Efficient Hedging: Cost versus Shortfall Risk

by

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Abstract. An investor faced with a contingent claim may eliminate risk by (super-) hedging in a financial market. As this is often quite expensive, we study partial hedges which require less capital and reduce the risk. In a previous paper we determined quantile hedges which succeed with maximal probability, given a capital constraint. Here we look for strategies which minimize the shortfall risk defined as the expectation of the shortfall weighted by some loss function. The resulting efficient hedges allow the investor to interpolate in a systematic way between the extremes of no hedge and a perfect (super-) hedge, depending on the accepted level of shortfall risk.

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1. Introduction

In a complete financial market a given contingent claim can be replicated by a self-financing trading strategy, and the cost of replication defines the price of the claim. In incomplete financial markets one can still stay on the safe side by using a "superhedging" strategy; cf. [ElQ (1995)] and [K (1997)]. But from a practical point of view the cost of superhedging is often too high. Also perfect (super-) hedging takes away the opportunity of making a profit together with the risk of a loss.

Suppose that the investor is unwilling to put up the initial amount of capital required by a perfect (super-) hedge and is ready to accept some risk. What is the optimal "partial hedge" which can be achieved with a given smaller amount of capital? In order to make this question precise we need a criterion expressing the investor's attitude towards the shortfall risk. In [FL (1998)] we introduced strategies of "quantile hedging" which maximize the probability that a hedge is successful. In that case the investor applies a dynamic version of the static Value at Risk concept. Just as the static VaR approach, the dynamic concept of quantile hedging does not take into account the size of the shortfall but only the probability of its occurrence.

In this paper we describe the investor's attitude towards the shortfall in terms of a loss function l. Convexity of l corresponds to risk aversion. The shortfall risk is defined as the expectation of the shortfall weighted by the loss function. Our aim is to minimize this shortfall risk, given some capital constraint. Instead we could prescribe a bound on the shortfall risk and minimize the cost. In other words, we are looking for hedges which are efficient with respect to the partial ordering defined by the shortfall risk and the initial capital. These efficient hedges allow the investor to interpolate in a systematic way between the extremes of a perfect hedge (no chance of making a profit) and no hedge (full risk of shortfall, full chance of profit) depending on the accepted level of shortfall risk.

In the special case $l(x) = x^p$ for $p \geq 1$, our approach can be viewed as a dynamic version of static risk analysis in terms of lower partial moments; see, e.g., [F (1977)], [Ba (1978)], [BaL (1977)], [HR (1989)]. A systematic analysis of "coherent" measures of risk in a static setting is given in [ADEH (1997)] where coherence is defined in terms of monotonicity, homogeneity, and translation invariance with respect to adding amounts of the riskless asset. But from the individual investor's point of view, it seems to make sense to relax these requirements to monotonicity and convexity, and in this case risk measures of the type considered above with a convex loss function l appear; see [L (1998)].

We begin in section 2 by defining our optimization problem for a given contingent claim H in a general semimartingale setting. Existence and essential uniqueness of the solution is shown in section 3. The optimal strategy consists in (super-) hedging a suitable modified claim $\widetilde{H} = \widetilde{\varphi} H$ where $\widetilde{\varphi}$ is some "randomized test" taking values in $[0, 1]$. In the special case $l(x) = x$, where we simply minimize the expected shortfall, we can construct the optimal test $\widetilde{\varphi}$ by applying the Neyman Pearson lemma in direct analogy to the case of quantile hedging; see section 4 and [FL (1998)]. In this case, the resulting claim $\widetilde{H}$ typically has the form of a knock-out option.

For a general convex loss function l the problem becomes more involved. In section 5 we consider the complete case where the equivalent martingale measure is unique. Using a method of [Ka (1959)], we show how the construction of the optimal test $\widetilde{\varphi}$ can again be reduced to an application of the Neyman Pearson lemma. Typically the resulting claim $\tilde{H}$ has a smoother structure than the knock-out options which occur in the case of quantile hedging and in the case of a linear loss function.

