

CONTRIBUTIONS TO MATHEMATICAL ECONOMICS

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Chapter 12

HEDGING OF NON-REDUNDANT CONTINGENT CLAIMS

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1. Introduction

Financial economics has undoubtedly achieved some of its most striking results in the theory of option pricing, starting with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). Another approach based on arbitrage methods was introduced independently by Ross; for basic results along this line cf. Cox and Ross (1976a). A further important development is due to Harrison and Kreps (1979): They analyzed the valuation of contingent claims in terms of martingale theory and thus clarified the mathematical structure of the problem.

There is, however, what Hakansson (1979, p. 722) called *The Catch 22 of Option Pricing*: "A security can be unambiguously valued by reference of the other securities in a perfect market if and only if the security being valued is redundant in that market." Indeed all preference-independent valuation formulas assume that the asset to be valued is *attainable*, i.e., that it can be perfectly duplicated by a dynamically adjusted portfolio of the existing assets. "But if this is the case, the option adds nothing new to the market and no social welfare can arise – the option is perfectly redundant So we find ourselves in the awkward position of being able to derive unambiguous values only for redundant assets and unable to value options which do have social value." [Hakansson (1979), p. 723].¹

¹Of course, as Hakansson points out, one can still use arbitrage arguments to put bounds on the value of a non-redundant asset as done, e.g., in Harrison and Kreps (1979) and Egle and Trautmann (1981).

In this paper, our purpose is to extend the martingale approach of Harrison and Kreps (1979) to contingent claims which are *non-redundant*. We are less concerned here with valuation formulas than with how to use the existing assets for an optimal hedge against the claim. To this end we introduce a class of admissible portfolio strategies which generate a given contingent claim at some terminal time T . Due to the underlying martingale assumptions, the expected terminal cost does not depend on the specific choice of the strategy. It is therefore natural to look for admissible strategies which *minimize risk* in a sequential sense. We show in Section 3 that this problem has a unique solution where the risk is reduced to what we call the intrinsic risk of the claim. This risk-minimizing strategy is *mean-self-financing*, i.e., the corresponding cost process is a martingale. A claim is attainable if and only if its intrinsic risk is zero. In that particular case, the risk-minimizing strategy becomes *self-financing*, i.e., the cost process is constant, and we obtain the usual arbitrage value of the claim. In Section 4 we study the dependence on the hedger's subjective beliefs: It is shown how the strategy changes under an absolutely continuous change of the underlying martingale measure.

In Section 5 we illustrate our results by computing explicitly the intrinsic risk and the risk-minimizing strategy for a call option where the underlying stock price follows a two-sided jump process. Contrary to jump processes already studied in the literature [see, e.g., Cox and Ross (1976b)], this model is not complete, and a typical call is non-redundant.

It is a pleasure to thank M. Schweizer who worked out a large part of the first example in Section 5; cf. Schweizer (1984).

2. Basic definitions

Let $(\Omega, \mathcal{F}, P^*)$ be a probability space, and let $(\mathcal{F}_t)_{0 \leq t \leq T}$ denote a right-continuous family of σ -algebras contained in \mathcal{F} ; \mathcal{F}_t is interpreted as the collection of events which are observable up to time t . A *stochastic process* $Z = (Z_t)_{0 \leq t \leq T}$ is given by a measurable function Z on $\Omega \times [0, T]$. Z is called *adapted* if Z_t is \mathcal{F}_t -measurable for each $0 \leq t \leq T$; it is called *predictable* if it is measurable with respect to the σ -algebra \mathcal{P} on $\Omega \times [0, T]$ which is generated by the adapted processes with left-continuous paths. We refer to Metivier (1982) for further details.

The evolution of *stock prices* will be described by a stochastic process $X = (X_t)_{0 \leq t \leq T}$ which is adapted and whose paths are right-continuous with limits X_{t-} from the left.² The process $Y = (Y_t)_{0 \leq t \leq T}$ of *bond prices* is fixed to

²For simplicity we only consider the one-dimensional case. The extension to an n -dimensional stock process $X = (X^1, \dots, X^n)$ is straightforward if the components of X are mutually orthogonal; see, e.g., Schweizer (1984). For difficulties which can arise otherwise, see Müller (1984).

