# A Representation of Excessive Functions as Expected Suprema


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*Dedicated to the memory of Kazimierz Urbanik*


**Abstract**

For a nice Markov process such as Brownian motion on a domain in $\mathbb{R}^d$, we prove a representation of excessive functions in terms of expected suprema. This is motivated by recent work of El Karoui [5] and El Karoui and Meziou [8] on the max-plus decomposition for supermartingales. Our results provide a singular analogue to the non-linear Riesz representation in El Karoui and Föllmer [6], and they extend the representation of potentials in Föllmer and Knispel [10] by clarifying the role of the boundary behavior and of the harmonic points of the given excessive function.



**Key words:** Markov processes, excessive functions, expected suprema

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## 1 Introduction

Consider a bounded superharmonic function $u$ on the open disk $S$. Such a function admits a limit $u(y)$ in almost all boundary points $y \in \partial S$ with respect to the *fine topology*, and we have

$$u(x) \geqslant \int u(y)\, \mu_x(dy),$$

where $\mu_x$ denotes the harmonic measure on the boundary. The right-hand side defines a harmonic function $h$ on $S$, and the difference $u - h$ can be represented as the potential of a measure on $S$. This is the classical Riesz representation of the superharmonic function $u$.

In probabilistic terms, $\mu_x$ may be viewed as the exit distribution of Brownian motion on $S$

starting in $x$, $u$ is an excessive function of the process, the fine limit can be described as a limit along Brownian paths to the boundary, and the Riesz representation takes the form

$$u(x) = E_x[\lim_{t \uparrow \zeta} u(X_t) + A_\zeta],$$

where $\zeta$ denotes the first exit time from $S$ and $(A_t)_{t \geqslant 0}$ is the additive functional generating the potential $u - h$; cf., e. g., Blumenthal and Getoor [4].

In this paper we consider an alternative probabilistic representation of the excessive function $u$ in terms of expected suprema. We construct a function $f$ on the closure of $S$ which coincides with the boundary values of $u$ on $\partial S$ and yields the representation

$$u(x) = E_x[\sup_{0 < t \leqslant \zeta} f(X_t)], \tag{1}$$

i. e.,

$$u(x) = E_x[\sup_{0 < t < \zeta} f(X_t) \vee \lim_{t \uparrow \zeta} u(X_t)]. \tag{2}$$

Instead of Brownian motion on the unit disk, we consider a general Markov process with state space $S$ and life time $\zeta$. Under some regularity conditions we prove in section 3 that an excessive function $u$ admits a representation of the form (1) in terms of some function $f$ on $S$. Under additional conditions, the limit in (2) can be identified as a boundary value $f(X_\zeta)$ for some function $f$ on the Martin boundary of the process, and in this case (2) can also be written in the condensed form (1).

The representing function $f$ is in general not unique. In section 4 we characterize the class of representing functions in terms of a maximal and a minimal representing function. These bounds are described in potential theoretic terms. They coincide in points where the excessive function $u$ is not harmonic, the lower bound is equal to zero on the set $H$ of harmonic points, and the upper bound is constant on the connected components of $H$.

Our representation (2) of an excessive function is motivated by recent work of El Karoui and Meziou [8] and El Karoui [5] on problems of portfolio insurance. Their results involve a representation of a given supermartingale as the process of conditional expected suprema of another process. This may be viewed as a singular analogue to a general representation for semimartingales in Bank and El Karoui [1], which provides a unified solution to various representation problems arising in connection with optimal consumption choice, optimal stopping, and multi-armed bandit problems. We refer to Bank and Föllmer [2] for a survey and to the references given there, in particular to El Karoui and Karatzas [7] and Bank and Riedel [3]; see also Kaspi and Mandelbaum [11].

In the context of probabilistic potential theory such representation problems take the following form: For a given function $u$ and a given additive functional $(B_t)_{t \geqslant 0}$ of the underlying Markov process we want to find a function $f$ such that

$$u(x) = E_x[\int_0^\zeta \sup_{0 < t \leqslant \zeta} f(X_t)\, dB_t].$$

In El Karoui and Föllmer [6] this potential theoretic problem is discussed for the smooth additive functional $B_t = t \wedge \zeta$ and for the case when $u$ has boundary behavior zero. The results are easily extended to the case where the random measure corresponding to the additive functional satisfies the regularity assumptions required in [1].

Our representation (2) corresponds to the singular case $B_t = 1_{[\zeta,\infty)}(t)$ where the random measure is given by the Dirac measure $\delta_\zeta$. This singular representation problem, which does not satisfy the regularity assumptions of [1], is discussed in Föllmer and Knispel [10] for the special case of a potential $u$. The purpose of the present paper is to consider a general excessive function $u$ and to clarify the impact of the boundary behavior on the representation of $u$ as an expected supremum. We concentrate on those proofs which involve explicitly the boundary behavior of $u$, and we refer to [10] whenever the argument is the same as in the case of a potential.

**Acknowledgement.** *While working on his thesis in probabilistic potential theory, a topic which is revisited in this paper from a new point of view, the first author had the great pleasure of attending the beautiful "Lectures on Prediction Theory" of Kazimierz Urbanik [12], given at the University of Erlangen during the winter semester 1966/67. We dedicate this paper to his memory.*

## 2 Preliminaries

Let $(X_t)_{t\geqslant 0}$ be a strong Markov process with locally compact metric state space $(S,d)$, shift operators $(\theta_t)_{t\geqslant 0}$, and life time $\zeta$, defined on a stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (P_x)_{x\in S})$ and satisfying the assumptions in [6] or [10]. In particular we assume that the excessive functions of the process are lower-semicontinuous. As a typical example, we could consider a Brownian motion on a domain $S \subset \mathbb{R}^d$.

