result in Proposition 2.14 for a strictly convex function f. □

The following Lemma was used in the proof of Theorem 3.11; it extends the arguments in Goll and Rüschendorf [21], Theorem 5.1.

Lemma 3.12. *Let P_0 be a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$, and let Q_0 be the reverse f-projection of P_0 on $\mathcal{Q}$. Then the contingent claim H_0 defined by* (45) *has the following properties:*

$$H_0 := I\left(\lambda_0 \frac{dP_0}{dQ_0}\right) \in L^1(P) \textit{ for all } P \in \mathcal{P}_0,$$

$$u(H_0) \in L^1(Q) \textit{ for all } Q \in \mathcal{Q}_0,$$

$$E_{P_0}[H_0] = \max_{P\in\mathcal{P}_0} E_P[H_0], \tag{47}$$

and

$$E_{Q_0}[u(H_0)] = \min_{Q\in\mathcal{Q}_0} E_Q[u(H_0)]. \tag{48}$$

If $P \approx Q_0$ for some $P \in \mathcal{P}_0$, then $P_0 \approx Q_0$. If in addition $Q_0 \approx R$, then for all $t \in [0,T]$ and $P \in \mathcal{P}_0$,

$$E_{P_0}[H_0|\mathcal{F}_t] \geq E_P[H_0|\mathcal{F}_t] \qquad R - a.s. \tag{49}$$

Proof. Take $P \in \mathcal{P}_0$, $\rho := dP^a/dQ_0$, and $\rho_0 := dP_0^a/dQ_0$. Due to our assumption $a = 0$ or $a = -\infty$ we have $f(P|Q_0) = f(P^a|Q_0)$ if $f(P|Q_0) < \infty$. Since P_0 is an f-projection of Q_0 on $\mathcal{P}$ and $f := v_{\lambda_0}$ is differentiable on $(0,\infty)$, a criterion in Rüschendorf [37], Theorem 5, for f-projections implies

$$E_{Q_0}\left[f'(\rho_0)(\rho - \rho_0)\right] \geq 0. \tag{50}$$

For the convenience of the reader we include the argument: Define $P_\alpha := \alpha P + (1-\alpha)P_0$ and $\rho_\alpha := dP_\alpha^a / dQ_0$. The function $\alpha \mapsto f(\rho_\alpha)$ is convex on $[0,1]$, and so

$$Z_\alpha := \frac{f(\rho_\alpha) - f(\rho_0)}{\alpha}$$

is increasing in α and decreasing to $Z_0 = f'(\rho_0)(\rho - \rho_0)$ as $\alpha \searrow 0$. By definition of $\mathcal{P}_0$ there is $\alpha_0 \in (0,1]$ such that $Z_{\alpha_0} \in L^1(Q_0)$, and Z_α is bounded by Z_{α_0} for $\alpha \leq \alpha_0$. By monotone convergence we obtain $Z_0 \in L^1(Q_0)$ and $E_{Q_0}[Z_0] \geq 0$, since $E_{Q_0}[Z_\alpha] = \alpha^{-1}\left(f(P_\alpha|Q_0) - f(P_0|Q_0)\right) \geq 0$ for any $\alpha > 0$.

In our situation we have $f'(x) = -\lambda_0 I(\lambda_0 x)$ and $f'(\rho_0)\rho_0 \in L^1(Q_0)$ by Lemma 3.6, hence $f'(\rho_0)\rho \in L^1(Q_0)$ and $H_0 \in L^1(P)$ since $Z_0 \in L^1(Q_0)$. Moreover, Inequality (50) and Assumption (27) allow us to conclude

$$E_P\left[f'\left(\frac{dP_0}{dQ_0}\right)\right] \geq E_{P_0}\left[f'\left(\frac{dP_0}{dQ_0}\right)\right], \tag{51}$$

