Convex risk measures and the dynamics of their penalty functions

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Abstract

We study various properties of a dynamic convex risk measure for bounded random variables which describe the discounted terminal values of financial positions. In particular we characterize time-consistency by a joint supermartingale property of the risk measure and its penalty function. Moreover we discuss the limit behavior of the risk measure in terms of asymptotic safety and of asymptotic precision, a property which may be viewed as a non-linear analogue of martingale convergence. These results are illustrated by the entropic dynamic risk measure.

Key words: Dynamic convex risk measures, conditional risk measures, robust representation, dynamic penalty functions, time-consistency, asymptotic safety, asymptotic precision, entropic risk measure.

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1 Introduction

Starting with the introduction of coherent risk measures in Artzner et al. [1], the problem of quantifying the risk associated to a financial position given the available information has emerged as a key topic in Mathematical Finance. The theory of coherent and, more generally, of convex risk measures is now well developed; see, e.g., Delbaen [7] and [8] for the coherent case and Föllmer and Schied [13], Frittelli and Rosazza Gianin [14] for the convex case.

Financial positions are usually described as random variables X on some probability space. A convex risk measure ρ is then defined as a real-valued convex functional on a suitable space of such positions. Under some regularity conditions, the duality theory of the Fenchel-Legendre yields a robust representation of the form

$$\rho(X) = \sup_{Q} \big(E_Q[-X] - \alpha(Q)\big).$$

Thus the risk of a position is evaluated as the worst expected loss, suitably modified, under a whole class of probabilistic models. These alternative models are described by probability

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measures Q on the underlying set of scenarios. But they are taken seriously at a different degree, and this is made precise by the non-negative penalty function $\alpha(Q)$.

In this formulation, however, the role of information is not yet visible. Suppose that the information available at time t is described by a σ-field $\mathcal{F}_t$. The updated risk assessment at time t is then described by a conditional risk measure ρ_t which associates to each position X an $\mathcal{F}_t$-measurable random variable $\rho_t(X)$. In this conditional setting, the expectations appearing in the robust representation of a convex risk measure are replaced by conditional expectations, the penalty function $\alpha(Q)$ becomes an $\mathcal{F}_t$-measurable random variable $\alpha_t(Q)$, and the supremum is understood as an essential supremum with respect to the reference measure P. Such representations for conditional risk measures were discussed in Riedel [18], Arztner et al. [2], Detlefsen [10], Detlefsen and Scandolo [11], Frittelli and Rosazza Gianin [15], Bion-Nadal [3], [4], Cheridito et al. [6], Burgert [5], and Klöppel and Schweizer [16].

In this paper we study a dynamic risk measure, given by a sequence $(\rho_t)_{t=0,1,\ldots}$ of conditional convex risk measures adapted to some filtration $(\mathcal{F}_t)_{t=0,1,\ldots}$ on the underlying probability space. In sections 2 and 3 we review and refine the robust representation of conditional convex risk measures. These two sections are mostly expository, but we include the proofs in order to introduce some technical modifications and to give a self-contained presentation.

A key question in the dynamical setting is how the conditional risk assessments at different times are related among each other. Several notions of time-consistency have been discussed in the literature; see [2], [11], [4], [6], [5] and references therein, and also Tutsch [19]. In section 4 we focus on the strong form of time-consistency which amounts to the recursion $\rho_t(-\rho_{t+1}) = \rho_t$. Our aim is to review and to clarify the corresponding properties of the process of penalty functions. In particular we show that time-consistency is equivalent to a combined supermartingale property of the risk measure and its penalty function under any reasonable model Q, in analogy to results of Föllmer and Kramkov [12] on the optional decomposition under convex constraints; see also Chapter 9 in Föllmer and Schied [13]. This extends results of [2] from the coherent to the convex case.

In section 5 we study the asymptotic behavior of a time-consistent dynamic risk measure. As shown by example 5.5, not every time-consistent sequence $(\rho_t)_{t=0,1,\ldots}$ is asymptotically safe in the sense that the limiting capital requirement $\rho_\infty(X)$ covers the actual final loss $-X$. Theorem 5.4 gives criteria for asymptotic safety in terms of the asymptotic behavior of acceptance sets and penalty functions. We also discuss the case where $\rho_\infty(X)$ is exactly equal to $-X$. This property of asymptotic precision may be viewed as a non-linear analogue of martingale convergence, and Proposition 5.11 provides a sufficient condition in terms of the initial risk measure ρ_0.

In the final section 6 we illustrate the general results of sections 4 and 5 by describing the corresponding properties of the entropic dynamic risk measure.

2 Conditional convex risk measures and their robust representation

We consider a discrete-time multiperiod information structure given by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, P)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{F}_T$, where the time horizon T might be finite or infinite. The set of all financial positions will be $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$. By L^∞_t we denote the set of all $\mathcal{F}_t$-measurable P-a.s. bounded random variables. All inequalities and equalities applied to random variables are meant to hold P-a.s. .

We define a conditional convex risk measure as in [11]:

Definition 2.1. *A map $\rho_t : L^\infty \to L^\infty_t$ is called a* conditional convex risk measure *if it satisfies the following properties for all $X, Y \in L^\infty$:*

- *Conditional cash invariance: For all $X_t \in L^\infty_t$ $\rho_t(X + X_t) = \rho_t(X) - X_t$*
- *Monotonicity: $X \leq Y \;\Rightarrow\; \rho_t(X) \geq \rho_t(Y)$*
- *Conditional convexity: For all $\lambda \in L^\infty_t$, $0 \leq \lambda \leq 1$:*

$$\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda \rho_t(X) + (1-\lambda)\rho_t(Y)$$

- *Normalization: $\rho_t(0) = 0$.*

A conditional convex risk measure is called a conditional coherent risk measure *if it has in addition the following property:*

- *Conditional positive homogeneity: For all $\lambda \in L^\infty_t$, $\lambda \geq 0$:*

$$\rho_t(\lambda X) = \lambda \rho_t(X).$$

Remark 2.2.
1. *For $t = 0$ we have $L^\infty_t = \mathbb{R}$, and so we recover the usual definition of a convex risk measure; cf. [13]*
2. *In [11] a conditional convex risk measure is called* regular, *if $\rho_t(I_A X) = I_A \rho_t(X)$ for all $A \in \mathcal{F}_t$ and $X \in L^\infty$. It was shown in [11] Corollary 1 that any normalized conditional convex risk measure is regular. In [6] a* local property *of a conditional convex risk measure is defined as $\rho_t(I_A X + I_{A^c} Y) = I_A \rho_t(X) + I_{A^c} \rho_t(Y)$ for all $A \in \mathcal{F}_t$ and all $X, Y \in L^\infty$. Proposition 3.3 of [6] shows that monotonicity and cash invariance imply this local property. For a normalized conditional convex risk measure regularity and the local property are equivalent, as shown in Proposition 1 in [11].*
3. *A weaker definition of a conditional convex risk measure is given in [16], where normalization is not required and conditional convexity is replaced by regularity and by convexity only for constant coefficients.*
4. *If ρ_t is a convex conditional risk measure, then $-\rho_t$ defines a monetary concave utility functional on L^∞ in the sense of [6], [16].*
5. *In the dynamic setting it is also possible to define risk measures for payoff streams, i.e. for stochastic processes instead of random variables, as it is done in [18], [6]. The results obtained in this more general setting clearly apply to our present situation.*

With a conditional convex risk measure ρ_t we associate its *acceptance set*

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; \rho_t(X) \leq 0 \right\}.$$

$\mathcal{A}_t$ is conditionally convex, solid and such that $\operatorname{ess\,inf}\left\{ X \in L_t^\infty \,\middle|\, X \in \mathcal{A}_t \right\} = 0$ and $0 \in \mathcal{A}_t$, as shown in Proposition 3 in [11]. Moreover, ρ_t is uniquely determined through its acceptance set, since

$$\rho_t(X) = \operatorname{ess\,inf}\left\{ Y \in L_t^\infty \,\middle|\, X + Y \in \mathcal{A}_t \right\}. \tag{1}$$

A conditional convex risk measure can thus be viewed as a conditional capital requirement needed to make a financial position acceptable at time t.

Conversely, one can use acceptance sets to define conditional convex risk measures: If $\mathcal{A}_t \subseteq L_t^\infty$ satisfies the conditions above, then the functional $\rho_t : L^\infty \to L_t^\infty$ defined via (1) is a conditional convex risk measure; cf. Proposition 3 in [11]. A characterization of acceptance sets in a more general setting can be found in Proposition 3.6 of [6].

By $\mathcal{M}_1(P)$ we denote the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P, and by $\mathcal{M}^e(P)$ the set of all probability measures on $(\Omega, \mathcal{F})$, which are equivalent to P on $\mathcal{F}$. It is well known that an unconditional convex risk measure which is continuous from above is of the form

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \left(E_Q[X] - \alpha(Q) \right)$$

with some penalty function $\alpha : \mathcal{M}_1(P) \to \mathbb{R} \cup \{+\infty\}$, see [13] for details.

Analogous representations for conditional convex risk measures were obtained in [10], [5], [11], [16] and in [6]. In the rest of this section we will state and prove a robust representation result from [11], introducing some technical modifications which we well need later on. Let us define the sets

$$\mathcal{P}_t := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q \approx P \text{ on } \mathcal{F}_t \right\}$$

and

$$\mathcal{Q}_t := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q = P \text{ on } \mathcal{F}_t \right\}.$$

The *penalty function* will be given by a map α_t from some set $\mathcal{P} \subseteq \mathcal{P}_t$ to the set of $\mathcal{F}_t$-measurable random variable with values in $\mathbb{R} \cup \{+\infty\}$ such that

$$\operatorname*{ess\,sup}_{Q \in \mathcal{P}} (-\alpha_t(Q)) = 0.$$

In our setting the typical form of a penalty function will be

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X \,|\, \mathcal{F}_t]. \tag{2}$$

Note that this penalty function is well defined for $Q \in \mathcal{P}_t$. We will say that ρ_t has a robust representation if

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}} (E_Q[-X \,|\, \mathcal{F}_t] - \alpha_t(Q)) \qquad \text{for all } X \in L^\infty$$

with some set $\mathcal{P} \subseteq \mathcal{P}_t$ and some penalty function α_t on $\mathcal{P}$.

The next theorem relates robust representations to some continuity properties of conditional convex risk measures. It is a version of Theorem 1 in [11] and Theorem 2.27 in [10], cf. also Theorem 3 in [3], Theorem 3.6 in [5], Theorem 3.16 in [16] and Theorem 3.16 in [6].

Theorem 2.3. *For a conditional convex risk measure ρ_t the following are equivalent:*

1. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}(E_Q[-X\,|\,\mathcal{F}_t\,] - \alpha_t^{\min}(Q)), \qquad X \in L^\infty, \tag{3}$$

where the penalty function $\alpha_t^{\min}$ is given by (2).

2. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{P}_t}(E_Q[-X\,|\,\mathcal{F}_t\,] - \alpha_t^{\min}(Q)), \qquad X \in L^\infty, \tag{4}$$

where the penalty function $\alpha_t^{\min}$ is given by (2).

3. *ρ_t has a robust representation.*

4. *ρ_t has the "Fatou-property": For any bounded sequence (X_n) which converges P-a.s. to some X,*

$$\rho_t(X) \le \liminf_{n\to\infty} \rho_t(X_n) \quad P\text{-a.s.}.$$

5. *ρ_t is continuous from above, i.e.*

$$X_n \searrow X \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X_n) \nearrow \rho_t(X) \;\; P\text{-a.s.}$$

for any sequence $(X_n) \subseteq L^\infty$ and $X \in L^\infty$.

Proof. 2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 4) Dominated convergence implies that $E_Q[X_n|\mathcal{F}_t] \to E_Q[X|\mathcal{F}_t]$ for each $Q \in \mathcal{P}_t$, and $\liminf \rho_t(X_n) \ge \rho_t(X)$ follows by using a robust representation of ρ_t as in the unconditional setting, see, e.g., Lemma 4.20 in [13].

4) $\Rightarrow$ 5) Monotonicity implies $\limsup \rho_t(X_n) \le \rho_t(X)$, and $\liminf \rho_t(X_n) \ge \rho_t(X)$ follows by 4).

5) $\Rightarrow$ 1) The inequality

$$\rho_t(X) \ge \operatorname*{ess\,sup}_{Q\in\mathcal{P}_t}(E_Q[-X\,|\,\mathcal{F}_t\,] - \alpha_t^{\min}(Q)) \ge \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}(E_Q[-X\,|\,\mathcal{F}_t\,] - \alpha_t^{\min}(Q)) \tag{5}$$

follows immediately from the definition of $\alpha_t^{\min}$ and $\mathcal{Q}_t \subseteq \mathcal{P}_t$.