For any measurable function $u \geqslant 0$ on $S$ and for any stopping time $T$ we use the notation

$$P_T u(x) := E_x[u(X_T); T < \zeta].$$

Recall that $u$ is *excessive* if $P_t u \leqslant u$ for any $t > 0$ and $\lim_{t\downarrow 0} P_t u(x) = u(x)$ for any $x \in S$. In that case the process $(u(X_t)1_{\{t<\zeta\}})_{t\geqslant 0}$ is a right-continuous $P_x$-supermartingale for any $x \in S$ such that $u(x) < \infty$, and this implies the existence of

$$u_\zeta := \lim_{t\uparrow\zeta} u(X_t) \quad P_x\text{–a. s.}.$$

Let us denote by $\mathcal{T}(x)$ the class of all exit times

$$T_U := \inf\{t \geqslant 0 | X_t \notin U\} \wedge \zeta$$

from open neighborhoods $U$ of $x \in S$, and by $\mathcal{T}_0(x)$ the subclass of all exit times from open neighborhoods of $x$ which are relatively compact. Note that $\zeta = T_S \in \mathcal{T}(x)$. For $T \in \mathcal{T}(x)$

and any measurable function $u \geqslant 0$ we introduce the notation

$$u_T := u(X_T)1_{\{T<\zeta\}} + \overline{\lim_{t\uparrow\zeta}}\, u(X_t)1_{\{T=\zeta\}}$$

and

$$\widetilde{P}_T u(x) := E_x[u_T] = P_T u(x) + E_x[\overline{\lim_{t\uparrow\zeta}}\, u(X_t); T=\zeta].$$

We say that a function $u$ belongs to class (D) if for any $x \in S$ the family $\{u(X_T)|T \in \mathcal{T}_0(x)\}$ is uniformly integrable with respect to $P_x$. Recall that an excessive function $u$ is *harmonic* on $S$ if $P_T u(x) = u(x)$ for any $x \in S$ and any $T \in \mathcal{T}_0(x)$. A harmonic function $u$ of class (D) also satisfies $u(x) = \widetilde{P}_T u(x)$ for all $T \in \mathcal{T}(x)$, and $u$ is uniquely determined by its boundary behavior:

$$u(x) = E_x[\lim_{t\uparrow\zeta} u(X_t)] = E_x[u_\zeta] \quad \text{for any } x \in S. \tag{3}$$

**Proposition 2.1** *Let $f \geqslant 0$ be an upper-semicontinuous function on $S$ and let $\phi \geqslant 0$ be $\mathcal{F}$-measurable such that $\phi = \phi \circ \theta_T$ $P_x$-a. s. for any $x \in S$ and any $T \in \mathcal{T}_0(x)$. Then the function $u$ on $S$ defined by the expected suprema*

$$u(x) := E_x[\sup_{0<t<\zeta} f(X_t) \vee \phi] \tag{4}$$

*is excessive, hence lower-semicontinuous. Moreover, $u$ belongs to class (D) if and only if $u$ is finite on $S$. In this case $u$ has the boundary behavior*

$$u_\zeta = \overline{\lim_{t\uparrow\zeta}}\, f(X_t) \vee \phi = f_\zeta \vee \phi \quad P_x - a.\,s., \tag{5}$$

*and $u$ admits a representation (2), i. e., a representation (4) with $\phi = u_\zeta$.*

*Proof.* It follows as in [10] that $u$ is an excessive function. If $u(x) < \infty$ then

$$\sup_{0<t<\zeta} f(X_t) \vee \phi \in \mathcal{L}^1(P_x).$$

Thus $\{u(X_T)|T \in \mathcal{T}_0(x)\}$ is uniformly integrable with respect to $P_x$, since

$$0 \leqslant u(X_T) = E_x[\sup_{T<t<\zeta} f(X_t) \vee (\phi \circ \theta_T)|\mathcal{F}_T] \leqslant E_x[\sup_{0<t<\zeta} f(X_t) \vee \phi|\mathcal{F}_T]$$

for all $T \in \mathcal{T}_0(x)$. Conversely, if $u$ belongs to class (D) then $u$ is finite on $S$ since by lower-semicontinuity

$$u(x) \leqslant E_x[\underline{\lim}_{n\uparrow\infty}\, u(X_{T_{\epsilon_n}})] \leqslant \underline{\lim}_{n\uparrow\infty}\, E_x[u(X_{T_{\epsilon_n}})] < \infty,$$

for $\epsilon_n \downarrow 0$, where $T_{\epsilon_n} \in \mathcal{T}_0(x)$ denotes the exit time from the open ball $U_{\epsilon_n}(x)$.

In order to verify (5), we take a sequence $(U_n)_{n\in\mathbb{N}}$ of relatively compact open neighborhoods of $x$ increasing to $S$ and denote by $T_n$ the exit time from $U_n$. Since $u$ is excessive and finite on $S$ we conclude that

$$\begin{aligned}
\overline{\lim_{t\uparrow\zeta}}\, f(X_t) \vee \phi &= \lim_{n\uparrow\infty} \sup_{T_n<s<\zeta} f(X_s) \vee (\phi \circ \theta_{T_n}) \\
&= \lim_{n\uparrow\infty} E_x[\sup_{T_n<s<\zeta} f(X_s) \vee (\phi \circ \theta_{T_n})|\mathcal{F}_{T_n}] \\
&= \lim_{n\uparrow\infty} u(X_{T_n}) = u_\zeta \quad P_x\text{–a. s.},
\end{aligned}$$

where the second identity follows from a martingale convergence argument.