be $Y_t = 1$.³ We assume that P^* is a *martingale measure* in the sense of Harrison and Kreps (1979); i.e., we assume

$$E^*[X_T^2] < \infty, \quad (1)$$

and

$$X_t = E^*[X_T | \mathcal{F}_t], \quad 0 \leq t \leq T, \quad (2)$$

where $E^*[\cdot | \mathcal{F}_t]$ denotes the conditional expectation under P^* with respect to the σ -algebra \mathcal{F}_t . This means that X is a *square-integrable martingale* under P^* .⁴ Let $\langle X \rangle = (\langle X \rangle_t)_{0 \leq t \leq T}$ be the corresponding *Meyer process*, i.e., the unique predictable process with $\langle X \rangle_0 = 0$ and right-continuous increasing paths such that $X^2 - \langle X \rangle$ is a martingale;⁵ cf. Metivier (1982). We denote by P_X^* the finite measure on $(\Omega \times [0, T], \mathcal{P})$ given by

$$P_X^*[A] = E^*\left[\int_0^T I_A(t, \omega) d\langle X \rangle_t(\omega)\right],$$

and by $L^2(P_X^*)$ the class of predictable processes Z which, viewed as \mathcal{P} -measurable functions on $\Omega \times [0, T]$, are square-integrable with respect to P_X^* . Two such processes will be considered as equal if they coincide P_X^* -almost surely.

A *trading strategy* will be of the form $\varphi = (\xi, \eta)$ where $\xi = (\xi_t)_{0 \leq t \leq T}$ and $\eta = (\eta_t)_{0 \leq t \leq T}$ describe the successive amounts invested into the stock and into the bond. Thus,

$$V_t = \xi_t X_t + \eta_t \quad (3)$$

is the *value of the portfolio* at time t . We need the following technical assumptions.

Definition 1. $\varphi = (\xi, \eta)$ is called a *strategy* if

- (a) ξ is a predictable process, and $\xi \in L^2(P_X^*)$,
- (b) η is adapted,
- (c) $V = \xi X + \eta$ has right-continuous paths and satisfies $V_t \in L^2(P^*)$, $0 \leq t \leq T$.

³As shown by Harrison and Kreps (1979), there is no loss of generality in making this assumption, since their method of standardizing the bond process to unity allows also stochastic interest rates.

⁴In this paper we leave aside problems of viability, i.e., we only consider martingale measures and do not study the case where the underlying probability distribution is only assumed to admit an equivalent martingale measure. For the relationship between viability, absence of arbitrage opportunities, and the existence of an equivalent martingale measure, we refer to Harrison and Kreps (1979), Harrison and Pliska (1981), and Müller (1984).

⁵E.g., if X is the Brownian Motion with variance σ^2 then $\langle X \rangle_t$ equals $\sigma^2 t$.

Condition (a) allows to calculate the *accumulated gain* obtained from the stock price fluctuation up to time t as the stochastic integral

$$\int_0^t \xi_s dX_s, \quad 0 \leq t \leq T. \quad (4)$$

For fixed t , the gain has expectation $E^*[\int_0^t \xi_s dX_s] = 0$ and variance

$$E^* \left[\left(\int_0^t \xi_s dX_s \right)^2 \right] = E^* \left[\int_0^t \xi_s^2 d\langle X \rangle_s \right]. \quad (5)$$

Viewed as a stochastic process, (4) defines a square-integrable martingale with right-continuous paths. The *accumulated cost* of the strategy up to time t can now be defined as

$$C_t = V_t - \int_0^t \xi_s dX_s. \quad (6)$$

$V = (V_t)_{0 \leq t \leq T}$ and $C = (C_t)_{0 \leq t \leq T}$ are adapted processes with right-continuous paths; they are called the *value process* and the *cost process*.

Remark 1. Consider a *simple strategy* where stock trading only occurs at finitely many times, $0 \leq t_0 < \dots < t_n \leq T$, i.e.,

$$\xi_t(\omega) = \sum_i \alpha_i(\omega) I_{(t_i, t_{i+1}]}(t), \quad (7)$$

where α_i is \mathcal{F}_{t_i} -measurable. Equation (7) means that the amount ξ_t is fixed just before the portfolio is actually changed, in accordance with predictability. Since

$$\int_0^t \xi_s dX_s = \sum_{j < i} \alpha_j (X_{t_{j+1}} - X_{t_j}) + \alpha_i (X_t - X_{t_i}),$$

for $t \in (t_i, t_{i+1}]$, the cost process (6) is given by

$$C_t = \eta_t + \alpha_i X_t - \sum_{j < i} \alpha_j (X_{t_{j+1}} - X_{t_j}), \quad (8)$$

for $t \in (t_i, t_{i+1}]$. Since η is only assumed to be adapted, not necessarily predictable, the value of η_t can be fixed after observing the situation at time t . In particular, η can be used to keep the value process V on a certain desired

level V^* . For the class of mean-self-financing admissible strategies introduced below, this level V^* will not depend on the specific choice of ξ . In that case we can justify definition (6), which is intuitive for simple strategies, by a passage to the limit. Indeed, for any predictable process $\xi \in L^2(P_X^*)$ there is a sequence of simple predictable processes ξ^n of the form (7) which converges to ξ in $L^2(P_X^*)$, and (5) together with a maximal inequality for martingales implies that

$$\sup_{0 \leq t \leq T} |C_t^n - C_t| = \sup_{0 \leq t \leq T} \left| \int_0^t \xi^n dX - \int_0^t \xi dX \right|$$

converges to zero in $L^2(P^*)$.