In order to prove the equality we will show that

$$E_P[\rho(X)] \le E_P[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}(E_Q[-X\,|\,\mathcal{F}_t\,] - \alpha_t^{\min}(Q))].$$

To this end, consider the map $\rho^P : L^\infty \to \mathbb{R}$ defined by $\rho^P(X) := E_P[\rho_t(X)],\; X \in L^\infty$. It is easy to check that ρ^P is a convex risk measure which is continuous from above. Hence Theorem 4.31 in [13] implies that ρ^P has the robust representation

$$\rho^P(X) = \sup_{Q\in\mathcal{M}_1(P)} (E_Q[-X] - \alpha(Q)) \qquad X \in L^\infty,$$

where the penalty function $\alpha(Q)$ is given by

$$\alpha(Q) = \sup_{X\in L^\infty, \rho^P(X)\le 0} E_Q[-X].$$

Next we will prove that $Q \in \mathcal{Q}_t$ if $\alpha(Q) < \infty$. Indeed, let $A \in \mathcal{F}_t$ and $\lambda > 0$. Then

$$-\lambda P[A] = E_P[\rho_t(\lambda I_A)] = \rho^P(\lambda I_A) \geq E_Q[-\lambda I_A] - \alpha(Q),$$

so

$$P[A] \leq Q[A] + \frac{1}{\lambda}\alpha(Q) \quad \text{for all} \quad \lambda > 0,$$

and hence $P[A] \leq Q[A]$ if $\alpha(Q) < \infty$. The same reasoning with $\lambda < 0$ implies $P[A] \geq Q[A]$, thus $P = Q$ on $\mathcal{F}_t$ if $\alpha(Q) < \infty$. Moreover,

$$E_P[\alpha_t^{\min}(Q)] \leq \alpha(Q) \tag{6}$$

holds for every $Q \in \mathcal{P}_t$, which can be seen as follows. As we will prove in Lemma 2.6 below,

$$E_P[\alpha_t^{\min}(Q)] = \sup_{Y \in \mathcal{A}_t} E_P[-Y].$$

Since $\rho^P(Y) \leq 0$ for all $Y \in \mathcal{A}_t$, inequality (6) follows from the definition of the penalty function $\alpha(Q)$.

Finally we obtain

$$\begin{aligned}
E_P[\rho_t(X)] = \rho^P(X) &= \sup_{Q \in \mathcal{M}_1(P), \alpha(Q) < \infty} (E_Q[-X] - \alpha(Q)) \\
&\leq \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} (E_Q[-X] - \alpha(Q)) \\
&\leq \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} E_P[E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)] \\
&\leq E_P\left[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} \big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\big)\right] \qquad (7) \\
&\leq E_P\left[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right],
\end{aligned}$$

proving equality (3).

1) $\Rightarrow$ 2) Follows immediately from inequality (5). $\square$

A closer look at the proof of Theorem 2.3 yields the following corollary, which will be useful later on.

Corollary 2.4. *A conditional convex risk measure ρ_t is continuous from above if and only if for any $P^* \in \mathcal{M}^e(P)$ it is representable in the form*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f(P^*)} (E_Q[-X \,|\, \mathcal{F}_t] - \alpha_t^{\min}(Q)), \qquad X \in L^\infty, \tag{8}$$

where

$$\mathcal{Q}_t^f(P^*) := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q = P^* \text{ on } \mathcal{F}_t,\ E_{P^*}[\alpha_t^{\min}(Q)] < \infty \right\}.$$

Proof. The inequality

$$\rho_t(X) \geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f(P)} (E_Q[-X \,|\, \mathcal{F}_t] - \alpha_t^{\min}(Q))$$

follows from (5) since $\mathcal{Q}_t^f(P) \subseteq \mathcal{P}_t$, and (7) proves the equality for $\mathcal{Q}_t^f(P)$. Moreover, since the definition of a conditional convex risk measure and the continuity property only depend on the zero sets of P, the same reasoning works for any $P^* \in \mathcal{M}^e(P)$. $\square$

In the *coherent* case we obtain the following representation result:

Corollary 2.5. *A conditional coherent risk measure ρ_t is continuous from above if and only if for any $P^* \in \mathcal{M}^e(P)$ it is representable in the form*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^0(P^*)} E_Q[-X \,|\, \mathcal{F}_t\,], \qquad X \in L^\infty, \tag{9}$$

where

$$\mathcal{Q}_t^0(P^*) := \left\{\, Q \in \mathcal{M}_1(P) \;\middle|\; Q = P^* \text{ on } \mathcal{F}_t,\ \alpha_t^{\min}(Q) = 0 \ Q\text{-a.s.} \right\}.$$

Proof. Due to positive homogeneity of ρ_t the penalty function $\alpha_t^{\min}(Q)$ can only take values 0 or ∞ for all $Q \in \mathcal{P}_t$. Indeed, for $A := \{\alpha_t^{\min}(Q) > 0\}, X \in \mathcal{A}_t$ and all $\lambda > 0$ we have $\lambda I_A X \in \mathcal{A}_t$, and hence

$$\begin{aligned} \alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \\ &\geq \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-\lambda I_A X|\mathcal{F}_t] \\ &= \lambda I_A \alpha_t^{\min}(Q), \end{aligned}$$

where the lower bound converges to ∞ with $\lambda \to \infty$ on A. Thus $\alpha_t^{\min}(Q) = \infty$ on A and $\alpha_t^{\min}(Q) = 0$ on A^c. If $Q \in Q_t^f(P^*)$ for some $P^* \approx P$, the inequality $E_{P^*}[\alpha_t^{\min}(Q)] < \infty$ implies $P[A] = 0$, hence $Q \in \mathcal{Q}_t^0(P^*)$. Thus (8) is equivalent to (9). $\square$

The following lemma was used in the proof of the Theorem 2.3. Similar arguments are used in the proofs of Theorem 2.27 in [10], Theorem 1 in [11], Theorem 3.5 in [5], Theorem 3.16 in [16], and Theorem 3.16 in [6].

Lemma 2.6. *For $Q \in \mathcal{P}_t$ and $0 \leq s \leq t$,*

$$E_Q[\alpha_t^{\min}(Q)|\mathcal{F}_s] = \operatorname*{ess\,sup}_{Y \in \mathcal{A}_t} E_Q[-Y|\mathcal{F}_s],$$

and in particular

$$E_Q[\alpha_t^{\min}(Q)] = \sup_{Y \in \mathcal{A}_t} E_Q[-Y].$$

Proof. First we claim that the set

$$\left\{\, E_Q[-X|\mathcal{F}_t] \;\middle|\; X \in \mathcal{A}_t \right\}$$

is directed upward for any $Q \in \mathcal{P}_t$. Indeed, for $X, Y \in \mathcal{A}_t$ we can define $Z := X I_A + Y I_{A^c}$, where $A := \{E_Q[-X|\mathcal{F}_t] \geq E_Q[-Y|\mathcal{F}_t]\} \in \mathcal{F}_t$. Conditional convexity of ρ_t implies that $Z \in \mathcal{A}_t$, and by definition of Z

$$E_Q[-Z|\mathcal{F}_t] = \max\left(E_Q[-X|\mathcal{F}_t], E_Q[-Y|\mathcal{F}_t]\right).$$

Hence there exists a sequence (X_n^Q) in $\mathcal{A}_t$ such that

$$\alpha_t^{\min}(Q) = \lim_n E_Q[-X_n^Q|\mathcal{F}_t] \qquad P\text{-a.s..}, \tag{10}$$

and by monotone convergence we get

$$\begin{aligned} E_Q[\alpha_t^{\min}(Q)|\mathcal{F}_s] &= \lim_n E_Q\left[\, E_Q[-X_n^Q|\mathcal{F}_t] \,\middle|\, \mathcal{F}_s\,\right] \\ &\leq \operatorname*{ess\,sup}_{Y\in\mathcal{A}_t} E_Q[-Y|\mathcal{F}_s]. \end{aligned}$$

The converse inequality follows directly from the definition of $\alpha_t^{\min}(Q)$. □

Remark 2.7. *The penalty function $\alpha_t^{\min}(Q)$ is minimal in the sense that any other penalty function α_t in a robust representation of ρ_t satisfies*

$$\alpha_t^{\min}(Q) \leq \alpha_t(Q) \;\; P\text{-a.s.}$$

for all $Q \in \mathcal{P}_t$. An alternative formula for the minimal penalty function is given by

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X\in L^\infty} \left(E_Q[-X\,|\,\mathcal{F}_t\,] - \rho_t(X)\right) \quad \textit{for all } Q \in \mathcal{P}_t. \tag{11}$$

This follows as in the unconditional case; see e.g. Theorem 4.15 and Remark 4.16 in [13].

3 Sensitivity

In this section we will show that under an assumption of *sensitivity* with respect to the reference measure P it is sufficient to use only equivalent probability measures in the robust representations of risk measures. This is more convenient for technical reasons, and it allows us to drop the dependence on time t for the representing set of measures.

Definition 3.1. *We call a conditional convex risk measure* sensitive *or* relevant*, if*

$$P[\rho_t(-\varepsilon I_A) > 0] > 0 \tag{12}$$

holds for all $\varepsilon > 0$ and for any $A \in \mathcal{F}$ such that $P[A] > 0$.

Remark 3.2.

1. *For coherent risk measures it is sufficient to require*

$$P[\rho_t(-I_A) > 0] > 0, \tag{13}$$

since (13) and (12) are equivalent under the assumption of positive homogeneity. This corresponds to the definition of relevance for coherent risk measures given in [7] for the unconditional case. For a convex risk measure, condition (12) is stronger than (13).

2. *Several slightly different definitions of relevance can be found in the literature. In [16] relevance is defined as in (13). In [6] the stronger property $A \subseteq \{\rho_t(-\varepsilon I_A) > 0\}$ for all $\varepsilon > 0$ is required in a more general setting. The arguments used in this section are similar to those in [16] and [6] up to some technical details.*

In the sequel we will assume that a conditional convex risk measure ρ_t has a robust representation. First we prove a "*σ-pasting property*" of the penalty functions which also appears in Lemma 3.12 of [16].

Lemma 3.3. *Let (Q_n) be a sequence in $\mathcal{Q}_t$ and (A_n) a sequence of pairwise disjoint events in $\mathcal{F}_t$ such that $\cup_n A_n = \Omega$ P-a.s.. Then*

$$\widetilde{Z} := \sum_{n=1}^{\infty} I_{A_n} \frac{dQ_n}{dP}$$

defines a density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$ such that

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n)$$

(here we define $I_{A_n}\alpha_t^{\min}(Q_n) := 0$, if $P[A_n] = 0$).

Proof. We will prove the first part of the lemma more generally for any sequence (λ_n) in L_t^{∞} with $0 \leq \lambda_n \leq 1$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ P-a.s.. Let $Z_n := dQ_n/dP$ and $\widetilde{Z} := \sum_{n=1}^{\infty} \lambda_n Z_n$. By monotone convergence,

$$E[\widetilde{Z}|\mathcal{F}_t] = \lim_n \sum_{k=1}^{n} \lambda_k E[Z_k|\mathcal{F}_t] = 1,$$

and so $\widetilde{Z}$ is indeed the density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$. Since

$$|\sum_{k=1}^{n} \lambda_k Z_k X| \leq \widetilde{Z}\, \|X\|_{\infty} \in L^1(P) \qquad \text{for all } n,$$

the dominated convergence theorem implies

$$E_{\widetilde{Q}}[X|\mathcal{F}_t] = \sum_{n=1}^{\infty} \lambda_n E_{Q_n}[X|\mathcal{F}_t] \tag{14}$$

for any $X \in L^{\infty}$. From the definition of the minimal penalty function we obtain immediately

$$\alpha_t^{\min}(\widetilde{Q}) \leq \sum_{n=1}^{\infty} \lambda_n \alpha_t^{\min}(Q_n).$$

In particular if $\lambda_n := I_{A_n}$ for a sequence (A_n) as above we obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_{\widetilde{Q}}[-X|\mathcal{F}_t] &= \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} \left(\sum_{n=1}^{\infty} I_{A_n} E_{Q_n}[-X|\mathcal{F}_t] \right) \\ &= \sum_{n=1}^{\infty} I_{A_n} \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_{Q_n}[-X|\mathcal{F}_t] \end{aligned}$$

so

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n).$$

□

In particular, for any $A \in \mathcal{F}_t$ and $Q_1, Q_2 \in \mathcal{Q}_t$,

$$\widetilde{Z} := I_A \frac{dQ_1}{dP} + I_{A^c} \frac{dQ_2}{dP}$$

defines a density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$ with

$$\alpha_t^{\min}(\widetilde{Q}) = I_A \alpha_t^{\min}(Q_1) + I_{A^c} \alpha_t^{\min}(Q_2). \tag{15}$$

This finite pasting property of the penalty functions, which corresponds to the local property of the risk measure, also appears in Remark 3.13 of [6].