In view of (5) we have

$$\{\phi \leqslant \sup_{0<t<\zeta} f(X_t)\} = \{u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)\} \quad P_x\text{−a. s.}$$

and $\phi = u_\zeta$ on $\{\phi > \sup_{0<t<\zeta} f(X_t)\}$ $P_x$-a. s.. Thus we can write

$$\begin{aligned} u(x) &= E_x[\sup_{0<t<\zeta} f(X_t); \phi \leqslant \sup_{0<t<\zeta} f(X_t)] + E_x[\phi; \phi > \sup_{0<t<\zeta} f(X_t)] \\ &= E_x[\sup_{0<t<\zeta} f(X_t); u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)] + E_x[u_\zeta; u_\zeta > \sup_{0<t<\zeta} f(X_t)] \\ &= E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta]. \end{aligned}$$ □

In the next section we show that, conversely, any excessive function $u$ of class (D) admits a representation of the form (2), where $f$ is some upper-semicontinuous function on $S$.

# 3 Construction of a representing function

Let $u \geqslant 0$ be an excessive function of class (D). In order to avoid additional technical difficulties, we also assume that $u$ is continuous. For convenience we introduce the notation $u^c := u \vee c$.

Consider the family of optimal stopping problems

$$Ru^c(x) := \sup_{T \in \mathcal{T}_0(x)} E_x[u^c(X_T)] \tag{6}$$

for $c \geqslant 0$ and $x \in S$. It is well known that the value function $Ru^c$ of the optimal stopping problem (6) can be characterized as the smallest excessive function dominating $u^c$. In particular, $Ru^c$ is lower-semicontinuous. Moreover,

$$Ru^c(x) \geqslant E_x[u^c(X_T); T < \zeta] + E_x[\lim_{t \uparrow \zeta} u^c(X_t); T = \zeta] = \widetilde{P}_T u^c(x) \tag{7}$$

for any stopping time $T \leqslant \zeta$, and equality holds for the first entrance time into the closed set $\{Ru^c = u^c\}$; cf. for example the proof of Lemma 4.1 in [6].

The following lemma can be verified by a straightforward modification of the arguments in [10]:

**Lemma 3.1** *1) For any $x \in S$, $Ru^c(x)$ is increasing, convex and Lipschitz-continuous in $c$, and*

$$\lim_{c \uparrow \infty} (Ru^c(x) - c) = 0. \tag{8}$$

*2) For any $c \geqslant 0$,*

$$Ru^c(x) = E_x[u^c_{D^c}] = \widetilde{P}_{D^c} u^c(x), \tag{9}$$

*where $D^c := \inf\{t \geqslant 0 \mid Ru^c(X_t) = u(X_t)\} \wedge \zeta$ is the first entrance time into the closed set $\{Ru^c = u\}$. Moreover, the map $c \mapsto D^c$ is increasing and $P_x$-a. s. left-continuous.*

Since the function $c \mapsto Ru^c(x)$ is convex, it is almost everywhere differentiable. The following identification of the derivatives is similar to Lemma 3.2 of [10].

**Lemma 3.2** *The left-hand derivative $\partial^- Ru^c(x)$ of $Ru^c(x)$ with respect to $c > 0$ is given by*

$$\partial^- Ru^c(x) = P_x[u_\zeta < c, D^c = \zeta].$$

*Proof.* For any $0 \leqslant a < c$, the representation (9) for the parameter $c$ combined with the inequality (7) for the parameter $a$ and for the stopping time $T = D^c$ implies

$$Ru^c(x) - Ru^a(x) \leqslant E_x[u^c(X_{D^c}) - u^a(X_{D^c}); D^c < \zeta] + E_x[u^c_\zeta - u^a_\zeta; D^c = \zeta].$$

Since

$$u(X_{D^c}) = Ru^c(X_{D^c}) \geqslant c > a \quad \text{on } \{D^c < \zeta\}$$

and $u^c_\zeta - u^a_\zeta \leqslant (c-a)1_{\{u_\zeta < c\}}$, the previous estimate simplifies to

$$Ru^c(x) - Ru^a(x) \leqslant (c-a)P_x[u_\zeta < c, D^c = \zeta].$$

This shows $\partial^- Ru^c(x) \leqslant P_x[u_\zeta < c, D^c = \zeta]$. In order to prove the converse inequality, we use the estimate

$$Ru^c(x) - Ru^a(x) \geqslant (c-a)P_x[u_\zeta < a, D^a = \zeta]$$

obtained by reversing the role of $a$ and $c$ in the preceding argument. This implies

$$\partial^- Ru^c(x) \geqslant \lim_{a \uparrow c} P_x[u_\zeta < a, D^a = \zeta] = P_x[u_\zeta < c, D^c = \zeta]$$

since $\bigcup_{a<c} \{D^a = \zeta\} = \{D^c = \zeta\}$ on $\{u_\zeta < c\}$, due to the Lipschitz-continuity of $Ru^c(x)$ in $c$.□

Let us now introduce the function $f^*$ defined by

$$f^*(x) := \sup\{c | x \in \{Ru^c = u\}\} \tag{10}$$

for any $x \in S$. Note that $f^*(x) \geqslant c$ is equivalent to $Ru^c(x) = u(x)$ due to the continuity of $Ru^c(x)$ in $c$. It follows as in [10], Lemma 3.3, that the function $f^*$ is upper-semicontinuous and satisfies $0 \leqslant f^* \leqslant u$.

We are now ready to derive a representation of the value functions $Ru^c$ in terms of the function $f^*$. In the special case of a potential $u$, where $u_\zeta = 0$ and $u^c_\zeta = c$ $P_x$-a. s., our representation (11) reduces to Theorem 3.1 of [10].