Definition 2. A strategy $\varphi = (\xi, \eta)$ is called *mean-self-financing* if the corresponding cost process $C = (C_t)_{0 \leq t \leq T}$ is a martingale.

Remark 2. A strategy $\varphi = (\xi, \eta)$ is called *self-financing* if the cost process has constant paths, i.e., if

$$C_t = C_0, \quad P^*\text{-a.s.}, \quad 0 \leq t \leq T. \quad (9)$$

Any self-financing strategy is clearly mean-self-financing. For a self-financing strategy, the value process is of the form

$$V_t = C_0 + \int_0^t \xi_s dX_s, \quad 0 \leq t \leq T, \quad (10)$$

hence a square-integrable martingale. Self-financing strategies are the key tool in the analysis of option pricing in "complete" security markets; cf. Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Müller (1984). But in many situations security markets are *incomplete* in the sense that there may not be any self-financing strategy which allows to realize a pre-assigned terminal value $V_T = H$. This is the reason why we introduce the broader concept of a mean-self-financing strategy. As stated in the following lemma, the value process of a mean-self-financing strategy is again a martingale. But in general we cannot expect that this martingale can be represented as a stochastic integral with respect to X as in (10).

Lemma 1. A strategy is mean-self-financing if and only if its value process is a square-integrable martingale.

Proof. The properties of a strategy as defined in Definition 1 imply that the process of accumulated gains

$$\int_0^t \xi_s dX_s, \quad 0 \leq t \leq T,$$

is a square-integrable martingale, and that V_t , $0 \leq t \leq T$, is square-integrable. Thus Lemma 1 is clear from (6). \square

3. The intrinsic risk of contingent claims

Let us fix a contingent claim $H \in \mathcal{L}^2(P^*)$. For example, H could be a call option of the form $H = (X_T - C)^+$.

Definition 3. A strategy is called *admissible* (with respect to H) if its value process has terminal value

$$V_T = H, \quad P^*\text{-a.s.} \quad (11)$$

For any admissible strategy $\varphi = (\xi, \eta)$, the terminal cost is given by

$$C_T = H - \int_0^T \xi_s dX_s. \quad (12)$$

In particular, the expected value,

$$E^*[C_T] = E^*[H],$$

does not depend on the specific choice of the strategy as long as it is admissible. We are now going to analyse which admissible strategies have minimal risk in a suitable sense. As a first step in that direction, let us determine all admissible strategies which

$$\text{minimize the variance } E^*[(C_T - E^*[H])^2]; \quad (13)$$

the second step will consist in replacing (13) by a sequential procedure.

In view of (13), let us write the claim H in the form

$$H = E^*[H] + \int_0^T \xi_s^* dX_s + H^*, \quad (14)$$

with $\xi^* \in L^2(P_X^*)$ where $H^* \in L^2(P^*)$ has expectations zero and is orthogo-

nal to the space $\{\int_0^t \xi_s dX_s | \xi \in L^2(P_X^*)\}$ of stochastic integrals with respect to X ; cf. Metivier (1982) for the existence and uniqueness of this representation.

Theorem 1. An admissible strategy $\varphi = (\xi, \eta)$ has minimal variance

$$E^*[(C_T - E^*[H])^2] = E^*[(H^*)^2], \quad (15)$$

if and only if $\xi = \xi^*$.

Proof. For an admissible strategy $\varphi = (\xi, \eta)$ we have, by (14),

$$\begin{aligned} C_T &= H - \int_0^T \xi_s dX_s \\ &= E^*[H] + \int_0^T (\xi_s^* - \xi_s) dX_s + H^*. \end{aligned}$$

Since H^* is orthogonal to the stochastic integral on the right side, we obtain

$$\begin{aligned} E^*[(C_T - E^*[H])^2] &= E^*\left[\left(\int_0^T (\xi_s^* - \xi_s) dX_s\right)^2\right] + E^*[(H^*)^2] \\ &= E^*\left[\int_0^T (\xi_s^* - \xi_s)^2 d\langle X \rangle_s\right] + E^*[(H^*)^2]. \end{aligned}$$

Thus, the minimum $E^*[(H^*)^2]$ is assumed if and only if $\xi = \xi^*$ in $L^2(P_X^*)$. \square

So far we can draw no conclusion concerning the process $\eta = (\eta_t)_{0 \leq t \leq T}$, except that it must make the strategy admissible, i.e.,

$$\eta_T = H - \xi_T^* X_T. \quad (16)$$

Example 1. One natural idea is to use a *self-financing* strategy during the interval $[0, T)$, and to make up the balance at the end, i.e., to put

$$\eta_t = E^*[H] + \int_0^t \xi_s^* dX_s - \xi_t^* X_t, \quad 0 \leq t \leq T, \quad (17)$$

in addition to (16) so that

$$\begin{aligned} C_t &= E^*[H] \quad \text{for } 0 \leq t < T, \\ &= E^*[H] + H^* \quad \text{for } t = T. \end{aligned} \quad (18)$$

This strategy would indeed realize the minimal variance $E^*[(C_T - E^*[H])^2] = E^*[(H^*)^2]$ in (15).