It follows from (15) that the set $\left\{ \alpha_t^{\min}(Q) \,\middle|\, Q \in \mathcal{Q}_t \right\}$ is downward directed, and hence there exists a sequence (Q_n) in $\mathcal{Q}_t$ such that

$$\alpha_t^{\min}(Q_n) \searrow \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_t} \alpha_t^{\min}(Q) = 0 \qquad P\text{-a.s.}. \tag{16}$$

For $\varepsilon > 0$ we consider the set

$$\mathcal{Q}_t^{\varepsilon} := \left\{ Q \in \mathcal{Q}_t \,\middle|\, \alpha_t^{\min}(Q) < \varepsilon \;\; P\text{-a.s.} \right\},$$

and we use the same notation for the corresponding set of densities:

$$\mathcal{Q}_t^{\varepsilon} = \left\{ \frac{dQ}{dP} \,\middle|\, Q \in \mathcal{Q}_t,\ \alpha_t^{\min}(Q) < \varepsilon \; P\text{-a.s.} \right\}.$$

We now show that the set $\mathcal{Q}_t^{\varepsilon}$ is non-empty. Moreover, it contains an equivalent probability measure as soon as the risk measure is sensitive; this part is similar to Lemma 3.22 in [6].

Lemma 3.4. *For any $\varepsilon > 0$ the set $\mathcal{Q}_t^{\varepsilon}$ is nonempty. For a sensitive conditional convex risk measure there exists a probability measure $P^* \approx P$ such that $P^* \in \mathcal{Q}_t^{\varepsilon}$.*

Proof. For $\varepsilon > 0$ and a sequence (Q_n) as in (16) with densities (Z_n) we define an $\mathcal{F}_t$-measurable $\mathbb{N}$-valued random variable

$$\tau^{\varepsilon} := \min \left\{ n \,\middle|\, \alpha_t^{\min}(Q_n) < \varepsilon \right\}.$$

It follows from (16) that $\tau^{\varepsilon} < \infty$ P-a.s.. Thus the sets $A_n := \{\tau^{\varepsilon} = n\}$ $(n = 1, 2, \dots)$ form a disjoint partition of Ω with $A_n \in \mathcal{F}_t$ for all n. By Lemma 3.3

$$Z_{\tau^{\varepsilon}} := \sum_{n=1}^{\infty} Z_n I_{\{\tau^{\varepsilon} = n\}}$$

defines a density of a probability measure $Q^{\varepsilon} \in \mathcal{Q}_t$ with

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n) < \varepsilon \quad P\text{-a.s.},$$

which proves $Q^{\varepsilon} \in \mathcal{Q}_t^{\varepsilon}$.
Next we use a standard exhaustion argument to conclude that $\mathcal{Q}_t^{\varepsilon}$ contains an equivalent measure P^* under the assumption of sensitivity. Let

$$c := \sup \left\{ P[Z > 0] \,\middle|\, Z \in \mathcal{Q}_t^{\varepsilon} \right\}$$

and take a sequence $(Z_n)_{n \in \mathbb{N}}$ in $\mathcal{Q}_t^{\varepsilon}$ such that $P[Z_n > 0] \to c$. Then

$$Z^* := \sum_{n=1}^{\infty} \frac{1}{2^n} Z_n$$

belongs to the set $\mathcal{Q}_t^\varepsilon$ by Lemma 3.3, and

$$\{Z^* > 0\} = \cup_n \{Z_n > 0\}.$$

Hence $P[Z^* > 0] = c$. Next we show that $c = 1$, and so the probability measure P^* defined via $dP^*/dP := Z^*$ has the desired properties. Suppose by way of contradiction that the set $A := \{Z^* = 0\}$ has positive probability. Sensitivity implies $P[\rho_t(-\varepsilon I_A) > 0] > 0$, where

$$\rho_t(-\varepsilon I_A) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[\varepsilon I_A | \mathcal{F}_t] - \alpha_t^{\min}(Q) \right).$$

Hence there exists $\widetilde{Q} \in \mathcal{Q}_t$ such that the set $B := \{\alpha_t^{\min}(\widetilde{Q}) < E_{\widetilde{Q}}[\varepsilon I_A|\mathcal{F}_t]\} \in \mathcal{F}_t$ satisfies $P[B] > 0$. In particular is $\alpha_t^{\min}(\widetilde{Q}) < \varepsilon$ on B. By $\widetilde{Z}$ we denote the density of $\widetilde{Q}$ with respect to P. Without loss of generality we assume that $\widetilde{Q} \in \mathcal{Q}_t^\varepsilon$; otherwise we can simply switch to a probability measure $\widehat{Q}$ defined via $d\widehat{Q}/dP := I_B \widetilde{Z} + I_{B^c} Z$, where Z is an arbitrary element of $\mathcal{Q}_t^\varepsilon$. Then $\widehat{Q}$ is in $\mathcal{Q}_t^\varepsilon$ by (15) and $\widehat{Q}$ and $\widetilde{Q}$ coincide on B.
Next we will show that the set $\{\widetilde{Z} > 0\} \cap A$ has positive probability. Indeed, it follows from the definition of B and $\alpha_t^{\min}(\widetilde{Q}) \geq 0$, that

$$E[\widetilde{Z} I_B I_A] = E_{\widetilde{Q}}[I_B I_A] = E_{\widetilde{Q}}[I_B E_{\widetilde{Q}}[I_A|\mathcal{F}_t]] > 0,$$

which implies $P[\{\widetilde{Z} > 0\} \cap A \cap B] > 0$ and in particular $P[\{\widetilde{Z} > 0\} \cap A] > 0$. Thus the probability measure $\widehat{Q}$ defined via

$$\frac{d\widehat{Q}}{dP} := \widehat{Z} := \frac{1}{2} Z^* + \frac{1}{2} \widetilde{Z},$$

belongs to $\widehat{Q} \in \mathcal{Q}_t^\varepsilon$, and we have

$$P[\widehat{Z} > 0] = P[Z^* > 0] + P[\{\widetilde{Z} > 0\} \cap A] > P[Z^* > 0],$$

in contradiction to the maximality of $P[Z^* > 0]$. $\square$

Our next aim is to obtain a robust representation for a conditional convex risk measure in terms of equivalent probability measures. The following lemma shows that this is possible if there exists some equivalent probability measure such that its penalty function is a.s. bounded. Similar arguments are used in Proposition 3.22 of [16] and Theorem 3.22 of [6]. In the second part of the lemma we reduce the class of the representing measures even further, and this reduced representation will be useful in our discussion of time-consistency.

Lemma 3.5. *Let ρ_t be a conditional convex risk measure that is continuous from above, and let P^* be a probability measure such that $P^* \approx P$ and $\alpha_t^{\min}(P^*) < \infty$ P-a.s.. Then*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right) \tag{17}$$

for all $X \in L^\infty$. Moreover, if $E_{P^*}\left[\alpha_t^{\min}(P^*)\right] < \infty$ then

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{f,e}(P^*)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right) \tag{18}$$

for all $X \in L^\infty$, where

$$\mathcal{Q}_t^{f,e}(P^*) := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; Q = P^* \text{ on } \mathcal{F}_t,\ E_{P^*}[\alpha_t^{\min}(Q)] < \infty \right\}.$$

Proof. By Z^* we denote the density of P^* with respect to P, and for $\varepsilon \in (0,1)$ and $Q \in \mathcal{Q}_t$ we define a probability measure Q_ε via

$$\frac{dQ_\varepsilon}{dP} := (1-\varepsilon)\frac{dQ}{dP} + \varepsilon \frac{Z^*}{E[Z^*|\mathcal{F}_t]}.$$

Then $Q_\varepsilon \in Q_t$, $Q_\varepsilon \in \mathcal{M}^e(P)$ and

$$E_{Q_\varepsilon}[X|\mathcal{F}_t] = (1-\varepsilon)E_Q[X|\mathcal{F}_t] + \varepsilon E_{P^*}[X|\mathcal{F}_t]$$

for all $X \in L^\infty$. By definition of the minimal penalty function we obtain

$$\alpha_t^{\min}(Q_\varepsilon) \leq (1-\varepsilon)\alpha_t^{\min}(Q) + \varepsilon\alpha_t^{\min}(P^*).$$

Thus

$$\begin{aligned}
\rho_t(X) &= \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t\cap\mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \left(E_{Q_\varepsilon}[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q_\varepsilon)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \left((1-\varepsilon)E_Q[-X|\mathcal{F}_t] + \varepsilon E_{P^*}[-X|\mathcal{F}_t] - (1-\varepsilon)\alpha_t^{\min}(Q) - \varepsilon\alpha_t^{\min}(P^*)\right) \\
&= (1-\varepsilon)\rho_t(X) + \varepsilon\left(E_{P^*}[-X|\mathcal{F}_t] - \alpha_t^{\min}(P^*)\right) \\
&\geq \rho_t(X) - \varepsilon\left(\rho_t(X) + \|X\|_\infty + \alpha_t^{\min}(P^*)\right), \qquad (19)
\end{aligned}$$

where the lower bound converges a.s. to ρ_t with $\varepsilon \to 0$. Hence

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t\cap\mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right).$$

On the other hand it follows from the representation (4) that

$$\begin{aligned}
\rho_t(X) &\geq \operatorname*{ess\,sup}_{Q\in\mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t\cap\mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right),
\end{aligned}$$

proving the representation (17).

If $E_{P^*}\left[\alpha_t^{\min}(P^*)\right] < \infty$ we define for $Q \in \mathcal{Q}_t^f(P^*)$ and $\varepsilon \in (0,1)$ a probability measure Q_ε via

$$\frac{dQ_\varepsilon}{dP^*} := (1-\varepsilon)\frac{dQ}{dP^*} + \varepsilon.$$

Then $Q_\varepsilon = P^*$ on $\mathcal{F}_t$, $Q_\varepsilon \in \mathcal{M}^e(P)$ and

$$E_{Q_\varepsilon}[X|\mathcal{F}_t] = (1-\varepsilon)E_Q[X|\mathcal{F}_t] + \varepsilon E_{P^*}[X|\mathcal{F}_t]$$

for all $X \in L^\infty$. This implies

$$\alpha_t^{\min}(Q_\varepsilon) \leq (1-\varepsilon)\alpha_t^{\min}(Q) + \varepsilon\alpha_t^{\min}(P^*)$$

and in particular $E_{P^*}\left[\alpha_t^{\min}(Q_\varepsilon)\right] < \infty$, so $Q_\varepsilon \in \mathcal{Q}_t^{f,e}(P^*)$. Thus we obtain using Corollary 2.4

$$
\begin{aligned}
\rho_t(X) &= \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^f} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^{f,e}} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\
&\geq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^f} \left(E_{Q_\varepsilon}[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q_\varepsilon)\right) \\
&\geq \rho_t(X) - \varepsilon\left(\rho_t(X) + \|X\|_\infty + \alpha_t^{\min}(P^*)\right)
\end{aligned}
$$

and the representation (18) follows. □

In view of Lemma 3.4 and Lemma 3.5 we obtain the following corollary:

Corollary 3.6. *Any sensitive conditional convex risk measure that is continuous from above is representable as in (17) and (18).*

4 Time-consistency

In this section we consider a sequence of conditional convex risk measures $(\rho_t)_{t=0,1,\ldots}$. Such a sequence (with a finite time horizon) is called a dynamic convex risk measure in [11] or (with opposite sign) a monetary utility functional process in [6].

A key question in the dynamic setting is how the risk assessments of a financial position in different periods of time are interrelated. Several notions of time-consistency of dynamic risk measures have been introduced in the literature; see [2], [11], [6], , [16], [5] and references therein, and also [19] for a systematic overview.

In this section we will focus on the strong notion of time-consistency defined as follows.