In particular we consider the case of lower partial moments, i.e. $l(x) = x^p$. Thus we introduce a scale for the attitude towards risk. As p increases from 1 to ∞, the efficient hedges interpolate smoothly between the knock-out option and a shifted claim $\widetilde{H} = (H - c)^+$. If H is a call, then $\widetilde{H}$ is a call at higher strike whose arbitrage-free price equals the given initial capital. In section 5.4 we also consider the case $p < 1$ where risk-averse behavior is replaced by risk-seeking behavior. As appetite for risk increases and p decreases from 1 to 0, the corresponding efficient hedges converge to the knock-out option which appears in the case of quantile hedging. Thus quantile hedging corresponds to the bottom-end of our scale.

Alternatively we can use methods of convex duality. In section 7 we use a variant of the methods of [CK (1998)] and of [KS (1997)] in order to describe the structure of the solution in the general case. In the incomplete case we rely on the basic duality theorem in [KS (1997)]. Even in the complete case, these methods provide additional information on the qualitative properties of the value function of our problem. In the linear case $l(x) = x$ and in a model driven by Brownian motion, similar results including constraints on the strategies and margin requirements appear in [CK (1998)] and [C (1998)]. In [P (1998)] convex duality methods are applied in a discrete time setting with $l(x) = x^p$.

In order to illustrate our approach we compute in section 6 the efficient hedges for a call option in the standard case of a geometric Brownian motion with known volatility and for the loss function $l(x) = x^p$. While in the case of quantile hedging the optimal strategy consists in replicating the option "knocked out" above a certain threshold, the option is "knocked in" above some threshold if we minimize the expected shortfall, i.e., in the case $p = 1$. In the case $p > 1$ of risk aversion, the modified options are no longer "knocked out/in" but exhibit continuous payoffs. Finally in section 8 we study an incomplete extension of the model where volatility is subject to a random jump.

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2. Formulation of the problem

The discounted price process of the underlying asset is described as a semimartingale $X = (X_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. For simplicity we assume that $\mathcal{F}_0$ is trivial. Let $\mathcal{P}$ denote the set of equivalent martingale measures. We assume absence of arbitrage in the sense that $\mathcal{P} \neq \emptyset$.

A self-financing strategy is given by an initial capital $V_0 \geq 0$ and by a predictable process ξ such that the resulting value process

$$V_t = V_0 + \int_0^t \xi_s dX_s \quad \forall t \in [0,T] \tag{2.1}$$

is well defined. A strategy (V_0, ξ) is called *admissible* if the corresponding value process V satisfies

$$V_t \geq 0 \quad \forall t \in [0,T]\,, \quad P - a.s. \tag{2.2}$$

Consider a contingent claim given by a $\mathcal{F}_T$-measurable, nonnegative random variable H. We assume

$$U_0 = \sup_{P^* \in \mathcal{P}} E^*[H] < \infty \quad , \tag{2.3}$$

where E^* denotes expectation with respect to P^*. The value U_0 is the smallest amount V_0 such that there exists an admissible strategy (V_0, ξ) whose value process satisfies $V_T \geq H$ $P - a.s.$ This is well known in the complete case where the equivalent martingale measure P^* is unique, and where $U_0 = E^*[H]$ is the unique arbitrage-free price of the contingent claim H. For the general case see, e.g., [ElQ (1995)], [Kr (1996)], [FKab (1998)].

As in the discussion of quantile hedging in [FL (1998)], we now ask what can be done if the investor is unwilling or unable to put up the initial capital U_0. What is the best hedge the investor can achieve with a given smaller amount $\widetilde{V}_0 < U_0$? In [FL (1998)] we took as our optimality criterion the probability that the hedge is successful. In other words, we were looking for an admissible strategy (V_0, ξ) which minimizes the probability of a shortfall $P[V_T \leq H]$ under the constraint $V_0 \leq \widetilde{V}_0$.