We are now going to show that a sharper formulation of problem (13) determines a unique admissible strategy $\varphi^* = (\xi^*, \eta^*)$ which has *minimal risk* in a sequential sense, and which will be different from the strategy considered in Example 1.

Consider any strategy $\varphi = (\xi, \eta)$. Just before time $t < T$ we have accumulated the cost C_t . The strategy tells us how to proceed at and beyond time t . In particular, it fixes the present cost C_t and determines the remaining cost $C_T - C_t$. Let us measure the *remaining risk* by

$$R_t^\varphi = E^*[(C_T - C_t)^2 | \mathcal{F}_t]. \quad (19)$$

In view of (19), we might want to compare φ to any other strategy $\tilde{\varphi}$ which coincides with φ at all times $< t$ and which leads to the same terminal value V_T . Let us call such a $\tilde{\varphi}$ an *admissible continuation* of φ at time t .

Definition 4. A strategy φ is called *risk-minimizing* if φ at any time minimizes the remaining risk, i.e., for any $0 \leq t < T$, we have

$$R_t^\varphi \leq R_t^{\tilde{\varphi}}, \quad P^*\text{-a.s.}, \quad (20)$$

for every admissible continuation $\tilde{\varphi}$ of φ at time t .

Remarks 3. (1) Any self-financing strategy φ is clearly risk-minimizing since $R_t^\varphi \equiv 0$.

(2) Suppose that $\varphi = (\xi, \eta)$ is a risk-minimizing strategy which is also admissible. Then φ is in particular a solution of problem (13). In fact, (20) with $t = 0$ implies that φ minimizes

$$E^*[(C_T - C_0)^2] = E^*[(C_T - E^*[C_T])^2] + (E^*[C_T] - C_0)^2.$$

Thus, ξ minimizes the variance of C_T and this implies $\xi = \xi^*$ according to Theorem 1. In addition we obtain the condition

$$\eta_0 = C_0 - \xi_0^* X_0 = E^*[H] - \xi_0^* X_0.$$

The sequential version of this second fact will be provided by Theorem 2 below.

Lemma 2. An admissible risk-minimizing strategy is mean-self-financing.

Proof. Consider a strategy $\varphi = (\xi, \eta)$ and a fixed time $0 \leq t_0 \leq T$. Define $\tilde{\eta}_t = \eta_t$, $t < t_0$, and

$$\tilde{\eta}_t \equiv \tilde{C}_t + \int_0^t \xi_s dX_s - \xi_t X_t, \quad t_0 \leq t \leq T,$$

where $(\tilde{C}_t)_{0 \leq t \leq T}$ denotes a right-continuous version of the martingale

$$\tilde{C}_t = E^*[C_T | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Then $\tilde{\varphi} \equiv (\xi, \tilde{\eta})$ is an admissible continuation of φ at time t_0 , and its remaining cost is given by

$$\tilde{C}_T - \tilde{C}_{t_0} = (C_T - C_{t_0}) + (C_{t_0} - \tilde{C}_{t_0}).$$

This implies

$$E^*[(C_T - C_{t_0})^2 | \mathcal{F}_{t_0}] = E^*[(\tilde{C}_T - \tilde{C}_{t_0})^2 | \mathcal{F}_{t_0}] + (\tilde{C}_{t_0} - C_{t_0})^2.$$

Thus, φ is risk-minimizing only if $C_{t_0} = \tilde{C}_{t_0}$ P^* -a.s. for any $t_0 \leq T$, i.e., if φ is mean-self-financing. \square

In order to formulate our final result, let us denote by $V^* = (V_t^*)$ a right-continuous version of the square-integrable martingale

$$V_t^* = E^*[H | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (21)$$

To the representation (14) of the claim H corresponds the following sequential representation of V^* :

$$V_t^* = V_0^* + \int_0^t \xi_s^* dX_s + N_t^*, \quad (22)$$

where $N_t^* = E^*[H^* | \mathcal{F}_t]$ is a square-integrable martingale with zero expectations which is orthogonal to X in the following sense.