**Theorem 3.1** *For any $c \geqslant 0$ and any $x \in S$,*

$$Ru^c(x) = E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u^c_\zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u^c_\zeta]. \tag{11}$$

*Proof.* By Lemma 3.2 and (8) we get

$$Ru^c(x) - c = \int_c^\infty -\tfrac{\partial}{\partial \alpha}\left(Ru^\alpha(x) - \alpha\right)\, d\alpha = \int_c^\infty \left(1 - P_x[u_\zeta < \alpha, D^\alpha = \zeta]\right)\, d\alpha.$$

Since

$$\{D^{c+\epsilon} < \zeta\} \subseteq \{\sup_{0\leqslant t<\zeta} f^*(X_t) > c\} \subseteq \{D^c < \zeta\}$$

for any $c \geqslant 0$ and for any $\epsilon > 0$,

$$\begin{aligned}
Ru^c(x) - c &= \int_c^\infty (1 - P_x[u_\zeta < \alpha, D^\alpha = \zeta])\, d\alpha \\
&\geqslant \int_c^\infty (1 - P_x[u_\zeta \leqslant \alpha,\ \sup_{0\leqslant t<\zeta} f^*(X_t) \leqslant \alpha])\, d\alpha \\
&\geqslant \int_c^\infty (1 - P_x[u_\zeta < \alpha + \epsilon, D^{\alpha+\epsilon} = \zeta])\, d\alpha \\
&= Ru^{c+\epsilon}(x) - (c+\epsilon).
\end{aligned}$$

By continuity of $c \mapsto Ru^c$ we obtain

$$\begin{aligned}
Ru^c(x) - c &\geqslant \int_c^\infty (1 - P_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta \leqslant \alpha])\, d\alpha \\
&\geqslant \lim_{\epsilon\downarrow 0} (Ru^{c+\epsilon}(x) - (c+\epsilon)) = Ru^c(x) - c,
\end{aligned}$$

hence

$$\begin{aligned}
Ru^c(x) &= \int_c^\infty P_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta > \alpha]\, d\alpha + c \\
&= E_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta - (\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta) \wedge c + c] \\
&= E_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta^c].
\end{aligned}$$

Moreover, we can conclude that

$$Ru^c(x) = \lim_{t\downarrow 0} P_t(Ru^c)(x) = \lim_{t\downarrow 0} E_x[\sup_{t\leqslant s<\zeta} f^*(X_s) \vee u_\zeta^c; t < \zeta] = E_x[\sup_{0<s<\zeta} f^*(X_s) \vee u_\zeta^c]$$

since $Ru^c$ is excessive, i. e., $Ru^c(x)$ also admits the second representation in equation (11).□

As a corollary we see that $f^*$ is a representing function for $u$.

**Corollary 3.1** *The excessive function $u$ admits the representations*

$$u(x) = E_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta] \tag{12}$$

*in terms of the upper-semicontinuous function $f^* \geqslant 0$ defined by (10). Moreover,*

$$f^*(x) \leqslant \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \quad P_x - a.\,s.$$

*for any $x \in S$.*

*Proof.* Note that $u = Ru^0$ since $u$ is excessive. Applying Theorem 3.1 with $c = 0$ we obtain

$$u(x) = Ru^0(x) = E_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta].$$

In particular we get

$$\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \quad P_x\text{–a. s.},$$

and this implies $f^*(x) \leqslant \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta$ $P_x$-a. s. for any $x \in S$. □

**Remark 3.1** *Under additional regularity conditions, the underlying Markov process admits a* Martin boundary $\partial S$*, i. e., a compactification of the state space such that* $\lim_{t\uparrow\zeta} u(X_t)$ *can be identified with the values* $f(X_\zeta)$ *for a suitable continuation of the function* $f$ *to the Martin boundary; cf.,e. g., [9], (4.12) and (5.7). In such a situation the general representation (12) may be written in the condensed form (1).*

Corollary 3.1 shows that $u$ admits a representing function which is regular in the following sense:

**Definition 3.1** *Let us say that a nonnegative function* $f$ *on* $S$ *is* regular *with respect to* $u$ *if it is upper-semicontinuous and satisfies the condition*

$$f(x) \leqslant \sup_{0<t<\zeta} f(X_t) \vee u_\zeta \quad P_x - a.\, s. \tag{13}$$

*for any* $x \in S$.

Note that a regular function $f$ also satisfies the inequality

$$f(X_T) \leqslant \sup_{T<t<\zeta} f(X_t) \vee u_\zeta \quad P_x\text{–a. s. on } \{T < \zeta\} \tag{14}$$

for any stopping time $T$, due to the strong Markov property.

# 4 The minimal and the maximal representation

Let us first derive an alternative description of the representing function $f^*$ in terms of the given excessive function $u$. To this end, we introduce the superadditive operator

$$\underline{D}u(x) := \inf\{c \geqslant 0 | \exists\, T \in \mathcal{T}(x) : \widetilde{P}_T u^c(x) > u(x)\}.$$

**Proposition 4.1** *The functions* $f^*$ *and* $\underline{D}u$ *coincide. In particular,* $x \mapsto \underline{D}u(x)$ *is regular with respect to* $u$.