In this paper we want to control the *size* of the shortfall $(H - V_T)^+$, not only the probability that some shortfall occurs. To this end we introduce a loss function l which describes the investor's attitude with respect to the shortfall. We assume that l is an increasing convex function defined on $[0, \infty)$, with $l(0) = 0$. We further assume that

$$E[l(H)] < \infty. \tag{2.4}$$

Definition 2.1. *The shortfall risk is defined as the expectation*

$$E[l((H - V_T)^+)] \tag{2.5}$$

of the shortfall weighted by the loss function l.

Our aim is to find an admissible strategy (V_0, ξ) which minimizes the shortfall risk while not using more capital than $\widetilde{V}_0$. Thus we consider the optimization problem

$$E[l((H - V_T)^+)] = E[l((H - V_0 - \int_0^T \xi_s dX_s)^+)] = \min \tag{2.6}$$

under the constraint

$$V_0 \leq \widetilde{V}_0. \tag{2.7}$$

In section 7.1 we show how this optimization problem can be reformulated in terms of maximizing the expectation of a suitable state-dependent utility function.

Remark 2.1. *Instead of minimizing the shortfall risk under a cost constraint, we could fix a bound on the shortfall risk and minimize the cost. The results in section 7 show that both versions of the problem are in fact equivalent.*

Remark 2.2. *A typical example of a loss function is $l(x) = x^p$ for some $p \geq 1$. This approach to measuring risk by lower partial moments is well known in the economics literature, see, e.g.,* [F (1977)], [Ba (1978)], [BaL (1977)], [HR (1989)]. *In* [F (1977)] *a mean-risk dominance model for distributions of returns is considered. Risk consists in falling short of a specified target return t and it is measured by a partial moment below t. Thus the risk of a random variable X with distribution μ is given by*

$$E[((t-X)^+)^p] = \int_{-\infty}^{t} (t-x)^p \mu(dx).$$

As pointed out in [F (1977)], *mean-risk dominance is congruent with maximizing expected utility for a utility function of the form*

$$U(x) = x - const\ ((t-x)^+)^p\ .$$

Note that U is linear (i.e. risk-neutral) above the target return and concave (i.e. risk-averse) below the target return.
There are several aspects in which we move beyond this setting. We consider a dynamic instead of a static problem and we allow for general loss functions. Moreover our investor is faced with a contingent claim instead of a fixed target return, i.e. the investor aims at a random target.

3. The optimal hedge

Let us reduce our problem to the search for an element $\widetilde{\varphi}$ in the class

$$\mathcal{R} = \{\varphi : \Omega \longrightarrow [0,1] \mid \varphi\ \mathcal{F}_T\text{-measurable}\}$$

of "randomized tests" which solves the following optimization problem.

Proposition 3.1. *There exists a solution $\widetilde{\varphi} \in \mathcal{R}$ to the problem*

$$\min_{\varphi \in \mathcal{R}} E[l((1-\varphi)H)] \tag{3.1}$$

under the constraint

$$\sup_{P^* \in \mathcal{P}} E^*[\varphi H] \leq \widetilde{V}_0\ . \tag{3.2}$$

If l is strictly convex, then any two solutions coincide $P-a.s.$ on $\{H > 0\}$.

Proof. 1) Let $\mathcal{R}_0$ consist of those elements of $\mathcal{R}$ that satisfy (3.2). Let (φ_n) be a minimizing sequence for (3.1) in $\mathcal{R}_0$. Using Lemma A.1.1. in [DS (1994)] we can choose functions $\widetilde{\varphi}_n \in \mathcal{R}_0$ belonging to the convex hull of $\{\varphi_n, \varphi_{n+1}, ...\}$ such that $(\widetilde{\varphi}_n)$ converges $P-$ a.s. to some $\widetilde{\varphi} \in \mathcal{R}$. Since $l(H) \in L^1(P)$ we can use dominated convergence to conclude that

$$E[l((1-\widetilde{\varphi}_n)H)] \longrightarrow E[l((1-\widetilde{\varphi})H)] = \min .$$

On the other hand

$$E^*[\widetilde{\varphi}H] \le \liminf E^*[\widetilde{\varphi}_n H] \le \widetilde{V}_0 \quad \forall P^* \in \mathcal{P}$$

by Fatou's lemma. Thus $\widetilde{\varphi} \in \mathcal{R}_0$.