Remark 4. Two square-integrable martingales M_1 and M_2 are called *orthogonal* if their product $M_1 M_2$ is again a martingale, and this is equivalent to the condition

$$\langle M_1, M_2 \rangle \equiv \frac{1}{2}(\langle M_1 + M_2 \rangle - \langle M_1 \rangle - \langle M_2 \rangle) = 0. \quad (23)$$

In view of (26) below the process $R^* = (R_t^*)$, defined as a right-continuous version of

$$R_t^* = E^*[(N_T^* - N_t^*)^2 | \mathcal{F}_t] = E^*[\langle N^* \rangle_T | \mathcal{F}_t] - \langle N^* \rangle_t, \quad (24)$$

will be called the *intrinsic risk process* of the claim H . The expectation $E^*[R_0^*]$ coincides with the minimal variance calculated in (15); let us call it the *intrinsic risk* of the claim.

Theorem 2. *There exists a unique admissible strategy φ^* which is risk-minimizing, namely*

$$\varphi^* = (\xi^*, V^* - \xi^* X). \quad (25)$$

For this strategy, the remaining risk at any time $t \leq T$ is given by

$$R_t^{\varphi^*} = R_t^*, \quad P^*\text{-a.s.} \quad (26)$$

Proof. (1) Since the value process of the strategy (25) is given by the martingale V^* in (21), the strategy is admissible. If φ is an admissible continuation of φ^* at time t , then its cost process satisfies

$$C_T - C_t = \int_t^T (\xi_s^* - \xi_s) dX_s + N_T^* - N_t^* + V_t^* - V_t,$$

due to (22). The orthogonality of X and N^* implies

$$\begin{aligned} E^*[(C_T - C_t)^2 | \mathcal{F}_t] &= E^*\left[\int_t^T (\xi_s^* - \xi_s)^2 d\langle X \rangle_s | \mathcal{F}_t\right] \\ &\quad + R_t^* + (V_T^* - V_t)^2, \end{aligned} \quad (27)$$

and in particular (26). This shows that φ^* is risk-minimizing.

(2) Let $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ be any admissible strategy which is risk-minimizing. This implies $\tilde{\xi} = \xi^*$, as pointed out in Remark 3. By Lemma 2, $\tilde{\varphi}$ is mean-self-financing, and so its value process \tilde{V} is a martingale. Since $\tilde{\varphi}$ is admissible, \tilde{V} must coincide with the martingale V^* defined in (21), and this implies $\tilde{\eta}^* = V^* - \xi^* X$. Thus, a risk-minimizing admissible strategy is uniquely determined by (25). \square

As a special case of Theorem 2 we obtain the following characterization of attainable contingent claims [cf. Harrison and Kreps (1979)].

Corollary 1. *The following statements are equivalent:*

- (1) *The risk-minimizing admissible strategy φ^* is self-financing.*
- (2) *The intrinsic risk of the contingent claim H is zero.*
- (3) *The contingent claim H is attainable, i.e.,*

$$H = E^*[H] + \int_0^T \xi_s^* dX_s, \quad P^*\text{-a.s.} \quad (28)$$

Proof. φ^* is self-financing if and only if the remaining risk at any time is given by $R_t^{\varphi^*} = 0$. By (26), this is equivalent to $R_t^* = 0$, $0 \leq t \leq T$, and this means that the intrinsic risk is zero. Remark 4 shows that $R_0^* = 0$ is equivalent to the condition $N_t^* = 0$, $0 \leq t \leq T$, in the representation (22), and this is equivalent to (28). \square

4. Changing the measure

Let us now see how the risk-minimizing strategy is affected by an absolutely continuous change of the underlying martingale measure.

Let P be any martingale measure which is absolutely continuous with respect to P^* . Thus, the process X is again a square-integrable martingale under P . Let us also assume that our contingent claim $H \in L^2(P^*)$ is again square-integrable under P . Then the representation (22) and Theorem 2, applied to P instead of P^* , show that the risk-minimizing strategy under P is given by $\varphi = (\xi, V - \xi X)$, with

$$V_t = E[H | \mathcal{F}_t] = V_0 + \int_0^t \xi_s dX_s + N_t. \quad (29)$$

Let us now describe how ξ is related to ξ^* . In order to simplify the exposition we add the technical assumption

$$\xi^* \in L^2(P_X). \quad (30)$$

While X is again a martingale under P , the martingale property of (N_t^*) in (22) may be lost. In general, we have the Doob decomposition

$$N^* = M + A, \quad (31)$$

where $M = (M_t)$ is a martingale under P and $A = (A_t)$ is a predictable process with $A_0 = 0$ and with right-continuous paths of bounded variation; cf. Metivier (1982). Let us introduce the predictable processes ξ^M and ξ^A defined by

$$\begin{aligned} \langle M, X \rangle_t &= \int_0^t \xi_s^M d\langle X \rangle_s \quad \text{and} \quad \langle M^A, X \rangle_t = \int_0^t \xi_s^A d\langle X \rangle_s, \\ 0 &\leq t \leq T, \end{aligned}$$