*Proof.* Recall that $f^*(x) \geqslant c$ is equivalent to $Ru^c(x) = u(x)$. Thus $f^*(x) \geqslant c$ yields

$$u(x) = Ru^c(x) \geqslant \widetilde{P}_T u^c(x)$$

for any $T \in \mathcal{T}(x)$ due to (7). This amounts to $\underline{D}u(x) \geqslant c$, and so we obtain $f^*(x) \leqslant \underline{D}u(x)$. In order to prove the converse inequality, we take $c > f^*(x)$ and define $T_c \in \mathcal{T}(x)$ as the first exit time from the open neighborhood $\{f^* < c\}$ of $x$. Then

$$\begin{aligned} u(x) < Ru^c(x) &= E_x[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u^c_\zeta] \\ &= E_x[\sup_{T_c\leqslant t<\zeta} f^*(X_t) \vee u_\zeta; T_c < \zeta] + E_x[u^c_\zeta; T_c = \zeta] \\ &= E_x[E_{X_{T_c}}[\sup_{0\leqslant t<\zeta} f^*(X_t) \vee u_\zeta] \vee c; T_c < \zeta] + E_x[u^c_\zeta; T_c = \zeta] \\ &= E_x[u^c(X_{T_c}); T_c < \zeta] + E_x[u^c_\zeta; T_c = \zeta] = \widetilde{P}_{T_c} u^c(x), \end{aligned}$$

hence $\underline{D}u(x) \leqslant c$. This shows $\underline{D}u(x) \leqslant f^*(x)$. □

**Remark 4.1** *A closer look at the preceding proof shows that*

$$\underline{D}u(x) = \inf\{c \geqslant 0 | \exists\, T \in \mathcal{T}(x) : u(x) - P_T u(x) < E_x[u_\zeta^c; T = \zeta]\}.$$

*For any potential $u$ of class (D) we have $u_\zeta = 0$ $P_x$-a. s., and so we get*

$$\underline{D}u(x) = \inf \frac{u(x) - P_T u(x)}{P_x[T = \zeta]},$$

*where the infimum is taken over all exit times $T$ from open neighborhoods of $x$ such that $P_x[T = \zeta] > 0$. Thus our general representation in Corollary 3.1 contains as a special case the representation of a potential of class (D) given in [10].*

We are now going to identify the maximal and the minimal representing function for the given excessive function $u$.

**Theorem 4.1** *Suppose that $u$ admits the representation*

$$u(x) = E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta]$$

*for any $x \in S$, where $f$ is regular with respect to $u$ on $S$. Then $f$ satisfies the bounds*

$$f_* \leqslant f \leqslant f^* = \underline{D}u,$$

*where the function $f_*$ is defined by*

$$f_*(x) := \inf\{c \geqslant 0 | \exists\, T \in \mathcal{T}(x) : \widetilde{P}_T u^c(x) \geqslant u(x)\}$$

*for any $x \in S$.*

*Proof.* Let us first show that $f \leqslant f^* = \underline{D}u$. If $f(x) \geqslant c$ then we get for any $T \in \mathcal{T}(x)$

$$\begin{aligned} u(x) &= E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta^c] \geqslant E_x[\sup_{T<t<\zeta} f(X_t) \vee u_\zeta^c; T < \zeta] + E_x[u_\zeta^c; T = \zeta] \\ &\geqslant E_x[E_x[\sup_{T<t<\zeta} f(X_t) \vee u_\zeta | \mathcal{F}_T] \vee c; T < \zeta] + E_x[u_\zeta^c; T = \zeta] = \widetilde{P}_T u^c(x) \end{aligned}$$

due to our assumption (13) on $f$ and Jensen's inequality. Thus $\underline{D}u(x) \geqslant c$, and this yields $f(x) \leqslant \underline{D}u(x)$. In order to verify the lower bound, take $c > f(x)$ and let $T_c \in \mathcal{T}(x)$ denote the first exit time from $\{f < c\}$. Since

$$c \leqslant f(X_{T_c}) \leqslant \sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta \quad P_x\text{-a. s. on } \{T_c < \zeta\}$$

due to property (14) of $f$, we obtain

$$\begin{aligned} \widetilde{P}_{T_c} u^c(x) &= E_x[u^c(X_{T_c}); T_c < \zeta] + E_x[u_\zeta^c; T_c = \zeta] \\ &= E_x[E_x[\sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta | \mathcal{F}_{T_c}] \vee c; T_c < \zeta] + E_x[u_\zeta^c; T_c = \zeta] \\ &= E_x[\sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta; T_c < \zeta] + E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta^c; T_c = \zeta] \\ &\geqslant E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta] = u(x), \end{aligned}$$

hence $c \geqslant f_*(x)$. This implies $f_*(x) \leqslant f(x)$. □

The following example shows that the representing function may not be unique, and that it is in general not possible to drop the limit $u_\zeta$ in the representation (2).

**Example 4.1** *Let $(X_t)_{t\geqslant 0}$ be a Brownian motion on the interval $S = (0,3)$. Then the function $u$ defined by*

$$u(x) = \begin{cases} \quad x \quad , & x \in (0,1) \\ \frac{1}{2}x + \frac{1}{2}, & x \in [1,2] \\ \frac{1}{4}x + 1, & x \in (2,3) \end{cases}$$

*is concave on $S$, hence excessive. Here the maximal representing function $f^*$ takes the form*

$$f^*(x) = \tfrac{1}{2}1_{[1,2)}(x) + 1_{[2,3)}(x),$$

*and $f_*$ is given by $f_*(x) = \frac{1}{2}1_{\{1\}}(x) + 1_{\{2\}}(x)$. In particular we get for any $x \in (2,3)$*

$$u(x) > E_x[\sup_{0<t<\zeta} f^*(X_t)].$$

*This shows that we have to include $u_\zeta$ into the representation of $u$. Moreover, for any $x \in S$*

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \geqslant f^*(x) \geqslant f_*(x) \quad P_x - a.\,s.,$$

*and so $f_*$ is a regular representing function for $u$. In particular, the representing function is not unique.*

We are now going to derive an alternative description of $f_*$ which will allow us to identify $f_*$ as the minimal representing function for $u$.