and this amounts to the inequality

$$E_P[H_0] \leq E_{P_0}[H_0]. \tag{52}$$

In order to verify (48) take $\hat{f}(x) := xf(1/x)$. Then Q_0 is the $\hat{f}$-projection of P_0 on $\mathcal{Q}$, and $\hat{f}'\left(dQ_0/dP_0\right) = u(H_0)$ due to (44). Note that due to our assumption $u(\infty) = 0$ or $u(\infty) = \infty$ we have $f(P_0|Q) = f(P_0|Q^a)$ for any $Q \in \mathcal{Q}$ with $f(P_0|Q) < \infty$. Q_0-integrability of $u(H_0)$ follows from Lemma 3.6. Now we apply the argument above in terms of $\hat{f}$, reversing the role of the sets $\mathcal{Q}$ and $\mathcal{P}$ to obtain

$$E_Q[u(H_0)] \geq E_{Q_0}[u(H_0)].$$

Q-integrability of $u(H_0)$ for $Q \in \mathcal{Q}_0$ follows as above.

In order to show that $P_0 \approx Q_0$ take $P \in \mathcal{P}_0$ with $P \approx Q_0$. If P_0 is not equivalent to Q_0, then $P\left(dP_0/dQ_0 = 0\right) > 0$ and hence $E_P[H_0] = \infty$ since $I(0) = \infty$. But in view of (52) this is a contradiction to $H_0 \in L^1(P_0)$.

In order to show the conditional estimate (49) for $P \in \mathcal{P}_0$ and $t \in (0,T)$, we write $\rho_0 = \rho_{0,t}\hat{\rho}_{0,t}$ where $\rho_{0,t} := dP_0^a/dQ_0|_{\mathcal{F}_t}$ and $\hat{\rho}_{0,t}$ is the conditional density with respect to $\mathcal{F}_t$. In the same way we define ρ_t, $\hat{\rho}_t$, $\rho_{\alpha,t}$ and $\hat{\rho}_{\alpha,t}$. Due to (31) we have on $\{\rho_{\alpha,t} > 0\}$

$$\begin{aligned}
f(\rho_{0,t}\hat{\rho}_{\alpha,t}) &= f\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\rho_{\alpha,t}\hat{\rho}_{\alpha,t}\right) \\
&\leq a\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right) f(\rho_{\alpha,t}\hat{\rho}_{\alpha,t}) + b\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right)(\rho_{\alpha,t}\hat{\rho}_{\alpha,t} + 1) \\
&= a\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right) f(\rho_\alpha) + b\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right)(\rho_\alpha + 1).
\end{aligned}$$

For $\alpha \in (0, \alpha_0]$ we have $E_{Q_0}[f(\rho_\alpha)|\mathcal{F}_t] < \infty$ Q_0-almost surely, and this implies that also $E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < \infty$ Q_0-almost surely on $\{\rho_{\alpha,t} > 0\}$. If $f(0) = 0$, then $f(\rho_{0,t}\hat{\rho}_{\alpha,t}) = 0$ on $\{\rho_{\alpha,t} = 0\}$ due to the definition of $\rho_{\alpha,t}$. If $f(0) = \infty$, then $\rho_{\alpha,t} > 0$ R-almost surely. Hence $E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < \infty$ Q_0-almost surely on Ω. Furthermore,

$$E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] \geq E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{0,t})|\mathcal{F}_t] \qquad Q_0 - a.s.$$

Indeed, the measure $\tilde{P}$ with density

$$\tilde{\rho} := \begin{cases} \rho_{0,t}\hat{\rho}_{\alpha,t} & \text{on } A \\ \rho_0 & \text{on } A^c \end{cases}$$

with $A := \{E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < E_{Q_0}[f(\rho_{0,t}\hat{\rho}_{0,t})|\mathcal{F}_t]\}$ belongs to $\mathcal{P}$, and $Q_0[A] > 0$ would imply