Definition 4.1. *A sequence of conditional risk measures $(\rho_t)_{t=0,1,\ldots}$ is called* time-consistent*, if for any $X, Y \in L^\infty$ and for all $t \geq 0$ the following condition holds:*

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) = \rho_t(Y) \;\; P\text{-a.s.}. \tag{20}$$

Proposition 4.2. *For a sequence of conditional convex risk measures time-consistency is equivalent to* recursiveness*, that is*

$$\rho_t = \rho_t(-\rho_{t+s}) \quad P\text{-a.s.}$$

for all $t, s \in \{0, 1, \ldots\}$.

Proof. We will prove that time-consistency implies recursiveness by induction on s. For $s = 1$ we have $\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X)$ by cash invariance and the claim follows from (20). Now we assume that the induction hypothesis holds for each t and all $k \leq s$ for some $s \geq 1$. Then we obtain

$$
\begin{aligned}
\rho_t(-\rho_{t+s+1}(X)) &= \rho_t(-\rho_{t+s}(-\rho_{t+s+1}(X))) \\
&= \rho_t(-\rho_{t+s}(X)) \\
&= \rho_t(X),
\end{aligned}
$$

where we have applied the induction hypothesis to the random variable $-\rho_{t+s+1}(X)$. Hence the claim follows.
The converse implication is obvious. □

Remark 4.3. *1. The equivalence of time-consistency and "one-step" recursiveness, that is*

$$\rho_t = \rho_t(-\rho_{t+1}) \ P\text{-a.s.} \quad \text{for all } t = 0, 1, \dots, \tag{21}$$

was already proved in Proposition 5 of [11].

2. *As explained in [2], recursiveness may be viewed as a version of the Bellman principle for dynamic risk measures.*

3. *The following definition of time-consistency is given in [2]:*

$$\rho_{t+1}(X) \le \rho_{t+1}(Y) \ P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) \le \rho_t(Y) \ P\text{-a.s.} \tag{22}$$

Using recursiveness it is easy to see that (22) is equivalent to (20).

4. *In [5] time-consistency is defined as in (22) but in terms of stopping times. For coherent risk measures it is shown in [5] that this is equivalent to recursiveness for stopping times.*

5. *A more general definition of time-consistency in terms of recursiveness for stopping times is given in [6] for risk measures on stochastic processes. Proposition 4.5 in [6] shows that this definition is equivalent to recursiveness in the sense of (21) if the time horizon is finite or if all risk measures are continuous from above.*

Proposition 4.4. *Let $(\rho_t)_{t=0,1,\dots}$ be a time-consistent sequence of conditional convex risk measures and let ρ_0 be sensitive. Then ρ_t is sensitive for all $t \ge 0$.*

Proof. Let $A \in \mathcal{F}$ with $P[A] > 0$ and $\varepsilon > 0$. Then by monotonicity $\rho_t(-\varepsilon I_A) \ge 0$ P-a.s.. Assume that $\rho_t(-\varepsilon I_A) = 0$ P-a.s.. Then recursiveness and normalization imply $\rho_0(-\varepsilon I_A) = \rho_0(-\rho_t(-\varepsilon I_A)) = 0$ in contradiction to the sensitivity of ρ_0. Hence $P[\rho_t(-\varepsilon I_A) > 0] > 0$. □

In the sequel we will give alternative characterizations of time-consistency. To this end we introduce some notation. If we restrict a conditional convex risk measure ρ_t to the space L^∞_{t+s} for some $s \ge 0$, the corresponding acceptance set is given by

$$\mathcal{A}_{t,t+s} := \left\{ X \in L^\infty_{t+s} \,\middle|\, \rho_t(X) \le 0 \right\}.$$

If ρ_t is continuous from above, then this property holds on L^∞_{t+s}, and thus the restriction of ρ_t to L^∞_{t+s} is representable with the minimal penalty function

$$\alpha^{\min}_{t,t+s}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_{t,t+s}} E_Q[-X \mid \mathcal{F}_t], \quad Q \in \mathcal{P}_t.$$

Note that $\mathcal{A}_{t,t} = L^\infty_+(\mathcal{F}_t)$ and $\alpha^{\min}_{t,t}(Q) = 0$ Q-a.s. for all $Q \in \mathcal{P}_t$.

In our next theorem we will assume that the set

$$\mathcal{Q}^* := \left\{ Q \in \mathcal{M}^e(P) \,\middle|\, \alpha^{\min}_0(Q) < \infty \right\}$$

is nonempty. In view of Lemma 3.4, this assumption is satisfied if ρ_0 is sensitive. We will show that the set $\mathcal{Q}^*$ yields a robust representation of a time-consistent dynamic convex risk measure.

The next theorem, and in particular the equivalence of 1) and 4), is the main result of this section.

Theorem 4.5. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above, and assume that $\mathcal{Q}^* \neq \emptyset$. Then the following conditions are equivalent:*

1. *$(\rho_t)_{t=0,1,\dots}$ is time-consistent.*
2. *$\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ for all $s, t = 0, 1, \dots$.*
3. *$\alpha_t^{\min}(Q) = \alpha_{t,t+s}^{\min}(Q) + E_Q[\,\alpha_{t+s}^{\min}(Q)\,|\,\mathcal{F}_t\,]$ for all $s, t = 0, 1, \dots$ and all $Q \in \mathcal{M}^e(P)$.*
4. *For all $Q \in \mathcal{Q}^*$ and all $X \in L^\infty$, the process*
$$V_t^Q(X) := \rho_t(X) + \alpha_t^{\min}(Q), \qquad t \geq 0$$
is a Q-supermartingale.

In each case the dynamic risk measure admits a robust representation in terms of the set $\mathcal{Q}^*$, i.e.,
$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right) \tag{23}$$
for all $X \in L^\infty$ and all $t \geq 0$.

Before we begin the proof let us compare Theorem 4.5 to the existing literature. The equivalence of 1) and 2) is already known: It was proved in a more general setting in Theorem 4.5 in [6] and also in Lemma 3.25 in [16].
For penalty functions some necessary and sufficient conditions for time-consistency are given in Theorems 4.19 and 4.22 of [6]. In the more general context of risk measures for stochastic processes, they involve concatenation of the representing dual functionals. In our setting of risk measures for random variables, it is natural to identify dual functionals with probability measures and to use 3) as a necessary and sufficient condition. With a slight modification of 3) and under the assumption that the risk measures are continuous from below, the equivalence of the first three properties also appears in [4].
The equivalence of recursiveness and the supermartingale property of the process $(\rho_t)_{t=0,1,\dots}$ was shown in [2] for dynamic *coherent* risk measures which are given in terms of the same representing class $\mathcal{Q}$; see also [5]. In the context of dynamic *convex* risk measures, the equivalence of time-consistency and the supermartingale property 4) seems to be new.

The proof of Theorem 4.5 will be given in several steps. Note that we may assume that $P \in \mathcal{Q}^*$; otherwise we can simply replace P by some $P^* \in \mathcal{Q}^*$.

The equivalence of 1) and 2) follows from the next lemma, which holds for any sequence of conditional convex risk measures; here we do not need robust representations and the set $\mathcal{Q}^*$. The equivalences between set inclusions and inequalities may serve as starting points for various extensions of the strong notion of time-consistency used in this paper; cf. [19] and [17].

Lemma 4.6. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures. Then the following equivalences hold for all $s, t \geq 0$ and all $X \in L^\infty$:*

$$X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff -\rho_{t+s}(X) \in \mathcal{A}_{t,t+s} \tag{24}$$
$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \leq \rho_t \quad P\text{-a.s.} \tag{25}$$
$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \geq \rho_t \quad P\text{-a.s.}. \tag{26}$$

Proof.

a) To prove "⇒" in (24) let $X = X_{t,t+s} + X_{t+s}$ with $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Then

$$\rho_{t+s}(X) = \rho_{t+s}(X_{t+s}) - X_{t,t+s} \leq -X_{t,t+s}$$

by cash invariance, and monotonicity implies

$$\rho_t(-\rho_{t+s}(X)) \leq \rho_t(X_{t,t+s}) \leq 0.$$

The converse direction follows immediately from $X = X + \rho_{t+s}(X) - \rho_{t+s}(X)$ and $X + \rho_{t+s}(X) \in \mathcal{A}_{t+s}$ for all $X \in L^\infty$.

b) In order to show ⇒" in (25), take $X \in L^\infty$. Since $X + \rho_t(X) \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$, we obtain

$$\rho_{t+s}(X) - \rho_t(X) = \rho_{t+s}(X + \rho_t(X)) \in -\mathcal{A}_{t,t+s},$$

by (24) and cash invariance. Hence

$$\rho_t(-\rho_{t+s}(X)) - \rho_t(X) = \rho_t(-(\rho_{t+s}(X) - \rho_t(X))) \leq 0 \quad P\text{-a.s..}$$

To prove "⇐" let $X \in \mathcal{A}_t$. Then $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by the right hand side of (25), and hence $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ by (24).

c) Let $X \in L^\infty$ and assume $\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Then

$$\rho_t(-\rho_{t+s}(X)) + X = \rho_t(-\rho_{t+s}(X)) - \rho_{t+s}(X) + \rho_{t+s}(X) + X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \subseteq \mathcal{A}_t.$$

Hence

$$\rho_t(X) - \rho_t(-\rho_{t+s}(X)) = \rho_t(X + \rho_t(-\rho_{t+s}(X))) \leq 0$$

by cash invariance, and this proves "⇒" in (26). For the converse direction let $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Since $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by (24), we obtain

$$\rho_t(X) \leq \rho_t(-\rho_{t+s}(X)) \leq 0,$$

hence $X \in \mathcal{A}_t$. □

Proof of 2) ⇒ 3) of Theorem 4.5: For $Q \in \mathcal{M}^e(P)$ we obtain using the definition of the minimal penalty function and Lemma 2.6:

$$\begin{aligned}
\alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{X_{t,t+s} \in \mathcal{A}_{t,t+s}} E_Q[-X_{t,t+s}|\mathcal{F}_t] + \operatorname*{ess\,sup}_{X_{t+s} \in \mathcal{A}_{t+s}} E_Q[-X_{t+s}|\mathcal{F}_t] \\
&= \alpha_{t,t+s}^{\min}(Q) + E_Q[\,\alpha_{t+s}^{\min}(Q)\,|\,F_t\,]
\end{aligned}$$

for all $s, t \geq 0$. □

Remark 4.7. *In particular it follows from the preceding proofs that time-consistency implies*

$$\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \tag{27}$$

and

$$\alpha_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,\mathcal{F}_t\,] \tag{28}$$

for all $t = 0, 1, \dots$ and all $Q \in \mathcal{M}^e(P)$. These "one-step" versions already imply the general properties 2) and 3) of Theorem 4.5. Indeed, applying (27) and (28) step by step to the time-consistent sequence $(\rho_n)_{t\le n\le t+s}$ on the space $L^\infty(\mathcal{F}_{t+s})$ for each $t \ge 0$ and $s \ge 1$ we obtain

$$\mathcal{A}_{t,t+s} = \sum_{n=t}^{t+s-1} \mathcal{A}_{n,n+1}$$

and

$$\alpha_{t,t+s}^{\min}(Q) = E_Q\left[\sum_{n=t}^{t+s-1} \alpha_{n,n+1}^{\min}(Q)\,\Big|\,\mathcal{F}_t\right]$$

for all all $t, s \ge 0$ and all $Q \in \mathcal{M}^e(P)$ (note that we have not used that the initial σ-field is trivial in the preceding proofs).

Remark 4.8. *It follows from property 3) of Theorem 4.5 that*

$$E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] \le \alpha_t^{\min}(Q) \quad P\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}^e(P).$$

This in turn implies that $E_Q[\alpha_t^{\min}(Q)] < \infty$ for all $t \ge 0$ and $Q \in \mathcal{Q}^*$. Thus the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$. Moreover, equation (28) yields an explicit description of its Doob decomposition in terms of the "one-step" penalty functions $\alpha_{t,t+1}^{\min}(Q)$.

In fact the supermartingale property of the minimal penalty function corresponds to some weaker notion of time-consistency, so called *weak time-consistency*; this was noted in Lemma 3.17 in [5].

Definition 4.9. *A sequence of conditional risk measures $(\rho_t)_{t=0,1,\dots}$ is called* weakly time-consistent*, if for any $X \in L^\infty$ and for all $t \ge 0$ the following condition holds:*

$$\rho_{t+1}(X) \le 0 \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) \le 0 \;\; P\text{-a.s.}. \tag{29}$$

Some characterizations of weak time-consistency are given in [19]. In terms of penalty functions we obtain the following criterion.