**Definition 4.1** *Let us say that a point $x_0 \in S$ is* harmonic *for $u$ if the mean-value property*

$$u(x_0) = E_{x_0}[u(X_{T_\epsilon})]$$

*holds for $x_0$ and for some $\epsilon > 0$, where $T_\epsilon$ denotes the first exit time from the ball $U_\epsilon(x_0)$. We denote by $H$ the set of all points in $S$ which are harmonic with respect to $u$.*

Under the regularity assumptions of [10], the set $H$ coincides with the set of all points $x_0 \in S$ such that $u$ is harmonic in some open neighborhood $G$ of $x_0$, i. e., the mean-value property

$$u(x) = E_x[u(X_{T_{U_\epsilon(x)}})]$$

holds for all $x \in G$ and all $\epsilon > 0$ such that $\overline{U_\epsilon(x)} \subset G$; cf. Lemma 4.1 in [10]. In particular, $H$ is an open set.

The following proposition extends Proposition 4.1 in [10] from potentials to general excessive functions.

**Proposition 4.2** *For any $x \in S$,*

$$f_*(x) = f^*(x) 1_{H^c}(x). \tag{15}$$

*In particular, $f_*$ is upper-semicontinuous.*

*Proof.* For $x \in H$ there exists $\epsilon > 0$ such that $\overline{U_\epsilon(x)} \subset S$ and $u(x) = E_x[u(X_{T_{U_\epsilon(x)}})] = \widetilde{P}_{T_{U_\epsilon(x)}} u^0(x)$, and this implies $f_*(x) = 0$. Now suppose that $x \in H^c$, i. e., $u$ is not harmonic in $x$. Let us first prove that

$$\widetilde{P}_T u(x) < u(x) \quad \text{for all } T \in \mathcal{T}(x). \tag{16}$$

Indeed, if $T$ is the first exit time from some open neighborhood $G$ of $x$ then

$$\begin{aligned} \widetilde{P}_T u(x) &= E_x[E_{X_{T_{U_\epsilon(x)}}}[u(X_T); T < \zeta] + E_{X_{T_{U_\epsilon(x)}}}[u_\zeta; T = \zeta]] \\ &\leqslant E_x[Ru^0(X_{T_{U_\epsilon(x)}})] = E_x[u(X_{T_{U_\epsilon(x)}})] < u(x) \end{aligned}$$

for any $\epsilon > 0$ such that $\overline{U_\epsilon(x)} \subseteq G$. In view of Theorem 4.1 we have to show $f_*(x) \geqslant f^*(x)$, and we may assume $f^*(x) > 0$. Choose $c > 0$ such that $f^*(x) > c$. Then there exists $\epsilon > 0$ such that $Ru^{c+\epsilon}(x) = u(x)$, i. e.,

$$\widetilde{P}_T u^{c+\epsilon}(x) \leqslant u(x) \tag{17}$$

for any $T \in \mathcal{T}(x)$ in view of (7). Fix $\delta \in (0, \epsilon)$ and $T \in \mathcal{T}(x)$. If

$$P_x[u(X_T) \leqslant c + \delta; T < \zeta] + P_x[u_\zeta \leqslant c + \delta; T = \zeta] > 0$$

we get the estimate

$$\widetilde{P}_T u^{c+\delta}(x) = E_x[u^{c+\delta}(X_T); T < \zeta] + E_x[u_\zeta^{c+\delta}; T = \zeta] < \widetilde{P}_T u^{c+\epsilon}(x) \leqslant u(x).$$

On the other hand, if $P_x[u(X_T) \leqslant c + \delta; T < \zeta] = P_x[u_\zeta \leqslant c + \delta; T = \zeta] = 0$ then

$$\widetilde{P}_T u^{c+\delta}(x) = E_x[u(X_T); T < \zeta] + E_x[u_\zeta; T = \zeta] = \widetilde{P}_T u(x) < u(x)$$

due to (16). Thus we obtain $u(x) > \widetilde{P}_T u^{c+\delta}(x)$ for any $T \in \mathcal{T}(x)$, hence $f_*(x) \geqslant c + \delta$. This concludes the proof of (15). Upper-semicontinuity of $f_*$ follows immediately since $f^*$ is upper-semicontinuous and $H^c$ is closed. □

Our next purpose is to show that $f^*$ is constant on connected components of $H$.

**Proposition 4.3** *For any $x \in H$,*

$$f^*(x) = \operatorname*{ess\,inf}_{P_x} f_T^*, \tag{18}$$

*where $T$ denotes the first exit time from the maximal connected neighborhood $H(x) \subseteq H$ of $x$. In particular, $f^*$ is constant on $H(x)$.*

*Proof.* 1) Let us first show that for a connected open set $U \subset S$ and for any $x, y \in U$, the measures $P_x$ and $P_y$ are equivalent on the $\sigma$-field describing the exit behavior from $U$:

$$P_x \approx P_y \quad \text{on } \widehat{\mathcal{F}}_U := \sigma(\{g_{T_U} | g \text{ measurable on } S\}). \tag{19}$$

Indeed, any $A \in \widehat{\mathcal{F}}_U$ satisfies $1_A \circ \theta_{T_\epsilon} = 1_A$ if $T_\epsilon$ denotes the exit time from some neighborhood $U_\epsilon(x)$ such that $\overline{U_\epsilon(x)} \subset U$. Thus

$$P_x[A] = E_x[1_A \circ \theta_{T_\epsilon}] = \int P_z[A]\, \mu_{x,\epsilon}(dz),$$

where $\mu_{x,\epsilon}$ is the exit distribution from $U_\epsilon(x)$. Since $\mu_{x,\epsilon} \approx \mu_{y,\epsilon}$ by assumption **A3)** of [10], we obtain $P_x \approx P_y$ on $\widehat{\mathcal{F}}_U$ for any $y \in U_\epsilon(x)$. For arbitrary $y \in U$ we can choose $x_0, \dots, x_n$ and $\epsilon_1, \dots, \epsilon_n$ such that $x_0 = x$, $x_n = y$, $x_k \in U_{\epsilon_k}(x_{k-1})$ and $\overline{U_{\epsilon_k}(x_{k-1})} \subset U$. Hence $P_{x_k} \approx P_{x_{k-1}}$ on $\widehat{\mathcal{F}}_U$, and this yields (19).