2) Let $\widetilde{\varphi}$ be a solution. For any $\varphi \in \mathcal{R}_0$ and for $\varepsilon \in [0,1]$ we define

$$\varphi_\varepsilon = (1-\varepsilon)\widetilde{\varphi} + \varepsilon\varphi \ .$$

By the convexity of l we get

$$E[l((1-\varphi_\varepsilon)H)] \le (1-\varepsilon)E[l((1-\widetilde{\varphi})H)] + \varepsilon E[l((1-\varphi)H)] \ .$$

If l is strictly convex, then the inequality is strict if

$$P[\{\varphi \neq \widetilde{\varphi}\} \cap \{H > 0\}] > 0 \ .$$

□

Let $\widetilde{\varphi}$ be the solution to the problem defined by (3.1) and (3.2). Without loss of generality we assume

$$\widetilde{\varphi} = 1 \text{ on } \{H = 0\} \ . \tag{3.3}$$

Let us introduce the modified claim

$$\tilde{H} = \widetilde{\varphi}H \ , \tag{3.4}$$

and let us define $\tilde{U}$ as a right-continuous version of the process

$$\tilde{U}_t = \operatorname*{ess.sup}_{P^* \in \mathcal{P}} E^*[\widetilde{\varphi}H \mid \mathcal{F}_t] \ . \tag{3.5}$$

$\tilde{U}$ is a $\mathcal{P}$-supermartingale, i.e. a supermartingale with respect to any equivalent martingale measure $P^* \in \mathcal{P}$. We can now apply the optional decomposition theorem, see [Kr (1996)], [FKab (1998)]. Thus there exists an admissible strategy $(\tilde{V}_0, \tilde{\xi})$ and an increasing optional process $\tilde{C}$ with $\tilde{C}_0 = 0$ such that

$$\tilde{U}_t = \tilde{V}_0 + \int_0^t \widetilde{\xi} dX - \tilde{C}_t \ , \tag{3.6}$$

Remark 3.1. *In the complete case where the equivalent martingale measure is unique,* $(\tilde{V}_0, \tilde{\xi})$ *is simply the duplicating strategy for the modified claim* $\tilde{H} = \widetilde{\varphi}H$, *i.e.*

$$E^*[\widetilde{\varphi}H \mid \mathcal{F}_t] = \tilde{V}_0 + \int_0^t \widetilde{\xi} dX \quad \forall t \in [0,T] \ , \quad P-a.s. \tag{3.7}$$

Definition 3.1. *For any admissible strategy* (V_0, ξ) *we define the corresponding success ratio as*

$$\varphi_{(V_0,\xi)} = \mathbf{1}_{\{V_T \geq H\}} + \frac{V_T}{H} \mathbf{1}_{\{V_T < H\}}. \tag{3.8}$$

Theorem 3.2. *The strategy* $(\tilde{V}_0, \tilde{\xi})$ *determined by the optional decomposition (3.6) of the modified claim* $\widetilde{H} = \widetilde{\varphi} H$ *solves the optimization problem (2.6), (2.7). Its success ratio coincides* $P - a.s.$ *with* $\tilde{\varphi}$*.*

Proof. 1) Let (V_0, ξ) be any admissible strategy with $V_0 \leq \widetilde{V}_0$, and denote by φ the corresponding success ratio. Since $\varphi H = V_T \wedge H$ the shortfall takes the form

$$(H - V_T)^+ = H - V_T \wedge H = (1 - \varphi)H. \tag{3.9}$$

For any $P^* \in \mathcal{P}$ the corresponding value process is a supermartingale under P^* and so we get

$$E^*[\varphi H] \leq E^*[V_T] \leq V_0 \leq \widetilde{V}_0.$$

Thus the success ratio satisfies the constraints (3.2) and so we have

$$E[l((H - V_T)^+)] = E[l((1 - \varphi)H)] \geq E[l((1 - \widetilde{\varphi})H)] \tag{3.10}$$

since $\tilde{\varphi}$ is optimal for the problem defined by (3.1) and (3.2).