$$f(\tilde{P}|Q_0) = E_{Q_0}[f(\tilde{\rho})] = E_{Q_0}[E_{Q_0}[f(\tilde{\rho})|\mathcal{F}_t]] < E_{Q_0}[f(\rho_0)] = f(P_0|Q_0)$$

which contradicts the minimality of P_0. We can now repeat the argument above, with

$$Z_{\alpha,t} := \frac{f(\rho_{0,t}\hat{\rho}_{\alpha,t}) - f(\rho_0)}{\alpha}$$

instead of Z_α, to obtain

$$\rho_{0,t}E_{Q_0}[f'(\rho_0)(\hat{\rho}_t - \hat{\rho}_{0,t})|\mathcal{F}_t] \geq 0 \qquad Q_0 - a.s.$$

Since $Q_0 \approx R$, $P_0 \approx R$ and hence $\rho_{0,t} > 0$ R-almost surely, the proof of (49) is complete.

□

Remark 3.13. *Equation* (47) *shows that the robust f-projection P_0 of $\mathcal{Q}$ on $\mathcal{P}$ is indeed a least favorable pricing measure for the optimal claim H_0. In the same manner, Equation* (48) *allows us to view Q_0 as a least favorable measure for the utility evaluation of H_0. If Q_0 minimizes the reverse f-divergence of P_0 over the set $\mathcal{Q}$ simultaneously for all convex functions f, then Q_0 is in fact a least favorable measure in the sense of Huber and Strassen [23]; see Schied [39] and [40] for a more detailed discussion of the connection between robust utility maximization, risk measures, and the robust Neyman-Pearson lemma.*

Clearly, the solution of the reduced problem provides the solution of the original optimization problem for the utility functional U defined in (25) as soon as $\mathcal{Q}_0 = \mathcal{Q}$. This condition is satisfied in the classical case where $\mathcal{Q}$ consists of a single measure. Recall from part (ii) of Remark 3.10 that it also holds if u is bounded from above.

Corollary 3.14. *Suppose that $\mathcal{Q}$ is weakly compact in the sense of Assumption 2.5 and satisfies $\mathcal{Q} = \mathcal{Q}_0$, and that the utility function u is finite on $\mathbb{R}$. Then the robust optimization problem defined by* (25) *and* (26) *has the unique solution*

$$H_0 := I\left(\lambda_0 \frac{dP_0}{dQ_0}\right),$$

where P_0 is a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$ and Q_0 is its reverse projection on $\mathcal{Q}$. The contingent claim H_0 is affordable with $\mathcal{P}_0$-limited downside risk if $P_0 \approx Q_0 \approx R$.

Proof. If u is finite on $\mathbb{R}$, then $a = -\infty$. This implies $\lim_{x\to\infty} f(x)/x = -a = \infty$, and the same property holds for $f = v_{\lambda_0}$. Recall from Remark 3.3 that the set $\mathcal{K}_{\mathcal{P}}$ is closed. Thus both parts of Assumption 2.5 are satisfied in our case. We can therefore apply Theorem 2.6 in order to obtain the existence of a pair $(P_0, Q_0) \in \mathcal{P} \times \mathcal{Q}$ that minimizes $f(P|Q)$ over the sets $\mathcal{P}$ and $\mathcal{Q}$. Thus the assumptions of Theorem 3.11 are verified, hence H_0 is the solution of the original optimization problem defined by (25) and (26). $\square$

Remark 3.15. *The compactness assumption on the set $\mathcal{Q}$ can be motivated as follows. Note first that our robust utility functional U, defined by* (25) *for some class of measures $Q \ll R$, remains unchanged if we pass to the weak closure of $\mathcal{K}_{\mathcal{Q}}$ in $L^1(R)$. Thus it is no loss of generality to assume that $\mathcal{Q}$ is weakly closed. Weak compactness of $\mathcal{Q}$ is now equivalent to uniform integrability of $\mathcal{K}_{\mathcal{Q}}$, and hence to the condition*

$$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \textit{such that } R[A] < \delta \implies Q[A] < \epsilon \;\; \forall\, Q \in \mathcal{Q}; \tag{53}$$

see, for example, Dellacherie and Meyer [8], Theorem II.19. This condition means that all models in $\mathcal{Q}$ agree that an event is highly unlikely if it has sufficiently small probability under the reference measure R.