2) For $x \in H$ let $c(x)$ be the right-hand side of equation (18). In order to verify $f^*(x) \leqslant c(x)$, we take a sequence of relatively compact open neighborhoods $(U_n(x))_{n \in I\!N}$ of $x$ increasing to $H(x)$ and denote by $T_n$ the first exit time from $U_n(x)$. Since $f^*$ is upper-semicontinuous on $S$, we get the estimate

$$\overline{\lim_{n \uparrow \infty}} f^*(X_{T_n}) \leqslant f^*(X_T) 1_{\{T<\zeta\}} + \overline{\lim_{t \uparrow \zeta}} f^*(X_t) 1_{\{T=\zeta\}} = f^*_T \quad P_x\text{–a.s.},$$

hence $P_x[\overline{\lim}_{n\uparrow\infty} f^*(X_{T_n}) < c] > 0$ for any $c > c(x)$. Thus, there exists $n_0$ such that $P_x[Ru^c(X_{T_{n_0}}) > u(X_{T_{n_0}})] = P_x[f^*(X_{T_{n_0}}) < c] > 0$, and this implies

$$u(x) = E_x[u(X_{T_{n_0}})] < E_x[Ru^c(X_{T_{n_0}})] \leqslant Ru^c(x)$$

since $Ru^c$ is excessive. But this amounts to $f^*(x) < c$, and taking the limit $c \searrow c(x)$ yields $f^*(x) \leqslant c(x)$.

3) In order to prove the converse inequality, we use the fact that for any $c < c(x)$

$$E_x[u^c(X_{\widetilde{T}})] \leqslant u(x) \quad \text{for all } \widetilde{T} \in \mathcal{T}_0(x), \tag{20}$$

which is equivalent to $Ru^c(x) = u(x)$. Thus we get $f^*(x) \geqslant c$ for all $c < c(x)$, hence $f^*(x) = c(x)$ in view of 2). Since $c(x) = c(y)$ for any $y \in H(x)$ due to (19), we see that $f^*$ is constant on $H(x)$.

It remains to verify (20). To this end, note that for any $y \in H(x)$ we have $c < c(x) = c(y) \leqslant f^*_T \quad P_y$–a. s. due to (19). Thus, $f^*(X_T) > c$ $P_y$-a. s. on $\{T < \zeta\}$ for any $y \in H(x)$, and this yields

$$u^c(X_T) \leqslant Ru^c(X_T) = u(X_T) \quad P_y\text{–a. s. on } \{T < \zeta\}.$$

Moreover, we get $c < f^*_\zeta \leqslant u_\zeta$ $P_y$-a. s. on $\{T = \zeta\}$. Let us now fix $\widetilde{T} \in \mathcal{T}_0(x)$. Since $X_{\widetilde{T}} \in H(x)$ on $\{\widetilde{T} < T\}$, we can conclude that

$$\begin{aligned} E_x[u^c(X_{\widetilde{T}}); \widetilde{T} < T] &= E_x[\widetilde{P}_T u(X_{\widetilde{T}}) \vee c; \widetilde{T} < T] \\ &\leqslant E_x[E_{X_{\widetilde{T}}}[u^c(X_T); T < \zeta] + E_{X_{\widetilde{T}}}[u^c_\zeta; T = \zeta]; \widetilde{T} < T] \\ &= E_x[E_{X_{\widetilde{T}}}[u(X_T); T < \zeta] + E_{X_{\widetilde{T}}}[u_\zeta; T = \zeta]; \widetilde{T} < T] \\ &= E_x[u_T; \widetilde{T} < T]. \end{aligned} \tag{21}$$

On the other hand, we have $\{T \leqslant \widetilde{T}\} \subseteq \{T < \zeta\}$, and by the $P_x$-supermartingale property of $(Ru^c(X_t) 1_{\{t<\zeta\}})_{t \geqslant 0}$ we get the estimate

$$\begin{aligned} E_x[u^c(X_{\widetilde{T}}); \widetilde{T} \geqslant T] &\leqslant E_x[Ru^c(X_{\widetilde{T}}); \widetilde{T} \geqslant T] \leqslant E_x[Ru^c(X_T); \widetilde{T} \geqslant T] \\ &= E_x[u(X_T); \widetilde{T} \geqslant T] = E_x[u_T; \widetilde{T} \geqslant T], \end{aligned}$$

where the first equality follows from $f^*(X_T) \geqslant c(x) > c$ $P_x$-a.s. on $\{T < \zeta\}$. In combination with (21) this yields

$$E_x[u^c(X_{\widetilde{T}})] \leqslant E_x[u_T] = u(x). \qquad \square$$

**Remark 4.2** *A point $x \in S$ is harmonic with respect to $u$ if and only if there exists $\epsilon > 0$ such that $f^*$ is constant on $U_\epsilon(x) \subset S$. Indeed, Proposition 4.3 shows that this condition is necessary. Conversely, take $x \in H^c$ and assume that there exists $\epsilon > 0$ such that $f^*$ is constant on $U_{2\epsilon}(x) \subset S$. Then the exit time $T := T_{U_\epsilon(x)}$ satisfies*

$$\widetilde{P}_T u(x) = E_x[u(X_T)] = E_x[\sup_{T<t<\zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta] = u(x)$$

*in contradiction to (16).*

Our next goal is to show that $f_*$ is the minimal representing function for $u$.