Proposition 4.10. *Let $(\rho_t)_{t=0,1,\dots}$ be a weakly time-consistent sequence of conditional convex risk measures such that each ρ_t is continuous from above . Then*

$$E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] \le \alpha_t^{\min}(Q) \tag{30}$$

holds for all $Q \in \mathcal{M}^e(P)$ and all $t = 0, 1, \dots$. In particular, $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$. Conversely, property (30) implies weak time-consistency if the representation (17) or (23) holds.

Proof. Since property (29) is equivalent to $\mathcal{A}_{t+1} \subseteq \mathcal{A}_t$, Lemma 2.6 implies

$$\begin{aligned} E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] &= \operatorname*{ess\,sup}_{X_{t+1}\in\mathcal{A}_{t+1}} E_Q[-X_{t+1}|\mathcal{F}_t] \\ &\le \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] = \alpha_t^{\min}(Q) \end{aligned}$$

for all $Q \in \mathcal{M}^e(P)$. If $\alpha_0^{\min}(Q) < \infty$ then it follows from (30) that $\alpha_t^{\min}(Q)$ is Q-integrable for all $t \geq 0$. Thus $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$. To prove the opposite direction, note that for $X \in \mathcal{A}_{t+1}$

$$E_Q[-X|\mathcal{F}_{t+1}] \leq \alpha_{t+1}^{\min}(Q) \quad P\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}^e(P)$$

by definition of the minimal penalty function. Using (30) we obtain

$$E_Q[-X|\mathcal{F}_t] \leq E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] \leq \alpha_t^{\min}(Q) \quad P\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}^e(P).$$

So if (17) holds,

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \leq 0,$$

and hence $X \in \mathcal{A}_t$. If ρ_t has a representation (23), then the supermartingale property for all $Q \in \mathcal{Q}^*$ is sufficient to prove (29). $\square$

Proof of 3) $\Rightarrow$ 4) of Theorem 4.5:

a) First we will show that the representations (17), (18) and (23) hold for any $t \geq 0$. Note that property 3) implies $E_{P^*}\left[\alpha_t^{\min}(P^*)\right] < \infty$ for $P^* \in \mathcal{Q}^*$, and so the representations (17) and (18) of Lemma 3.5 hold for any $P^* \in \mathcal{Q}^*$. Now take $Q \in \mathcal{M}^e(P)$ such that $Q = P$ on $\mathcal{F}_t$ and $E_Q[\alpha_t^{\min}(Q)] < \infty$, that is $Q \in \mathcal{Q}_t^{f,e}(P)$. Using 3) we obtain

$$\begin{aligned}
\alpha_0^{\min}(Q) &= E_Q\left[\alpha_{0,t}^{\min}(Q)\right] + E_Q[\alpha_t^{\min}(Q)] \\
&= E_P\left[\alpha_{0,t}^{\min}(P)\right] + E_Q[\alpha_t^{\min}(Q)] \\
&\leq \alpha_0^{\min}(P) + E_Q[\alpha_t^{\min}(Q)] < \infty,
\end{aligned}$$

hence $Q \in \mathcal{Q}^*$. Thus it follows from (18) that

$$\rho_t(X) \leq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \quad \text{for all } X \in L^\infty.$$

The converse inequality "$\geq$" follows from (17) of Lemma 3.5.

b) In the next step we fix $\widetilde{Q} \in \mathcal{Q}^*$ and apply Lemma 3.3 to the set

$$\mathcal{Q}_{t+1}^{f,e}(\widetilde{Q}) = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; Q = \widetilde{Q} \text{ on } \mathcal{F}_{t+1},\, E_{\widetilde{Q}}[\alpha_{t+1}^{\min}(Q)] < \infty \right\}.$$

For $Q_1, Q_2 \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q})$ and $B \in \mathcal{F}_{t+1}$ we define

$$\widehat{Z} := I_B \frac{dQ_2}{d\widetilde{Q}} + I_{B^c} \frac{dQ_1}{d\widetilde{Q}}.$$

Then by Lemma 3.3 the probabilty measure $\widehat{Q}$ defined via $d\widehat{Q}/d\widetilde{Q} := \widehat{Z}$ satisfies $\widehat{Q} = \widetilde{Q}$ on $\mathcal{F}_{t+1}$ and

$$\alpha_{t+1}^{\min}(\widehat{Q}) = \alpha_{t+1}^{\min}(Q_1)\, I_{B^c} + \alpha_{t+1}^{\min}(Q_2)\, I_B,$$

hence $\widehat{Q} \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q})$.

c) Using b) and the same reasoning as in the proof of Lemma 2.6 we can deduce that the set

$$\left\{ E_Q[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q) \;\middle|\; Q \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q}) \right\}$$

is directed upward for all $X \in L^\infty$. Since ρ_{t+1} can be represented as essential supremum over this set by a), there exists a sequence $(Q_n) \subseteq \mathcal{Q}^{f,e}_{t+1}(\widetilde{Q})$ depending on $\widetilde{Q}$ and X such that

$$E_{Q_n}[-X|\mathcal{F}_{t+1}] - \alpha^{\min}_{t+1}(Q_n) \nearrow \rho_{t+1}(X) \quad P\text{-a.s.} \quad \text{with } n \to \infty.$$

The monotone convergence theorem implies

$$\begin{aligned} E_{\widetilde{Q}}[\rho_{t+1}(X)|\mathcal{F}_t] &= \lim_{n\to\infty} E_{\widetilde{Q}}[\, E_{Q_n}[-X|\mathcal{F}_{t+1}] - \alpha^{\min}_{t+1}(Q_n) \,|\, \mathcal{F}_t] \\ &= \lim_{n\to\infty} \left(E_{Q_n}[-X|\mathcal{F}_t] - E_{Q_n}[\alpha^{\min}_{t+1}(Q_n)|\mathcal{F}_t] \right), \end{aligned}$$

where we have used that Q_n and $\widetilde{Q}$ coincide on $\mathcal{F}_{t+1}$. Moreover, the same reasoning as in a) implies that $Q^{f,e}_{t+1}(\widetilde{Q}) \subseteq \mathcal{Q}^*$, and applying 3) to Q_n we obtain

$$\begin{aligned} E_{Q_n}[\alpha^{\min}_{t+1}(Q_n)|\mathcal{F}_t] &= \alpha^{\min}_t(Q_n) - \alpha^{\min}_{t,t+1}(Q_n) \\ &= \alpha^{\min}_t(Q_n) - \alpha^{\min}_{t,t+1}(\widetilde{Q}) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

d) In the final step we obtain for $\widetilde{Q} \in \mathcal{Q}^*$ and $X \in L^\infty$

$$\begin{aligned} E_{\widetilde{Q}}[V^{\widetilde{Q}}_{t+1}(X)|\mathcal{F}_t] &= E_{\widetilde{Q}}[\rho_{t+1}(X) + \alpha^{\min}_{t+1}(\widetilde{Q})|\mathcal{F}_t] \\ &= E_{\widetilde{Q}}[\rho_{t+1}(X)|\mathcal{F}_t] - \alpha^{\min}_{t,t+1}(\widetilde{Q}) + \alpha^{\min}_t(\widetilde{Q}) \\ &= \lim_{n\to\infty} \left(E_{Q_n}[-X|\mathcal{F}_t] - \alpha^{\min}_t(Q_n) \right) + \alpha^{\min}_t(\widetilde{Q}) \\ &\leq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha^{\min}_t(Q) \right) + \alpha^{\min}_t(\widetilde{Q}) \\ &= \rho_t(X) + \alpha^{\min}_t(\widetilde{Q}) \\ &= V^{\widetilde{Q}}_t(X) \qquad t = 0, 1, \dots, \end{aligned}$$

where we have used 3), c), a) and $Q_n \in \mathcal{Q}^*$ for all n. Moreover, $(V^{\widetilde{Q}}_t(X))_{t=0,1,\dots}$ is adapted and integrable for all $\widetilde{Q} \in \mathcal{Q}^*$, and thus a $\widetilde{Q}$-supermartingale.

Proof of 4) $\Rightarrow$ 1) of Theorem 4.5:

In the first step we will show again that the representation (23) holds for all $t \geq 0$. Indeed, 4) implies $E_{P^*}\left[\alpha^{\min}_t(P^*)\right] < \infty$ for all $t \geq 0$ and $P^* \in \mathcal{Q}^*$, since $\rho_t(X) + \alpha^{\min}_t(P^*)$ is P^*-integrable and $\rho_t(X) \in L^\infty_t$ for all $X \in L^\infty$ and $t \geq 0$. Hence the representation (18) of Lemma 3.5 holds for all $t \geq 0$ and $P^* \in \mathcal{Q}^*$. Moreover, for $Q \in \mathcal{Q}^{f,e}_t(P)$ and $X \in \mathcal{A}_0$ we obtain

$$\begin{aligned} E_Q[-X] &\leq E_Q[-X - \rho_t(X)] + E_Q[\rho_t(X) + \alpha^{\min}_t(P)] \\ &\leq E_Q[\alpha^{\min}_t(Q)] + E_P[\rho_t(X) + \alpha^{\min}_t(P)] \\ &\leq E_Q[\alpha^{\min}_t(Q)] + \rho_0(X) + \alpha^{\min}_0(P) \\ &\leq E_Q[\alpha^{\min}_t(Q)] + \alpha^{\min}_0(P), \end{aligned}$$

where we have used representation (11) for $\alpha^{\min}_t(Q)$, $Q \in \mathcal{Q}^{f,e}_t(P)$, $P \in \mathcal{Q}^*$, 4), and $X \in \mathcal{A}_0$. Hence

$$\alpha^{\min}_0(Q) \leq E_Q[\alpha^{\min}_t(Q)] + \alpha^{\min}_0(P) < \infty$$

which implies $Q \in \mathcal{Q}^*$. Now we can argue as in part a) of the proof 3) $\Rightarrow$ 4) to obtain representation (23).

In the next step we will prove time-consistency. To this end let $X, Y \in L^\infty$ such that $\rho_{t+1}(X) \leq \rho_{t+1}(Y)$ P-a.s.. Using 4) we obtain for all $Q \in \mathcal{Q}^*$:

$$\begin{aligned}
\rho_t(Y) + \alpha_t^{\min}(Q) &\geq E_Q[\rho_{t+1}(Y) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&\geq E_Q[\rho_{t+1}(X) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&\geq E_Q[E_Q[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&= E_Q[-X|\mathcal{F}_t].
\end{aligned}$$

Thus

$$\rho_t(Y) \geq E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)$$

for all $Q \in \mathcal{Q}^*$, and hence

$$\rho_t(Y) \geq \rho_t(X) \quad P\text{-a.s.},$$

proving time-consistency of the sequence (ρ_t) as characterized by (22). $\square$

In the *coherent* case the characterization of time-consistency is already well understood; see Theorem 5.1. in [2], Theorem 6.2 in [9], Lemma 3.29 in [16], Korollar 3.18 in [5], and section 4.4 in [6]. Let us show how the main results can be obtained as special cases of our discussion of the general convex case. This involves the following stability property for the representing set of measures, sometimes called *fork convexity* as in [9] and *multiplicative stabilty* or *m-stabilty* as in [2]. It is equivalent to Definition 6.44 in [13] and stronger than the *weak m-stabilty* in Definition 3.27 of [16].

Definition 4.11. *We call a set $\mathcal{Q} \subseteq \mathcal{M}^e(P)$* stable *if it has the following property: For any $Q^1, Q^2, Q^3 \in \mathcal{Q}$, any $t \geq 0$ and any $A_t \in \mathcal{F}_t$ the probability measure Q given by*

$$Q[A] = E_{Q^1}\big[\, I_{A_t}\, Q^2[\, A \,|\, \mathcal{F}_t\,] + I_{A_t^c}\, Q^3[\, A \,|\, \mathcal{F}_t\,]\,\big], \tag{31}$$

called the pasting *of Q^1, Q^2 and Q^3 in t via A_t, belongs again to the set $\mathcal{Q}$.*

Note that the density of the pasting Q is given by

$$Z_T := I_{A_t} \frac{Z_t^1}{Z_t^2} Z_T^2 + I_{A_t^c} \frac{Z_t^1}{Z_t^3} Z_T^3, \tag{32}$$

where Z^i denotes the density process of Q^i with respect to P for $i = 1, 2, 3$.