Clearly, Corollary 3.14 includes the case of exponential utility functions. But it does not cover the remaining cases considered in Remark 3.7 where $a = 0$, hence $\lim_{x\to\infty} f(x)/x = 0$. In order to formulate an existence result for such cases we are now going to extend our setting and in particular the notion of a martingale measure.

4 Extended Martingale Measures

In this section we enlarge our initial probability space by introducing an additional *default time* ζ, defined as the second coordinate $\zeta(\omega, s) := s$ on the product space $\bar{\Omega} := \Omega \times (0, \infty]$.

Let

$$\bar{\mathcal{F}} := \sigma(\{A \times (t, \infty] : A \in \mathcal{F}_t, t \geq 0\})$$

denote the predictable σ-field on $\bar{\Omega}$, where $\mathcal{F}_t := \mathcal{F}_T$ for $t > T$; the predictable filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ is defined in the same manner. An adapted process $Y = (Y_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ will be identified with the adapted process $\bar{Y} = (\bar{Y}_t)_{t \geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t \geq 0})$ defined by $\bar{Y}_t := Y_t I_{\{\zeta > t\}}$, i.e.,

$$\bar{Y}_t(\omega, s) := Y_t(\omega) 1_{(t,\infty]}(s) \qquad (t \geq 0).$$

To a probability measure Q on $(\Omega, \mathcal{F})$ corresponds the probability measure $\bar{Q} := Q \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$. Conversely, for any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ we define its projections Q^t on $(\Omega, \mathcal{F}_t)$ by

$$Q^t[A] := \bar{Q}[A \times (t, \infty]] \qquad (A \in \mathcal{F}_t).$$

In order to introduce the class $\bar{\mathcal{P}}$ of extended martingale measures, let us denote by $\mathcal{V}(x_0)$ the class of all non-negative value processes $V = (V_t)_{t \geq 0}$ of the form (22) with $V_t := V_T$ for $t \geq T$, i.e.,

$$V_t = x_0 + \int_0^{t \wedge T} \xi_s dX_s \geq 0 \qquad (t \geq 0),$$

and by $\bar{\mathcal{V}}(x_0)$ the class of the corresponding processes $\bar{V} = (\bar{V}_t)_{t \geq 0}$.

Definition 4.1. *A probability measure $\bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ will be called an* extended martingale measure *if*

(i) $P^t \ll R$ on $\mathcal{F}_t$ $(t \geq 0)$,

(ii) Under $\bar{P}$, any $\bar{V} \in \bar{\mathcal{V}}(x_0)$ is a supermartingale with respect to $(\bar{\mathcal{F}}_t)_{t \geq 0}$.

We denote by $\bar{\mathcal{P}}$ the class of all extended martingale measures.

Clearly, for any martingale measure $P \in \mathcal{P}$ the corresponding measure $\bar{P} := P \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ belongs to $\bar{\mathcal{P}}$.

We are going to use the representation of a right-continuous non-negative supermartingale $Z = (Z_t)_{t \geq 0}$ with $Z_0 = 1$ as a probability measure $\bar{P}^Z$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that

$$\bar{P}^Z[A \times (t, \infty]] = E_R[Z_t; A] \tag{54}$$

for $A \in \mathcal{F}_t$ and $t \geq 0$; see Föllmer [12]. This requires a regularity assumption on the underlying filtration, for instance in the following form.