where M^A denotes a right-continuous version of the martingale

$$M_t^A = E[A_T | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Theorem 3. *The risk-minimizing strategy under P is given by $\varphi = (\xi, V - \xi X)$ with*

$$\xi = \xi^* + \xi^M + \xi^A, \quad (32)$$

and

$$V_t = V_t^* + M_t^A - A_t, \quad 0 \leq t \leq T. \quad (33)$$

Proof. Consider the representation (14) of the contingent claim, i.e.,

$$H = E^*[H] + \int_0^T \xi_s^* dX_s + H^*, \quad (34)$$

P^* -a.s., hence P -a.s.. Since

$$H^* = N_T^* = M_T + A_T, \quad P\text{-a.s.},$$

(30) and (34) imply $E[H] = E^*[H] + E[A_T]$, hence

$$\begin{aligned} H &= E[H] + \int_0^T \xi_s^* dX_s + M_T + A_T - E[A_T] \\ &= E[H] + \int_0^T (\xi_s^* + \xi_s^M + \xi_s^A) dX_s + \tilde{N}_T, \quad P\text{-a.s.}, \end{aligned} \quad (35)$$

where we put

$$\tilde{N}_t \equiv M_t + M_t^A - \int_0^t (\xi_s^M + \xi_s^A) dX_s - E[A_T], \quad 0 \leq t \leq T.$$

By the definition of ξ^M and ξ^A ,

$$\begin{aligned} \langle \tilde{N}, X \rangle_t &= \langle M, X \rangle_t + \langle M^A, X \rangle_t - \int_0^t (\xi_s^M + \xi_s^A) d\langle X \rangle_s \\ &= 0, \quad 0 \leq t \leq T, \end{aligned}$$

i.e., \tilde{N} is orthogonal to X . Thus, (32) follows from (35), and \tilde{N} coincides with

the orthogonal martingale N introduced in (29). Moreover, (35) implies that

$$\begin{aligned} V_t^* + M_t^A - A_t &= E^*[H] + \int_0^t \xi_s^* dX_s + M_t + M_t^A \\ &= E[H] + \int_0^t (\xi_s^* + \xi_s^M + \xi_s^A) dX_s + \tilde{N}_t, \\ &\quad 0 \leq t \leq T, \end{aligned}$$

is a right-continuous version of the martingale

$$V_t = E[H | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

and this determines $\eta = V - \xi X$. \square

Corollary 2. *If both M and M^A are orthogonal to X , then we have $\xi = \xi^*$.*

Proof. By Remark 4 we get $\langle M, X \rangle = \langle M^A, X \rangle = 0$, hence $\xi^M = \xi^A = 0$ P_X -a.s.. \square

Remarks 5. (1) If X is a martingale with *continuous paths*, then $\langle M, X \rangle$ can be evaluated pathwise as a quadratic variation and coincides with $\langle N^*, X \rangle = 0$ P^* -a.s., hence P -a.s. This implies $\xi^M = 0$ P_X -a.s., hence

$$\xi = \xi^* + \xi^A. \quad (36)$$

(2) If P is a martingale measure in the stricter sense that it also preserves the martingale property of N^* , then we have $A = 0$, hence

$$\xi = \xi^* + \xi^M, \quad (37)$$

and

$$V_t = V_t^*. \quad (38)$$

If X has continuous paths then we can conclude, due to step (1), that the risk-minimizing strategy is completely preserved.

(3) An example in Section 5 will show that $\xi = \xi^*$ may occur even if the martingale property of N^* is lost under P .

5. Two examples

We illustrate the preceding results by two examples where the process X is a two-sided jump process. In both cases, the stock process X will be defined in terms of two independent Poisson processes N^+ and N^- with parameter

$\lambda^* > 0$ on some probability space $(\Omega, \mathcal{F}, P^*)$; $(\mathcal{F}_t)_{0 \leq t \leq T}$ will denote the smallest right-continuous family of σ -algebras which makes N^+ and N^- adapted.