It is also easy to see that a probabilty measure Q is a pasting of Q^1, Q^2 and Q^3 at time t via A_t iff it has the following property:

$$E_Q[X|\mathcal{F}_s] = \begin{cases} E_{Q^1}\big[\, I_{A_t}\, E_{Q^2}[\, X \,|\, \mathcal{F}_t\,] + I_{A_t^c}\, E_{Q^3}[\, X \,|\, \mathcal{F}_t\,] \,|\, \mathcal{F}_s\,\big] & ; \quad s < t \\ I_{A_t}\, E_{Q^2}[\, X \,|\, \mathcal{F}_s\,] + I_{A_t^c}\, E_{Q^3}[\, X \,|\, \mathcal{F}_s\,] & ; \quad s \geq t. \end{cases} \tag{33}$$

for all $s \geq 0$. In particular we have $Q = Q^1$ on $\mathcal{F}_t$.

If the initial risk measure ρ_0 is coherent then the penalty function $\alpha_0^{\min}(Q)$ can only take values 0 or ∞. Hence the set $\mathcal{Q}^*$ takes the form

$$\mathcal{Q}^* = \big\{\, Q \in \mathcal{M}^e(P) \;\big|\; \alpha_0^{\min}(Q) = 0 \big\}\,.$$

Corollary 4.12. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above. Assume that $\mathcal{Q}^* \neq \emptyset$ and that the initial risk measure ρ_0 is coherent. Then the following conditions are equivalent:*

1. $(\rho_t)_{t=0,1,\ldots}$ *is time-consistent.*

2. *The representation*
$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_t] \tag{34}$$
holds for all $X \in L^\infty$ *and all* $t \geq 0$, *and the set* $\mathcal{Q}^*$ *is stable.*

3. *The representation (34) holds for all* $X \in L^\infty$ *and all* $t \geq 0$, *and the process* $(\rho_t(X))_{t=0,1,\ldots}$ *is a* Q*-supermartingale for all* $Q \in \mathcal{Q}^*$ *and all* $X \in L^\infty$.

In each case $(\rho_t)_{t=0,1,\ldots}$ *is a dynamic* coherent *risk measure.*

Proof. As in the proof of Theorem 4.5 we may assume that $P \in \mathcal{Q}^*$.

1) $\Rightarrow$ 2) Time-consistency implies property 3) of Theorem 4.5, and we will show that this implies property 2) of Corollary 4.12. Indeed, $\alpha_0^{\min}(Q) = 0$ implies $\alpha_t^{\min}(Q) = 0$ for all $t \geq 0$ due to property 3). Hence the representation (23) reduces to (34). To prove stability of the set $\mathcal{Q}^*$, take $Q^1, Q^2, Q^3 \in \mathcal{Q}^*$, $t \geq 0$, $A_t \in \mathcal{F}_t$ and define Q via (31). Using (33) we obtain $\alpha_{0,t}^{\min}(Q) = \alpha_{0,t}^{\min}(Q^1) = 0$ and

$$\begin{aligned}
\alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \\
&= I_A \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_{Q^2}[-X|\mathcal{F}_t] + I_{A^c} \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_{Q^3}[-X|\mathcal{F}_t] \\
&= I_A \alpha_t^{\min}(Q^2) + I_{A^c}\alpha_t^{\min}(Q^3) = 0,
\end{aligned}$$

hence $\alpha_0^{\min}(Q) = \alpha_{0,t}^{\min}(Q) + E_Q[\alpha_t^{\min}(Q)] = 0$, and thus $Q \in \mathcal{Q}^*$.

2) $\Rightarrow$ 3) We have to show that 2) implies

$$E_{\widetilde{Q}}[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t\,] \leq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_t] \tag{35}$$

for all $t \geq 0$ and $\widetilde{Q} \in \mathcal{Q}^*$. To this end note first that the set

$$\left\{\, E_Q[-X|\mathcal{F}_{t+1}] \,\middle|\, Q \in \mathcal{Q}^* \right\}$$

is directed upward due to the stability of the set $\mathcal{Q}^*$ and our assumption $P \in \mathcal{Q}^*$. Indeed, for any $Q^1, Q^2 \in \mathcal{Q}^*$ the pasting Q of P, Q^1 and Q^2 in $t+1$ via $A_t := \left\{E_{Q^1}[-X|\mathcal{F}_{t+1}] > E_{Q^2}[-X|\mathcal{F}_{t+1}]\right\}$ with the density

$$Z_T := I_{A_t}\frac{Z_T^1}{Z_{t+1}^1} + I_{A_t^c}\frac{Z_T^2}{Z_{t+1}^2}$$

belongs to $\mathcal{Q}^*$ and

$$E_Q[-X|\mathcal{F}_{t+1}] = \max(E_{Q^1}[-X|\mathcal{F}_{t+1}], E_{Q^2}[-X|\mathcal{F}_{t+1}]).$$

Hence the same argument as in the proof of Lemma 2.6 implies

$$E_{\widetilde{Q}}[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_{\widetilde{Q}}[E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t].$$

Moreover, the pasting of $\widetilde{Q}$ and Q in $t+1$ via $A_{t+1} = \Omega$ belongs to $\mathcal{Q}^*$, and hence we have

$$\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_{\widetilde{Q}}[E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t] \leq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_t],$$

and this proves (35).

3) $\Rightarrow$ 1) We show that property 3) of Corollary 4.12 implies property 4) of Theorem 4.5. Indeed, for $X \in \mathcal{A}_{t+1}$ representation (34) implies $E_Q[-X|\mathcal{F}_{t+1}] \leq 0$ for all $Q \in \mathcal{Q}^*$. Hence $E_Q[-X|\mathcal{F}_t] \leq 0$ for all $Q \in \mathcal{Q}^*$ and $X \in \mathcal{A}_t$ by (34). Thus the sequence $(\rho_t)_{t=0,1,\dots}$ is weakly time-consistent, and the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a non-negative Q-supermartingale for all $Q \in \mathcal{Q}^*$ by Proposition 4.10. Moreover, since $\alpha_0^{\min}(Q) = 0$ we obtain $\alpha_t^{\min}(Q) = 0$ for all $t \geq 0$. Hence the process

$$\rho_t(X) = \rho_t(X) + \alpha_t^{\min}(Q), \qquad t \geq 0$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$, and so we have verified property 4) of Theorem 4.5. $\square$

5 Asymptotic safety and asymptotic precision

Consider a time-consistent sequence $(\rho_t)_{t=0,1,\dots}$ of conditional convex risk measures with infinite time horizon $T = \infty$. We assume that $\mathcal{F} = \mathcal{F}_\infty := \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$ and that $\mathcal{Q}^* \neq \emptyset$.

For $Q \in \mathcal{Q}^*$ and $X \in L^\infty$, the process $(V_t^Q(X))_{t=0,1,\dots}$ is a Q-supermartingale due to Theorem 4.5, and the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a non-negative Q-supermartingale by Remark 4.8. Moreover, $(V_t^Q(X))_{t=0,1,\dots}$ is bounded from below since

$$V_t^Q(X) \geq E_Q[-X|\mathcal{F}_t] \quad Q\text{-a.s.}$$

due to the robust representation (23) of the risk measure ρ_t. Hence $(V_t^Q(X))_{t=0,1,\dots}$ and $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ are both Q-a.s. convergent to some finite limits $\alpha_\infty^{\min}(Q)$ and $V_\infty^Q(X)$. In particular, the limit

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) = V_\infty^Q(X) - \alpha_\infty^{\min}(Q) \tag{36}$$

exists P-a.s..

Lemma 5.1. *The functional $\rho_\infty : L^\infty \to L^\infty$ defined by (36) is normalized, monotone, conditionally convex and conditionally cash invariant with respect to $\mathcal{F}_t$ for any $t \geq 0$, and it satisfies*

$$\rho_\infty(X) \geq -X - \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^*} \alpha_\infty^{\min}(Q) \quad P\text{-a.s..}$$

Proof. Normalization, monotonicity, conditional convexity and conditional cash invariance w.r.t. any $\mathcal{F}_{t_0}$ follow from the corresponding properties of ρ_t for $t \geq t_0$. Since

$$\rho_t(X) \geq E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)$$

for all t, we obtain

$$\rho_\infty(X) \geq -X - \alpha_\infty^{\min}(Q) \quad Q\text{-a.s.} \tag{37}$$

by martingale convergence for any $Q \in \mathcal{Q}^*$. $\square$

Clearly, ρ_∞ is a conditional convex risk measure if and only if it reduces to the trivial monetary risk measure

$$\rho_\infty(X) = -X, \tag{38}$$

since this is equivalent to cash invariance w.r.t. $\mathcal{F}_\infty = \mathcal{F}$. But this property does not always hold as shown by examples 5.5 and 5.10 below.

Let us first focus on the weaker property

$$\rho_\infty(X) \geq -X,$$

i.e., the asymptotic capital requirement ρ_∞ is enough to cover the actual final loss $-X$:

Definition 5.2. *We say that the sequence $(\rho_t)_{t=0,1,\dots}$ is* asymptotically safe *if the limit ρ_∞ defined by (36) satisfies*

$$\rho_\infty(X) \geq -X$$

for any $X \in L^\infty$.

In order to characterize asymptotic safety we recall that the classes

$$\mathcal{A}_{0,t} = \mathcal{A}_0 \cap L^\infty_t \qquad t = 0, 1, \dots$$

and the corresponding penalty functions

$$\alpha^{\min}_{0,t}(Q) = \sup_{X \in \mathcal{A}_{0,t}} E_Q[-X] \qquad t = 0, 1, \dots$$

satisfy the relations

$$\mathcal{A}_0 = \mathcal{A}_{0,t} + \mathcal{A}_t$$

and

$$\alpha^{\min}_0(Q) = \alpha^{\min}_{0,t}(Q) + E_Q[\alpha^{\min}_t(Q)],$$

for all $Q \in \mathcal{Q}^*$. In particular, $\alpha^{\min}_{0,t}(Q)$ is increasing in t by Remark 4.7 and bounded from above by $\alpha^{\min}_0(Q)$ for $Q \in \mathcal{Q}^*$. Thus the limit

$$\alpha^{\min}_{0,\infty}(Q) := \lim_{t\to\infty} \alpha^{\min}_{0,t}(Q) \leq \alpha^{\min}_0(Q) \tag{39}$$

exists for all $Q \in \mathcal{Q}^*$.

Definition 5.3. *Let us say that $X \in L^\infty$ is* predictably acceptable *if there exists a uniformly bounded and P-a.s. convergent sequence $(X_t) \subseteq L^\infty$ such that $X_t \in \mathcal{A}_{0,t}$ for all $t \geq 0$ and*

$$X \geq \lim_{t\to\infty} X_t.$$

We denote by $\mathcal{A}_{0,\infty}$ the class of all predictably acceptable positions X.

Note that

$$\mathcal{A}_{0,\infty} \subseteq \mathcal{A}_0, \tag{40}$$

since $X \geq \lim_t X_t$ implies

$$\rho_0(X) \leq \rho_0(\lim_t X_t) \leq \liminf_t \rho_0(X_t) \leq 0$$

by monotonicity and by the Fatou property of the unconditional risk measure ρ_0.

Theorem 5.4. *The following properties are equivalent:*

1. $\bigcap_{t\geq 0} \mathcal{A}_t = L^\infty_+$.

2. $\mathcal{A}_{0,\infty} = \mathcal{A}_0$.

3. $\lim_{t\to\infty} \alpha_{0,t}^{\min}(Q) = \alpha_0^{\min}(Q)$ *for all* $Q \in \mathcal{Q}^*$.

4. $\lim_{t\to\infty} \alpha_t^{\min}(Q) = 0$ *Q-a.s. and in* $L^1(Q)$ *for all* $Q \in \mathcal{Q}^*$.

5. $\lim_{t\to\infty} \alpha_t^{\min}(Q) = 0$ *Q-a.s. and in* $L^1(Q)$ *for at least one* $Q \in \mathcal{Q}^*$.

6. $(\rho_t)_{t=0,1,\ldots}$ *is asymptotically safe.*

Proof. 1) $\Rightarrow$ 2) In view of (40) we have to show that property 1 implies $\mathcal{A}_0 \subseteq \mathcal{A}_{0,\infty}$. For $X \in \mathcal{A}_0$ define $X_t := -\rho_t(X)$. Then $X_t \in \mathcal{A}_{0,t}$ by property (24) of Lemma 4.6. Moreover, for $0 \le n \le t$ we have

$$X + \rho_t(X) \in \mathcal{A}_n,$$

since $\rho_t(X + \rho_t(X)) = 0$ and thus $\rho_n(X + \rho_t(X)) = 0$ for all $n \le t$ by time-consistency. Using the Fatou property of ρ_n we obtain

$$\rho_n(X + \rho_\infty(X)) \le \liminf_{t\to\infty} \rho_n(X + \rho_t(X)) = 0$$

for any $n \ge 0$, hence

$$X + \rho_\infty(X) \in \bigcap_{n\ge 0} \mathcal{A}_n = L_+^\infty.$$

Thus $\lim_t X_t = -\rho_\infty(X) \le X$ P-a.s., and this shows $X \in \mathcal{A}_{0,\infty}$.