Assumption 4.2. *$(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous modification of a standard system $(\mathcal{F}_t^0)_{t\geq 0}$ in the sense of Parthasarathy [35] V, i.e.,* (i) *each $(\Omega, \mathcal{F}_t^0)$ is a standard Borel space, and* (ii) *any decreasing sequence of atoms A_i of $\mathcal{F}_{t_i}$ for $0 \leq t_1 \leq t_2 \leq ...$ has a non-void intersection.*

Remark 4.3. (i) *For any probability measure $\bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ whose projections satisfy Condition* (i) *of Definition 4.1, the adapted process $Z = (Z_t)_{t\geq 0}$ defined by*

$$Z_t := \frac{dP^t}{dR} \qquad (t \geq 0) \tag{55}$$

is a right-continuous non-negative supermartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, R)$ with $Z_0 = 1$. Conversely, any such supermartingale induces a probability measure $\bar{P}^Z$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ via (54) *if the underlying filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ is rich enough, for example in the sense of Assumption 4.2; see Föllmer [12] and also Föllmer [11], Meyer [33], Azéma and Jeulin [2], and Stricker [42]. For any supermartingale $Y = (Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, R)$, the process $\bar{U} = (\bar{U}_t)_{t\geq 0}$ defined by*

$$\bar{U}_t(\omega, s) := \frac{Y_t}{Z_t} 1_{\{Z_t \neq 0\}} 1_{(t,\infty]}(s)$$

is a $\bar{P}^Z$-supermartingale. Conversely, if the process $\bar{U}$ with $\bar{U}_t = U_t 1_{\{\zeta > t\}}$ is a supermartingale under $\bar{P}^Z$, then $Y := UZ$ is an R-supermartingale; see Föllmer [11], Proposition 4.2.

(ii) *Let $\bar{P} = \bar{P}^Z$ be a probability measure on $(\bar{\Omega}, \bar{F})$ such that* (54) *holds. It follows from part* (i) *that $\bar{P}$ is an extended martingale measure if and only if*

$$ZV \text{ is an } R\text{-supermartingale for any } V \in \mathcal{V}(x_0). \tag{56}$$

Thus our class $\bar{\mathcal{P}}$ of extended martingale measures corresponds exactly to the class of supermartingales which appear in the duality approach of Kramkov and Schachermayer to the problem of maximizing expected utility in incomplete financial markets; see [30], page 6.

Lemma 4.4. *Let $(\bar{P}_n)_{n\geq 1}$ be a sequence in the set $\bar{\mathcal{P}}$. Then there is a sequence $\bar{P}_{n,0} \in$ conv$(\bar{P}_n, \bar{P}_{n+1}, ...)$ $(n = 1, 2, ...)$ and a measure $\bar{P}_0 \in \bar{\mathcal{P}}$ such that*

$$\left.\frac{dP_{n,0}^T}{dR}\right|_{\mathcal{F}_T} \longrightarrow \left.\frac{dP_0^T}{dR}\right|_{\mathcal{F}_T} \qquad R - a.s. \tag{57}$$

Proof. Let Z^n be the supermartingale which corresponds to $\bar{P}_n$ via (55). By Föllmer and Kramkov [15], Lemma 5.2, there are processes

$$Z^{n,0} \in \ \text{conv}(Z^n, Z^{n+1}, ...) \quad (n = 1, 2, ...)$$

and a right-continuous non-negative supermartingale Z such that $Z^{n,0}$ is Fatou convergent to Z on the set of rational points, i.e.,

$$Z_t = \limsup_{s\downarrow t, s\in\mathbb{Q}} \limsup_{n\to\infty} Z_s^{n,0} = \liminf_{s\downarrow t, s\in\mathbb{Q}} \liminf_{n\to\infty} Z_s^{n,0}$$

R-almost surely for $t \geq 0$. In particular $Z_T^{n,0}$ converges to Z_T R-almost surely because $Z^{n,0}$ is constant for $t \geq T$ for every $n \geq 1$. Furthermore, $VZ^{n,0}$ is Fatou convergent to the supermartingale VZ for every $V \in \mathcal{V}(x_0)$. Thus, part (ii) of Remark 4.3 shows that the probability measure $\bar{P}_0 := \bar{P}^Z$ belongs to $\bar{\mathcal{P}}$, and this completes the proof. $\square$