**Theorem 4.2** *Let $f$ be an upper-semicontinuous function on $S$ such that $f_* \leqslant f \leqslant f^*$. Then $f$ is a regular representing function for $u$. In particular we obtain the representation*

$$u(x) = E_x[\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta],$$

*and $f_*$ is the minimal regular function yielding a representation of $u$.*

*Proof.* Let us show that

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \quad P_x\text{-a.s.} \tag{22}$$

for any $x \in S$. To this end, suppose first that $x \in H$. We denote by $T_c$ the exit time from the open set $\{f^* < c\}$. Since $0 \leqslant f_* \leqslant f \leqslant f^*$, it is enough to show that for fixed $c \geqslant f^*(x)$

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta \geqslant c \quad P_x\text{-a.s. on } \{T_c < \zeta\}. \tag{23}$$

By (15) we see that

$$\sup_{0<t<\zeta} f_*(X_t) \geqslant f_*(X_{T_c}) = f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } \{T_c < \zeta, X_{T_c} \in H^c\}.$$

On the set $A := \{T_c < \zeta, X_{T_c} \in H\}$ we use the inequality

$$f^*(X_{T_c}) \leqslant f_T^* \quad P_x\text{-a.s. on } A \tag{24}$$

for $T := T_c + T_H \circ \theta_{T_c}$ which follows from Proposition 4.3 combined with the strong Markov property. Using (15) and (24) we obtain

$$\sup_{0<t<\zeta} f_*(X_t) \geqslant f_*(X_T) = f^*(X_T) \geqslant f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } A \cap \{T < \zeta\}$$

and

$$u_\zeta \geqslant f_\zeta^* \geqslant f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } A \cap \{T = \zeta\},$$

hence $\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta \geqslant c$ $P_x$-a. s. on $A$. This concludes the proof of (23) for $x \in H$, and so (22) holds for any $x \in H$. In particular, we have

$$\sup_{\widetilde{T}<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{\widetilde{T}<t<\zeta} f^*(X_t) \vee u_\zeta \quad P_x\text{-a. s. on } \{\widetilde{T} < \zeta, X_{\widetilde{T}} \in H\} \tag{25}$$

for any stopping time $\widetilde{T}$, due to the strong Markov property.

Let us now fix $x \in H^c$ and denote by $\widehat{T}$ the first exit time from $H^c$. Since the functions $f_*$ and $f^*$ coincide on $H^c$ due to Proposition 4.2, the identity (22) follows immediately on the set $\{\widehat{T} = \zeta\}$. On the other hand, using again Proposition 4.2, we get

$$\begin{aligned} \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta &= \sup_{0<t\leqslant\widehat{T}} f^*(X_t) \vee \sup_{\widehat{T}<t<\zeta} f^*(X_t) \vee u_\zeta \\ &= \sup_{0<t\leqslant\widehat{T}} f_*(X_t) \vee \sup_{\widehat{T}<t<\zeta} f^*(X_t) \vee u_\zeta \quad \text{on } \{\widehat{T} < \zeta\}. \end{aligned} \tag{26}$$

By definition of $\widehat{T}$, on $\{\widehat{T} < \zeta\}$ there exists a sequence of stopping times $\widehat{T} < T_n < \zeta$, $n \in \mathbb{N}$, decreasing to $\widehat{T}$ such that $X_{T_n} \in H$. Thus,

$$\begin{aligned} \sup_{\widehat{T}<t<\zeta} f^*(X_t) \vee u_\zeta &= \lim_{n\uparrow\infty} \sup_{T_n<t<\zeta} f^*(X_t) \vee u_\zeta \\ &= \lim_{n\uparrow\infty} \sup_{T_n<t<\zeta} f_*(X_t) \vee u_\zeta \\ &= \sup_{\widehat{T}<t<\zeta} f_*(X_t) \vee u_\zeta \quad P_x\text{-a. s. on } \{\widehat{T} < \zeta\} \end{aligned}$$

due to (25). Combined with (26) this yields (22) on $\{\widehat{T} < \zeta\}$. Thus we have shown that (22) holds as well for any $x \in H^c$.

In particular, $f$ is a representing function for $u$. Moreover,

$$f(x) \leqslant f^*(x) \leqslant \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta \quad P_x\text{-a. s.}$$

for any $x \in S$ due to (22), and so $f$ is a regular function on $S$ with respect to $u$. In view of Theorem 4.1 we see that $f_*$ is the minimal regular representing function for $u$. □

**Remark 4.3** *Suppose that $u$ admits a representation of the form*

$$u(x) = E_x[\sup_{0<t<\zeta} f(X_t)] \tag{27}$$

*for all $x \in S$ and for some regular function $f$ on $S$. Then $f$ satisfies the bounds $f_* \leqslant f \leqslant f^*$, due to Theorem 4.1 combined with proposition 2.1 for $\phi = 0$. Clearly such a reduced representation, which does not involve explicitly the boundary behavior of $u$, holds if and only if $u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)$ $P_x$-a. s.. In particular, this is the case for a potential $u$ where $u_\zeta = 0$, in accordance with the results in [10]. Example 4.1 shows that a reduced representation (27) is not possible in general. If $u$ is harmonic on $S$, (27) would in fact imply that $u$ is constant on $S$. Indeed, harmonicity of $u$ on $S$ implies that $f^* = c$ on $S$ for some constant $c$ due to Proposition 4.3, hence*

$$E_x[\sup_{0<t<\zeta} f(X_t)] \leqslant c \leqslant E_x[u_\zeta] = u(x)$$

*due to $f \leqslant f^* \leqslant u$ and (3), and so (27) would imply $u(x) = c$ for all $x \in S$.*

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