Remark 6. The underlying stochastic model can be characterized as follows: The paths of N^\pm are right-continuous and piecewise constant with jumps of size 1, and under P^* the two processes

$$M_t^\pm \equiv N_t^\pm - \lambda^* t, \quad 0 \leq t \leq T, \quad (39)$$

are square-integrable martingales with

$$\langle M^\pm \rangle_t = \lambda^* t, \quad \langle M^+, M^- \rangle = 0. \quad (40)$$

It is also well-known that M^+ and M^- form a basis, i.e., any square-integrable martingale with respect to P^* and (\mathcal{F}_t) is of the form

$$M_t = M_0 + \int_0^t \xi_s^+ dM_s^+ + \int_0^t \xi_s^- dM_s^-, \quad (41)$$

where ξ^\pm is the unique predictable process in $L^2(P^* \times dt)$ such that

$$\langle M, M^\pm \rangle_t = \lambda^* \int_0^t \xi_s^\pm ds. \quad (42)$$

In our first example we suppose that the stock process X is of the form

$$X = x_0 + N^+ - N^- = x_0 + M^+ - M^-. \quad (43)$$

Thus, X is a square-integrable martingale with

$$\langle X \rangle_t = 2\lambda^* t, \quad 0 \leq t \leq T, \quad (44)$$

whose paths are piecewise constant with jumps of size ± 1 .

Now let $H \in \mathcal{L}^2(P^*)$ be a contingent claim of the form

$$H = h(X_T). \quad (45)$$

The Markov property of X implies that the value process is of the form

$$V_t^*(\omega) = E^*[H | \mathcal{F}_t](\omega) = v^*(X_t(\omega), t), \quad (46)$$

with

$$\begin{aligned} v^*(x, t) &= \sum_{y \in \mathbb{Z}} h(x+y) P^*[(N_T^+ - N_t^+) - (N_T^- - N_t^-) = y] \\ &= \sum_{y \in \mathbb{Z}} h(x+y) \sum_{\substack{k, l \geq 0 \\ k-l=y}} e^{-2\lambda^*(T-t)} \frac{[\lambda^*(T-t)]^{k+l}}{k!l!}. \end{aligned} \quad (47)$$

Putting

$$\Delta^\pm(x, t) \equiv v^*(x \pm 1, t) - v^*(x, t), \quad (48)$$

we obtain the following proposition.

Proposition 1. The risk-minimizing strategy (25) is determined by

$$\xi_t^* = \frac{\Delta^+ - \Delta^-}{2}(X_{t-}, t), \quad (49)$$

and the intrinsic risk process is given by

$$R_t^* = \frac{\lambda^*}{2} E^* \left[\int_t^T (\Delta^+ + \Delta^-)^2(X_{s-}, x) ds | \mathcal{F}_t \right]; \quad (50)$$

note that R_t^* can be calculated explicitly in the manner of (47).

Proof. (1) The process $Z \equiv M^+ + M^-$ is a square-integrable martingale with $\langle Z \rangle_t = 2\lambda^* t$, and Z is orthogonal to X since

$$X \cdot Z = (M^+ - M^-)(M^+ + M^-) = (M^+)^2 - (M^-)^2$$

is a martingale due to (39). In the representation (41), the basis (M^+, M^-) can thus be replaced by the basis (X, Z) . In particular, we can represent the process $V_t^* = E^*[H | \mathcal{F}_t]$, $0 \leq t \leq T$, in the form

$$V_t^* = V_0^* + \int_0^t \xi_s^* dX_s + \int_0^t \zeta_s^* dZ_s, \quad (51)$$

where ξ^* and ζ^* are determined by the equations

$$\langle V^*, X \rangle_t = 2\lambda^* \int_0^t \xi_s^* ds, \quad (52)$$

and

$$\langle V^*, Z \rangle_t = 2\lambda^* \int_0^t \xi_s^* ds. \quad (53)$$

By (22) and (24), the risk process is given by

$$\begin{aligned} R_t^* &= E^* \left[\int_t^T (\xi_s^*)^2 d\langle Z \rangle_s | \mathcal{F}_t \right] \\ &= 2\lambda^* E^* \left[\int_t^T (\xi_s^*)^2 ds | \mathcal{F}_t \right]. \end{aligned} \quad (54)$$

(2) In order to calculate ξ^* and ζ^* via (52) and (53), we introduce the quadratic variation processes

$$[V^*, X]_t \equiv \sum_{0 \leq s \leq t} (\Delta V^*)_s (\Delta X)_s,$$

and

$$[V^*, Z]_t \equiv \sum_{0 \leq s \leq t} (\Delta V^*)_s (\Delta Z)_s,$$

where we put

$$(\Delta X)_t(\omega) \equiv X_t(\omega) - X_{t-}(\omega), \text{ etc.}$$

The process $\langle V^*, X \rangle$ can be characterized as the unique predictable process such that $[V^*, X] - \langle V^*, X \rangle$ is a martingale; cf. Metivier (1982). But since

$$\begin{aligned} [V^*, X]_t &= \sum_{0 \leq s \leq t} (\Delta^+(X_{s-}, s) \Delta_s N^+ - \Delta^-(X_{s-}, s) \Delta_s N^-) \\ &= \int_0^t \Delta^+(X_{s-}, s) dM_s^+ - \int_0^t \Delta^-(X_{s-}, s) dM_s^- \\ &\quad + \lambda^* \int_0^t (\Delta^+ - \Delta^-)(X_{s-}, s) ds, \end{aligned}$$