Let us now formulate a general projection result for the class $\bar{\mathcal{P}}$ of extended martingale measures and for the class

$$\bar{\mathcal{Q}} := \{Q \times \delta_\infty : Q \in \mathcal{Q}\}.$$

Let $f : (0, \infty) \to \mathbb{R}$ be a strictly convex function such that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0. \tag{58}$$

In this case the definition of $f(\cdot,\cdot)$ in (9) simplifies to

$$f(x,y) := \begin{cases} 0 & \text{if } y = 0 \\ yf\left(\frac{x}{y}\right) & \text{if } y > 0, \end{cases}$$

$f(\cdot,\cdot)$ is continuous on $(0,\infty)\times[0,\infty)$, and the f-divergence of $\bar{P} \in \bar{\mathcal{P}}$ with respect to $\bar{Q} \in \bar{\mathcal{Q}}$ is given by

$$f(\bar{P}|\bar{Q}) = E_Q\left[f\left(\frac{d(P^\infty)^a}{dQ}\right)\right] = E_Q\left[f\left(\frac{d(P^T)^a}{dQ}\right)\right] = f(P^T|Q)$$

due to Remark 2.2 and our assumption $\mathcal{F}_T = \mathcal{F}$, where $(P^T)^a$ is the absolutely continuous part of P^T with respect to Q.

Theorem 4.5. *Let $\mathcal{Q}$ be weakly compact in the sense of Assumption 2.5, and let f satisfy Condition* (58)*. Then there exist a robust f-projection $\bar{P}_0$ of $\bar{\mathcal{Q}}$ on $\bar{\mathcal{P}}$ and its reverse f-projection $\bar{Q}_0$, i.e.,*

$$f(\bar{P}_0|\bar{Q}_0) = f(\bar{\mathcal{P}}|\bar{\mathcal{Q}}) = \inf_{\bar{P}\in\bar{\mathcal{P}}} \inf_{Q\in\mathcal{Q}} f(P^T|Q).$$

Proof. Let $(Q_n)_{n\geq 1} \subseteq \mathcal{Q}$ and $(\bar{P}_n)_{n\geq 1} \subseteq \bar{\mathcal{P}}$ be such that $f(\bar{P}_n|\bar{Q}_n)$ converges to the infimum of the values $f(\bar{P}|\bar{Q})$ for $\bar{P} \in \bar{\mathcal{P}}$ and $Q \in \mathcal{Q}$, and define

$$\psi_n := \frac{dQ_n}{dR}.$$

By Delbaen and Schachermayer [7], Lemma A1.1, we can choose

$$\psi_{n,0} \in \operatorname{conv}(\psi_n, \psi_{n+1}, ...) \quad (n = 1, 2, ...)$$

and a function ψ_0 such that

$$\psi_{n,0} \longrightarrow \psi_0 \qquad R - a.s.$$

Since the set $\mathcal{K}_{\mathcal{Q}}$ is weakly compact we have $\psi_0 \in \mathcal{K}_{\mathcal{Q}}$, i.e., ψ_0 is the density of some measure $Q_0 \in \mathcal{Q}$. Due to Lemma 4.4 we can also choose

$$\bar{P}_{n,0} \in \ \operatorname{conv}(\bar{P}_n, \bar{P}_{n+1}, ...) \quad (n = 1, 2, ...)$$

and $\bar{P}_0 \in \bar{\mathcal{P}}$ such that (57) holds.