2) $\Rightarrow$ 3) If $X \in \mathcal{A}_0 = \mathcal{A}_{0,\infty}$, then there exists a bounded convergent sequence $X_t \in \mathcal{A}_{0,t},\ t \ge 0$, such that $\lim_t X_t \le X$ P-a.s.. For any $Q \in \mathcal{Q}^*$ we have

$$\alpha_0^{\min}(Q) \ge \alpha_{0,\infty}^{\min}(Q) = \lim_{t\to\infty} \alpha_{0,t}^{\min}(Q) \ge \liminf_{t\to\infty} E_Q[-X_t] \ge E_Q[-X],$$

where we have used (39), the definition of $\alpha_{0,t}^{\min}(Q)$ and Lebesgue's convergence theorem for Q. But

$$\alpha_0^{\min}(Q) = \sup_{X\in\mathcal{A}_0} E_Q[-X],$$

and this implies the equality $\alpha_0^{\min}(Q) = \alpha_{0,\infty}^{\min}(Q)$.

3) $\Rightarrow$ 4) Note that property 3 in Theorem 4.5 implies

$$\alpha_0^{\min}(Q) = \alpha_{0,t}^{\min}(Q) + E_Q[\alpha_t^{\min}(Q)]$$

for $Q \in \mathcal{Q}^*$. Thus the convergence of $\alpha_{0,t}^{\min}(Q)$ to $\alpha_0^{\min}(Q)$ implies that the Q-expectation of $\alpha_t^{\min}(Q)$ converges to 0 as $t \to \infty$. This yields our claim since $(\alpha_t^{\min}(Q))_{t=0,1,\ldots}$ is a non-negative Q-supermartingale by Remark 4.8.

4) $\Rightarrow$ 5) This is obvious.

5) $\Rightarrow$ 6) Property 5) and Lemma 5.1 imply $\rho_\infty(X) \ge -X$ P-a.s..

6) $\Rightarrow$ 1) We have to show that the inequality $\rho_\infty(X) \ge -X$ implies $\bigcap_{t\ge 0} \mathcal{A}_t \subseteq L_+^\infty$. Indeed,

$$\begin{aligned} X \in \bigcap_{t\ge 0} \mathcal{A}_t \quad &\Rightarrow \quad \rho_t(X) \le 0 \quad \text{for all } t \ge 0 \\ &\Rightarrow \quad -X \le \rho_\infty(X) \le 0 \\ &\Rightarrow \quad X \in L_+^\infty. \end{aligned}$$

$\square$

Not every time-consistent sequence of conditional convex risk measures is asymptotically safe, as illustrated by the following example.

Example 5.5. Let P denote Lebesgue measure on the unit interval $\Omega := (0,1]$, and let $\mathcal{F}_t$ denote the finite σ-field generated by the t-th dyadic partition into the intervals $J_{t,k} := (k2^{-t}, (k+1)2^{-t}]$ $(k = 0, \ldots, 2^t - 1)$. Take a set $A \in \mathcal{F} := \sigma(\cup_{t \geq 0} \mathcal{F}_t)$ such that $P[A] > 0$ and $P[A^c \cap J_{t,k}] \neq 0$ for any dyadic interval, for example

$$A^c = \bigcup_{t=1}^{\infty} \bigcup_{k=1}^{2^t-1} U_{\varepsilon_t}(k2^{-t})$$

with $\varepsilon_t \in (0, 2^{-2t}]$. For any $t \geq 0$ we fix the same acceptance set

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; X \geq -I_A \right\}.$$

The corresponding conditional convex risk measure ρ_t is given by

$$\rho_t(X) = -\operatorname{ess\,sup} \left\{ m \in L_t^\infty \;\middle|\; m \leq X + I_A \right\}.$$

Note that ρ_t is indeed normalized since $m \leq 0$ for any $m \in L_t^\infty$ such that $m \leq I_A$, due to our assumption that $P[A^c \cap J_{t,k}] > 0$ for any atom of the σ-field $\mathcal{F}_t$. The corresponding penalty function is given by

$$\alpha_t^{\min}(Q) = E_Q[\, I_A \,|\, \mathcal{F}_t \,].$$

Since $\alpha_0^{\min}(Q) = Q[A]$, we have $\mathcal{Q}^* = \mathcal{M}^e(P)$, and in particular $\mathcal{Q}^* \neq \emptyset$ as required in Theorem 4.5.

The sequence $(\rho_t)_{t=0,1,\ldots}$ is *time-consistent*. Indeed, $\mathcal{A}_{t,t+s} = L_+^\infty(\mathcal{F}_{t+s})$ for $t \geq 0$ and $s \geq 0$, and so we have

$$\mathcal{A}_t = \mathcal{A}_{t+s} = \mathcal{A}_{t+s} + L_+^\infty(\mathcal{F}_{t+s}) = \mathcal{A}_{t+s} + \mathcal{A}_{t,t+s}$$

in accordance with property 2 of Theorem 4.5. On the other hand, the sequence $(\rho_t)_{t=0,1,\ldots}$ decreases to

$$\rho_\infty(X) = -\operatorname{ess\,sup} \left\{ m \in \bigcup_{t=0}^{\infty} L_t^\infty \;\middle|\; m \leq X + I_A \right\},$$

and

$$\rho_\infty(-I_A) = 0 \not\geq I_A,$$

i.e., the sequence $(\rho_t)_{t=0,1,\ldots}$ is not asymptotically safe. In order to illustrate the criteria of Theorem 5.4, note that

$$\bigcap_{t \geq 0} \mathcal{A}_t = \mathcal{A}_0 \neq L_+^\infty,$$

that

$$\alpha_{0,t}^{\min}(Q) \equiv 0 \neq \alpha_0^{\min}(Q),$$

and that

$$\alpha_\infty^{\min}(Q) = \lim_{t \to \infty} \alpha_t^{\min}(Q) = I_A \neq 0.$$

Remark 5.6. *Every dynamic conditional* coherent *risk measure that satisfies the conditions of Theorem 4.5 is asymptotically safe. Indeed, property 4) of Theorem 5.4 is clearly satisfied, since $\alpha_t^{\min}(Q) = 0$ for all $Q \in \mathcal{Q}^*$ as shown in the proof of Corollary 4.12.*

Lemma 5.7. *Asymptotic safety holds if the initial risk measure* ρ_0 *satisfies the condition*

$$\rho_0\left(E_{P^X}[X|\mathcal{F}_t]\right) \le \rho_0(X) \tag{41}$$

for any $X \in L^\infty$*, all* $t \ge 0$ *and for some measure* $P^X \approx P$*.*

Proof. Let us verify that condition (41) implies property 2) of Theorem 5.4. Indeed, for any $X \in \mathcal{A}_0$ the sequence $X_t := E_{P^X}[X|\mathcal{F}_t] \in L_t^\infty$ ($t \ge 0$) is uniformly bounded and P-a.s. convergent to X. Moreover, $X_t \in \mathcal{A}_{0,t}$ for all $t \ge 0$ since $\rho_0(X_t) \le \rho_0(X) \le 0$ due to (41). □

Remark 5.8. *Condition (41) is satisfied for* $P^X = P$ *if* ρ_0 *is* law-invariant *w.r.t.* P*; see Corollary 4.59 in [13].*

Let us now return to the question whether the asymptotic capital requirement ρ_∞ is exactly equal to the actual final loss.

Definition 5.9. *We say that the sequence* $(\rho_t)_{t=0,1,\dots}$ *is* asymptotically precise *if the limit* ρ_∞ *defined by (36) satisfies*

$$\rho_\infty(X) = -X$$

for any $X \in L^\infty$*.*

The following example shows that the sequence $(\rho_t)_{t=0,1,\dots}$ may be asymptotically safe without being asymptotically precise.

Example 5.10. In the situation of example 5.5 we now define the acceptance sets

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; X \ge 0 \right\}$$

and the corresponding conditional coherent risk measures

$$\rho_t(X) = -\operatorname{ess\,sup}\left\{ m \in L_t^\infty \;\middle|\; m \le X \right\}.$$

The sequence $(\rho_t)_{t=0,1,\dots}$ is time-consistent and satisfies the conditions of Theorem 4.5. Moreover, it is asymptotically safe due to Remark 5.6. But it is not asymptotically precise, since the set A defined in example 5.5 satisfies $\rho_t(I_A) = 0$ for all $t \ge 0$, hence $\rho_\infty(I_A) = 0 \ne -I_A$.

Let us now formulate a simple sufficient condition for asymptotic precision.

Proposition 5.11. *Suppose that the time-consistent sequence* $(\rho_t)_{t=0,1,\dots}$ *is asymptotically safe, and that the supremum in the robust representation of the initial risk measure* ρ_0 *is in fact a maximum, i.e.,*

$$\rho_0(X) = E_{Q^X}[-X] - \alpha_0^{\min}(Q^X) \tag{42}$$

for any $X \in L^\infty$ *and for some* $Q^X \approx P$*. Then the sequence* $(\rho_t)_{t=0,1,\dots}$ *is asymptotically precise.*

Proof. Let us fix $X \in L^\infty$. Since we are assuming asymptotic safety, it remains to show $\rho_\infty(X) \le -X$. Due to time-consistency as characterized by property 4) of Theorem 4.5, the process

$$U_t^Q := V_t^Q(X) + E_Q[X|\mathcal{F}_t], \qquad t \ge 0,$$

is a non-negative Q-supermartingale for any $Q \in \mathcal{Q}^*$. For $Q = Q^X$, we have $Q \in \mathcal{Q}^*$ and

$$U_0^Q = \rho_0(X) + \alpha_0(Q) + E_Q[X] = 0$$

due to (42). This implies $U_t^Q = 0$ for any $t > 0$ and

$$U_\infty^Q = \rho_\infty(X) + \alpha_\infty(Q) + X = 0,$$

hence $\rho_\infty(X) \le -X$, P-a.s.. □

6 Example: The entropic dynamic risk measure

Suppose that preferences are characterized by an exponential utility function $u(x) = 1 - \exp(-\gamma x)$ with $\gamma > 0$. At time t the conditional expected utility of a financial position $X \in L^\infty$ is then given by the $\mathcal{F}_t$-measurable random variable

$$U_t(X) = E[1 - e^{-\gamma X}|\mathcal{F}_t].$$

The set

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; U_t(X) \geq U_t(0) \right\} = \left\{ X \in L^\infty \;\middle|\; E[e^{-\gamma X}|\mathcal{F}_t] \leq 1 \right\}$$

satisfies the necessary conditions for an acceptance set, and hence we can define a sequence of conditional convex risk measures $(\rho_t)_{t=0,1,\dots}$ via

$$\begin{aligned} \rho_t(X) &:= \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; Y + X \in \mathcal{A}_t \right\} \\ &= \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; E[e^{-\gamma X}|\mathcal{F}_t] \leq e^{\gamma Y} \right\} \\ &= \frac{1}{\gamma} \log E[e^{-\gamma X}|\mathcal{F}_t]. \end{aligned} \tag{43}$$

We call a risk measure defined via (43) a *conditional entropic risk measure*. These risk measures are also discussed in section 4 of [11].

It is easy to see that a conditional entropic risk measure is continuous from above and hence representable for all $t \geq 0$. To identify the minimal penalty function in the robust representation we will need the notion of *conditional relative entropy*.

Recall that the *relative entropy* of $Q \in \mathcal{M}_1(P)$ with respect to P on the σ-field $\mathcal{F}_t$ is defined as

$$H_t(Q|P) := E_Q[\log Z_t] = E_P[Z_t \log Z_t] \in [0, \infty],$$

where Z_t denotes a density of Q with respect to P on $\mathcal{F}_t$. By Jensen's inequality we have $H_t(Q|P) \geq 0$, with equality iff $Q = P$ on $\mathcal{F}_t$.

Definition 6.1. *For $Q \in \mathcal{M}^e(P)$ we define the* conditional relative entropy *of Q with respect to P at time $t \geq 0$ as the $\mathcal{F}_t$-measurable random variable*

$$\begin{aligned} \widehat{H}_t(Q|P) &:= E_Q\left[\log \frac{Z_T}{Z_t} \,\middle|\, \mathcal{F}_t\right] \\ &= E_P\left[\frac{Z_T}{Z_t} \log \frac{Z_T}{Z_t} \,\middle|\, \mathcal{F}_t\right] I_{\{Z_t > 0\}} \end{aligned}$$

(note that $Z_t > 0$ Q-a.s.).