we see that

$$\langle V^*, X \rangle_t = \lambda^* \int_0^t (\Delta^+ - \Delta^-)(X_{s-}, s) ds,$$

and this together with (52) implies (49). Since

$$[V^*, Z]_t = \sum_{0 \leq s \leq t} (\Delta^+(X_{s-}, s) \Delta_s N^+ + \Delta^-(X_{s-}, s) \Delta_s N^-),$$

we obtain in the same way

$$\xi_t^* = \frac{\Delta^+ + \Delta^-}{2}(X_{t-}, t), \quad (55)$$

and this implies (50) due to (54). \square

Remarks 7. (1) To replace P^* by an equivalent martingale measure P means that we replace λ^* by any $\lambda > 0$. Under P , the process $Z = M^+ + M^-$ is now of the form $Z = N + B$ with

$$N_t = N_t^+ + N_t^- - 2\lambda t, \quad B_t = 2(\lambda - \lambda^*)t.$$

The martingale M is orthogonal to X and this implies $\xi^M = 0$ since $M_t = \int_0^t \xi_s^* dN_s$ is also orthogonal to X . Thus, (32) reduces to

$$\xi = \xi^* + \xi^A, \quad (56)$$

just as in the continuous case (36) with $A_t = 2(\lambda - \lambda^*) \int_0^t \xi_s^* ds$.

(2) By (56), the risk-minimizing strategy remains unchanged if and only if the martingale

$$M_t^A = 2(\lambda - \lambda^*) E \left[\int_0^T \xi_s^* ds | \mathcal{F}_t \right]$$

is orthogonal to X . Consider, for example, the special case $H = X_T^2$. Since

$$E^* [X_T^2 | \mathcal{F}_t] = X_t^2 + 2\lambda^*(T - t),$$

we obtain $\Delta^\pm(x, t) = \pm 2x + 1$, hence $\xi_t^*(\omega) \equiv 1$ by (55). In particular, M^B is orthogonal to X , and the risk-minimizing strategy,

$$\xi_t(\omega) = 2X_{t-}(\omega), \quad (57)$$

does not depend on the specific choice of $\lambda > 0$. The value process $V_t = X_t^2 + 2\lambda(T - t)$ does depend on λ , and so does the intrinsic risk process $R_t = 2\lambda(T - t)$.

In our *second example* we assume that the security process X is governed by the stochastic differential equation

$$dX = \delta X_- (dN^+ - dN^-), \quad (58)$$

with some $\delta > 0$. For a given initial value $x_0 > 0$, (58) implies

$$X_t = x_0 (1 + \delta)^{N_t^+} (1 - \delta)^{N_t^-}, \quad 0 \leq t \leq T. \quad (59)$$

By (58), X is a square-integrable martingale with

$$\langle X \rangle_t = 2\delta^2 \lambda \int_0^t X_{s-}^2 ds, \quad 0 \leq t \leq T.$$

The value process V^* associated to the contingent claim (45) is again of the form (46), now with the function

$$v^*(x, t) = \sum_{k, l \geq 0} h(x(1 + \delta)^k (1 - \delta)^l) e^{-2\lambda^*(T-t)} \frac{[\lambda^*(T-t)]^{k+l}}{k!l!}. \quad (60)$$

Defining Δ^\pm as before in (48), we obtain the following proposition.

Proposition 2. The risk-minimizing strategy (24) is determined by

$$\xi_t^* = \frac{(\Delta^+ - \Delta^-)(X_{t-}, t)}{2\delta X_{t-}}, \quad (61)$$

and the intrinsic risk process is given by

$$R_t^* = 2\lambda E^* \left[\int_t^T (\xi_s^*)^2 ds \mid \mathcal{F}_t \right] \quad (62)$$

where

$$\xi_t^* = \frac{(\Delta^+ + \Delta^-)(X_{t-}, t)}{2\delta Z_{t-}}, \quad (63)$$

and

$$Z_t = (1 + \delta)^{N_t^+ + N_t^-} e^{-2\delta t}. \quad (64)$$

Proof. We have $dX = \delta X_- (dM^+ - dM^-)$. The process Z defined in (64) is a solution of

$$dZ = \delta Z_- (dM^+ + dM^-),$$

and this implies that X and Z are two orthogonal martingales which may be used as a basis. Proceeding exactly as in the proof of Proposition 1, we obtain

$$\langle V^*, X \rangle_t = \int_0^t \delta X_{s-} (\Delta^+ - \Delta^-)(X_{s-}, s) ds,$$

and

$$\langle V^*, Z \rangle_t = \int_0^t \delta Z_{s-} (\Delta^+ + \Delta^-)(X_{s-}, s) ds,$$

hence (61) and (63). \square

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