Define $\phi_{n,0} := dP^T_{n,0}/dR\big|_{\mathcal{F}_T}$ and $\phi_0 := dP^T_0/dR\big|_{\mathcal{F}_T}$. Note first that

$$\begin{aligned} f(\bar{P}_0|\bar{Q}_0) &= E_R \left[f\left(\phi_0, \psi_0\right)\right] \\ &= E_R \left[\lim_{\epsilon \to 0} f\left(\phi_0 + \epsilon, \psi_0\right)\right] \\ &= \lim_{\epsilon \to 0} E_R \left[f\left(\phi_0 + \epsilon, \psi_0\right)\right] \end{aligned}$$

by monotone convergence, since $f(\cdot, y)$ is continuous and decreasing on $[0, \infty)$ and

$$E_R \left[f\left(\phi_0 + \epsilon, \psi_0\right)\right] = E_{Q_0} \left[f\left(\frac{\phi_0 + \epsilon}{\psi_0}\right)\right] \geq f(1+\epsilon) > -\infty$$

by Jensen's inequality. For any $\epsilon > 0$ if follows as in Schied and Wu [41], Lemma 3.6, that the set $\{f^-(\phi + \epsilon, \psi) : \phi \in \mathcal{K}_{\bar{\mathcal{P}}},\ \psi \in \mathcal{K}_{\mathcal{Q}}\}$ is uniformly integrable, where $\mathcal{K}_{\bar{\mathcal{P}}} := \left\{dP^T/dR : \bar{P} \in \bar{\mathcal{P}}\right\}$. This implies

$$\begin{aligned} E_R \left[f\left(\phi_0 + \epsilon, \psi_0\right)\right] &= E_R \left[\lim_{n \to \infty} f(\phi_{n,0} + \epsilon, \psi_{n,0})\right] \\ &= E_R \left[\lim_{n \to \infty} f^+(\phi_{n,0} + \epsilon, \psi_{n,0})\right] - E_R \left[\lim_{n \to \infty} f^-(\phi_{n,0} + \epsilon, \psi_{n,0})\right] \\ &\leq \liminf_{n \to \infty} E_R[f(\phi_{n,0} + \epsilon, \psi_{n,0})] \\ &\leq \liminf_{n \to \infty} E_R[f(\phi_{n,0}, \psi_{n,0})] \\ &\leq \liminf_{n \to \infty} E_R[f(\phi_n, \psi_n)] = f(\bar{\mathcal{P}}|\bar{\mathcal{Q}}). \end{aligned}$$

The first equality follows from the continuity of $f(\cdot + \epsilon, \cdot)$ on $[0, \infty)^2$, the first inequality follows from Fatou's lemma (applied to the first term) and Lebesgue's theorem (applied to the second term) and the last one from the convexity of $f(\cdot, \cdot)$. This shows that $f(\cdot|\cdot)$ attains its minimum in $(\bar{P}_0, \bar{Q}_0)$. □

Remark 4.6. *Uniqueness of the density dP_0^T/dQ_0 holds as in Proposition 2.14 if the function f is strictly convex.*

Let us now return to the utility maximization problem. In view of Corollary 3.14 we assume that the utility function u is given on $(0,\infty)$. Thus the convex conjugate function v of u as defined in (29) satisfies Condition (58) due to (30).

Since $\{u > -\infty\} \subseteq [0,\infty)$, a contingent claim is relevant for our utility maximization problem only if it is non-negative. In this case affordability reduces to the price constraint (24) for $\mathcal{P}_0 = \mathcal{P}_e$ as explained in Remark 3.5. In fact the price constraint also includes the class $\bar{\mathcal{P}}$ of extended martingale measures:

Lemma 4.7. *For a contingent claim $H \geq 0$ the following conditions are equivalent:*

(i) $\sup_{P\in\mathcal{P}_e} E_P[H] \leq x_0$.

(ii) There exists a value process $V \in \mathcal{V}(x_0)$ such that $V_T \geq H$ R-almost surely.