If Ω is a polish space, then for all $Q \in \mathcal{P}_t$ there exists a regular conditional probability of Q given $\mathcal{F}_t$, that is a probability kernel $Q_t : \Omega \times \mathcal{F} \to [0, 1]$ such that $Q_t(\cdot, B) = Q[B|\mathcal{F}_t]$ Q-a.s. for all $B \in \mathcal{F}$. In this case the conditional relative entropy can be calculated pointwise as the relative entropy of $Q_t(\omega, \cdot)$ with respect to $P_t(\omega, \cdot)$.

The next lemma is a version of Proposition 4 in [11]; see also [3].

Lemma 6.2. *For all $t \geq 0$ the conditional entropic risk measure ρ_t has the robust representations (4) with the minimal penalty function*

$$\alpha_t^{\min}(Q) = \frac{1}{\gamma} \widehat{H}_t(Q|P), \qquad Q \in \mathcal{P}_t.$$

Proof. To calculate the minimal penalty function we use formula (11):

$$\begin{aligned}
\alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X \in L^\infty} \left(E_Q[-X|\mathcal{F}_t] - \rho_t(X)\right) \\
&= \operatorname*{ess\,sup}_{X \in L^\infty} \left(E_Q[-X|\mathcal{F}_t] - \frac{1}{\gamma}\log E_P[e^{-\gamma X}|\mathcal{F}_t]\right) \\
&= \frac{1}{\gamma}\operatorname*{ess\,sup}_{Y \in L^\infty} \left(E_Q[-Y|\mathcal{F}_t] - \log E_P[e^{Y}|\mathcal{F}_t]\right), \qquad Q \in \mathcal{P}_t.
\end{aligned}$$

Now we use the conditional version of a well-known variational formula for relative entropy:

$$\operatorname*{ess\,sup}_{Y \in L^\infty} \left(E_Q[-Y|\mathcal{F}_t] - \log E_P[e^{Y}|\mathcal{F}_t]\right) = \widehat{H}_t(Q|P)$$

for $Q \in \mathcal{P}_t$. This follows as in the unconditional case; see, e.g., Lemma 3.29 in [13] and Lemma 2 in [11]. $\square$

An easy calculation shows that the sequence of conditional entropic risk measures $(\rho_t)_{t=0,1,\dots}$. is *time-consistent*:

$$\begin{aligned}
\rho_t(X) &= \frac{1}{\gamma}\log E_P[e^{-\gamma X}|\mathcal{F}_t] \\
&= \frac{1}{\gamma}\log E_P[e^{-\gamma(\frac{1}{\gamma}\log E_P[e^{-\gamma X}|\mathcal{F}_{t+1}]}|\mathcal{F}_t] \\
&= \rho_t(-\rho_{t+1}(X)) \qquad t \geq 0.
\end{aligned}$$

Martingale convergence w.r.t. P shows that the sequence $(\rho_t)_{t=0,1,\dots}$ is asymptotically precise. Moreover, the set

$$\mathcal{Q}^* = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; H_T(Q|P) < \infty \right\}$$

is obviously not empty, and so we could apply Theorem 4.5 and Theorem 5.4. But let us rather illustrate the main criteria for time-consistency and asymptotic precision by verifying them directly in our present case.

To this end we introduce the "one-step" conditional entropy

$$\widehat{H}_{t,t+1}(Q|P) := E_Q\left[\log \frac{Z_{t+1}}{Z_t} \mid \mathcal{F}_t\right],$$

i.e., the conditional entropy at time t if Q and P are regarded as measures on $\mathcal{F}_{t+1}$. Clearly,

$$\widehat{H}_t(Q|P) = \widehat{H}_{t,t+1}(Q|P) + E_Q[\widehat{H}_{t+1}(Q|P)|\mathcal{F}_t],$$

and this illustrates property 3 in Theorem 4.5. Let us now prove directly the basic supermartingale property 4 of the process

$$V_t^Q(X) = \rho_t(X) + \alpha_t^{\min}(Q), \qquad t \geq 0$$

in the entropic case. Moreover, we clarify the structure of the corresponding Doob decomposition, i.e., we identify the increasing predictable process $(A_t^Q(X))$ such that

$$V_t^Q(X) - A_t^Q(X), \qquad t \geq 0$$

is a martingale under Q.

Theorem 6.3. *For any $Q \in \mathcal{M}_1(P)$ such that $H_T(Q|P) < \infty$ and for any $X \in L^\infty$ the process*

$$V_t^Q(X) = \frac{1}{\gamma} \log E_P[e^{-\gamma X}|\mathcal{F}_t] + \frac{1}{\gamma}\widehat{H}_t(Q|P), \qquad t \geq 0$$

is a supermartingale under Q. Its Doob decomposition is given by the predictable increasing process

$$A_t^Q(X) := \frac{1}{\gamma}\sum_{s=0}^{t-1} \widehat{H}_{s,s+1}(Q|P^X), \qquad t \geq 0, \tag{44}$$

where $P^X \in \mathcal{M}^e(P)$ is defined by

$$\frac{dP^X}{dP} := \frac{e^{-\gamma X}}{E_P[e^{-\gamma X}]}.$$

The process $(V_t^Q(X))_{t=0,1,\dots}$ is in fact a martingale iff $Q = P^X$. Moreover, $V_T^Q(X) = -X$ for $T < \infty$, and

$$\lim_{t\to\infty} \rho_t(X) = \lim_{t\to\infty} V_t^Q(X) = -X \quad Q\text{-a.s. and in } L^1(Q)$$

for $T = \infty$. In particular,

$$\lim_{t\to\infty} \alpha_t^{\min}(Q) = 0 \quad Q\text{-a.s. and in } L^1(Q).$$

Proof. Since $P^X \approx P$, we can write

$$\begin{aligned}\widehat{H}_t(Q|P) &= \widehat{H}_t(Q|P^X) + E_Q\left[\log \frac{e^{-\gamma X}}{E_P[e^{-\gamma X}|\mathcal{F}_t]} \,\Big|\, \mathcal{F}_t\right] \\ &= \widehat{H}_t(Q|P^X) - \gamma E_Q[X|\mathcal{F}_t] - \gamma\rho_t(X),\end{aligned}$$

and this implies

$$V_t^Q(X) = E_Q[-X|\mathcal{F}_t] + \frac{1}{\gamma}\widehat{H}_t(Q|P^X).$$

Lemma 6.4, applied to P^X instead of P, shows that $(V_t^Q(X))_{t=0,1,\dots}$ is a supermartingale under Q. It also shows that the increasing predictable process $(A_t^Q(X))$ defined by (44) is such that

$$V_t^Q(X) - A_t^Q(X), \qquad t \geq 0$$

is a Q-martingale. In particular, $(V_t^Q(X))_{t=0,1,\dots}$ is a Q-martingale iff $\widehat{H}_{t,t+1}(Q|P^X) = 0$ Q-a.s. for all $t \geq 0$, and this is the case iff $Q = P^X$ on $\mathcal{F} = \mathcal{F}_T$. □

The following lemma was used in the preceding proof.

Lemma 6.4. *For any $Q \in \mathcal{M}_1(P)$ such that $H_T(Q|P) < \infty$, the process of conditional relative entropies*

$$\widehat{H}_t(Q|P), \quad t \geq 0$$

is a supermartingale under Q, and it is in fact a potential in the sense that $\widehat{H}_T(Q|P) = 0$ for $T < \infty$ and

$$\lim_{t\to\infty} \widehat{H}_t(Q|P) = 0 \quad Q\text{-a.s. and in } L^1(Q) \tag{45}$$

for $T = \infty$. Its Doob decomposition is given by the predictable increasing process

$$A_t := \sum_{s=0}^{t-1} \widehat{H}_{s,s+1}(Q|P), \qquad t \geq 0, \tag{46}$$

i.e., the process $\widehat{H}_t(Q|P) + A_t$, $t \geq 0$ is a martingale under Q.

Proof. We have

$$\begin{aligned} \widehat{H}_{t+1}(Q|P) &= E_Q\left[\log \frac{Z_T}{Z_{t+1}} \,\middle|\, \mathcal{F}_{t+1}\right] \\ &= E_Q\left[\log \frac{Z_T}{Z_t} \,\middle|\, \mathcal{F}_{t+1}\right] - \log \frac{Z_{t+1}}{Z_t}, \end{aligned}$$

hence

$$E_Q[\,\widehat{H}_{t+1}(Q|P)\,|\,\mathcal{F}_t\,] = \widehat{H}_t(Q|P) - \widehat{H}_{t,t+1}(Q|P).$$

Since $\widehat{H}_{t,t+1}(Q|P) \geq 0$ Q-a.s. by Jensen's inequality, it follows that $(\widehat{H}_t(Q|P))_{t\geq 0}$ is a supermartingale under Q, and that the predictable increasing process in its Doob decomposition is given by (46). Moreover, (45) follows from

$$H_T(Q|P) = H_t(Q|P) + E_Q[\widehat{H}_t(Q|P)],$$

since $\lim_{t\to\infty} H_t(Q|P) = H_T(Q|P)$. Indeed, we have $H_t(Q|P) \leq H_T(Q|P)$ by Jensen's inequality, and the convergence follows by Fatou's lemma applied to the P-a.s. convergent sequence $(u(Z_t))_{t=0,1,\ldots}$ with $u(x) = x \log x$. $\square$

References

[1] Artzner, P., Delbaen, F., Eber, J., Heath, D.; *Coherent measures of risk.* Mathematical Finance 9(3), 203-228 (1999).

[2] Artzner, P., Delbaen, F., Eber, J., Heath, D., Ku, H.; *Coherent multiperiod risk adjusted values and Bellman's principle.* Forthcoming in Annals of Operations Research. (2004).

[3] Bion-Nadal, J.; *Conditional risk measure and robust representation of convex conditional risk measures.* CMAP preprint # 557 (2004).

[4] Bion-Nadal, J.; *Dynamic risk measuring: discrete time in a context of uncertainty, and continuous time on a probability space.* CMAP preprint # 596 (2006).

[5] Burgert, C.; *Darstellungssätze fuer statische und dynamische Risikomaße mit Anwendungen.* Dissertation, U. Freiburg (2005).

[6] Cheridito, P., Delbaen, F., Kupper, M.; *Dynamic monetary risk measures for bounded discrete-time processes.* Electronic Journal of Probability, Volume 11(3), 57-106 (2006).

[7] Delbaen, F.; *Coherent risk measures on general probability spaces.* Essays in Honour of Dieter Sondermann, Springer Verlag (2002).

[8] Delbaen, F.; *Coherent Risk Measures.* Cattedra Galileiana. Scuola Normale Superiore di Pisa (2000).

[9] Delbaen, F.; *The structure of m-stable sets and in particular of the set of risk-neutral measures.* Preprint, ETH Zürich (2003).

[10] Detlefsen, K; *Bedingte und mehrperiodige Risikomaße.* Diploma Thesis, HU Berlin (2003).

[11] Detlefsen, K., Scandolo, G.; *Conditional and dynamic convex risk measures.* Finance and Stochastics 9(4), 539-561 (2005).

[12] Föllmer, H., Kramkov, D.; *Optional decompositions under constraints.* Probability Theory and Related Fields 109, 1-25 (1997).

[13] Föllmer, H., Schied, A.; *Stochastic Finance - An Introduction in Discrete Time, 2nd Edition.* De Gruyter Studies in Mathematics (2004).

[14] Frittelli, M., Rosazza Gianin, E.; *Putting order in risk measures.* Journal of Banking Finance 26, 1473-1486 (2002).

[15] Frittelli, M., Rosazza Gianin, E.; *Dynamic convex risk measures.* New risk measures in the 21st Century, Wiley Finance, 227-248 (2004).

[16] Klöppel, S., Schweizer, M.; *Dynamic utility indifference valuation via convex risk measures.* Preprint, ETH Zürich (2005).

[17] Penner, I.; *Dynamic risk measures.* In preparation.

[18] Riedel, F.; *Dynamic coherent risk measures.* Stochastic Processes and their Applications, 112(2), 185-200 (2004).

[19] Tutsch, S.; *Konsistente und konsequente dynamische Risikomaße und das Problem der Aktualisierung.* Dissertation, HU Berlin (2006).

[20] Weber, S.; *Distribution-invariant risk measures, information, and dynamic consistency.* Mathematical Finance 16(2), 419-441 (2006).