(iii) The corresponding claim $\bar{H} := H1_{\{\zeta>T\}}$ satisfies the constraint

$$\sup_{\bar{P}\in\bar{\mathcal{P}}} E_{\bar{P}}[\bar{H}] \leq x_0.$$

Proof. The equivalence of (i) and (ii) is a key result in the theory of superhedging as recalled in Remark 3.5. To check that (ii) implies (iii) note that for any $V \in \mathcal{V}(x_0)$ the process $(\bar{V}_t)$ is a $\bar{P}$-supermartingale with $\bar{V}_T \geq \bar{H}$ $\bar{P}$-almost surely because $\bar{P}[\bar{V}_T \geq \bar{H}] = P^T[V_T \geq H]$ and $P^T \ll R$. Since $P \times \delta_\infty \in \bar{\mathcal{P}}$ for any $P \in \mathcal{P}_e$, (iii) implies (i). $\square$

As in Section 3 we denote by $\lambda_0 > 0$ a minimizer of $v_\lambda(\bar{\mathcal{P}}|\bar{\mathcal{Q}}) + \lambda x_0$ and define the class $\mathcal{Q}_0$ as in (38). Our aim is to maximize the robust utility

$$U_0(H) = \inf_{Q\in\mathcal{Q}_0} E_Q[u(H)] \tag{59}$$

over all contingent claims $H \geq 0$ such that $\bar{H} := H1_{\{\zeta>T\}}$ satisfies the constraint

$$\sup_{\bar{P}\in\bar{\mathcal{P}}} E_{\bar{P}}[\bar{H}] \leq x_0. \tag{60}$$

Corollary 4.8. *Let $\mathcal{Q}$ be weakly compact in the sense of* (2.5)*. Then there exists a solution to the utility maximization problem defined by* (59) *and* (60)*. It is given by*

$$H_0 := I\left(\lambda_0 \frac{dP_0^T}{dQ_0}\right),$$

where $\bar{P}_0$ is the robust v_{λ_0}-projection of $\bar{\mathcal{Q}}$ on $\bar{\mathcal{P}}$ and $\bar{Q}_0 = Q_0 \times \delta_\infty$ is its reverse v_{λ_0}-projection. H_0 is affordable in the sense that it satisfies the conditions of Lemma 4.7.

Proof. For $\bar{P} \in \bar{\mathcal{P}}$, $\bar{Q} = Q \times \delta_\infty \in \bar{\mathcal{Q}}$, and $H \in L^1(P^T)$ such that $E_{\bar{P}}[\bar{H}] = E_{P^T}[H] \le x_0$ we obtain

$$\begin{aligned} E_Q[u(H)] &\le E_Q\left[u(H) - \lambda H \frac{d(P^T)^a}{dQ}\right] - \lambda E_{(P^T)^s}[H] + \lambda x_0 \\ &\le v_\lambda(\bar{P}|\bar{Q}) + \lambda x_0 \end{aligned}$$

in analogy to (36). Since $\lim_{x\to\infty} v(x)/x = a = 0$, Theorem 4.5 ensures the existence of a robust v_{λ_0}-projection $\bar{P}_0$ of $\bar{\mathcal{Q}}$ on $\bar{\mathcal{P}}$. We can now continue as in the proof of Theorem 3.11 to conclude that $E_{\bar{P}_0}[\bar{H}_0] = E_{P_0^T}[H_0] = x_0$ and

$$\begin{aligned} U_0(H) &\le v_{\lambda_0}(\bar{P}_0|\bar{Q}_0) + \lambda_0 x_0 \\ &= E_{Q_0}[u(H_0)]. \end{aligned}$$

It follows from Lemma 3.12 and Remark 3.10 that

$$E_{\bar{P}_0}[\bar{H}_0] = \max_{\bar{P}\in\bar{\mathcal{P}}} E_{\bar{P}}[\bar{H}_0],$$

that $E_{\bar{Q}_0}[u(\bar{H}_0)] = \min_{\bar{Q}\in\bar{\mathcal{Q}}_0} E_{\bar{Q}}[u(\bar{H}_0)]$. Thus H_0 solves the optimization problem defined by (59) and (60). □

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