2) The strategy $(\tilde{V}_0, \tilde{\xi})$ is admissible since the corresponding value process satisfies

$$\tilde{V}_t \geq \tilde{V}_t - \tilde{C}_t = \operatorname*{ess.sup}_{P^* \in \mathcal{P}} E^*[\widetilde{\varphi} H \mid \mathcal{F}_t] \geq 0.$$

Its success ratio $\varphi_{(\tilde{V}_0, \tilde{\xi})}$ satisfies

$$\varphi_{(\tilde{V}_0,\tilde{\xi})} H = \tilde{V}_T \wedge H \geq \widetilde{\varphi} H \quad P - a.s. \quad \text{on} \quad \{H > 0\},$$

hence

$$\varphi_{(\tilde{V}_0,\tilde{\xi})} H = \widetilde{\varphi} H \quad P - a.s. \quad \text{on} \quad \{H > 0\}$$

due to (3.10). Moreover we have

$$\varphi_{(\tilde{V}_0,\tilde{\xi})} = \widetilde{\varphi} = 1 \quad \text{on} \quad \{H = 0\} \ ,$$

and so the success ratio coincides P-a.s. with $\widetilde{\varphi}$. In particular we have

$$(H - \tilde{V}_T)^+ = (1 - \tilde{\varphi})H$$

due to (3.9). Thus the inequality (3.10) shows that the strategy $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem defined by (2.6) and (2.7). $\square$

4. Minimizing the expected shortfall

In this section we consider the case of a linear loss function $l(x) = x$. Thus we want to minimize the expected shortfall

$$E[(H - V_T)^+] \tag{4.1}$$

under the constraint

$$V_0 \le \widetilde{V}_0. \tag{4.2}$$

Theorem 3.2 shows that this is equivalent to the optimization problem

$$E[\varphi H] = \max \tag{4.3}$$

under the constraint that $\varphi \in \mathcal{R}$ satisfies

$$\sup_{P^* \in \mathcal{P}} E^*[\varphi H] \le \widetilde{V}_0. \tag{4.4}$$

This takes the form

$$\int \varphi dQ = \max \tag{4.5}$$

under the constraints

$$\int \varphi dQ^* \le \alpha(P^*) = \tilde{V}_0 / E^*[H] \quad \forall P^* \in \mathcal{P}\ , \tag{4.6}$$

where the measures Q and Q^* are defined by

$$\frac{dQ}{dP} = \frac{H}{E[H]} \quad , \quad \frac{dQ^*}{dP^*} = \frac{H}{E^*[H]}\ .$$

Thus the solution $\widetilde{\varphi}_1$ is identified as the optimal randomized test in a problem of testing the compound hypothesis

$$\{Q^* \mid P^* \in \mathcal{P}\}$$

parametrized by the class of equivalent martingale measures against the simple alternative Q, where the significance level varies with the parameter $P^* \in \mathcal{P}$, see, e.g., [W (1985)].

In the complete case the Neyman-Pearson lemma provides an explicit solution:

Proposition 4.1. *Assume that $\mathcal{P} = \{P^*\}$. Then the optimal randomized test $\tilde{\varphi}_1 \in \mathcal{R}$ is given by*

$$\tilde{\varphi}_1 = \mathbf{1}_{\{\frac{dP}{dP^*} > \tilde{a}\}} + \gamma \mathbf{1}_{\{\frac{dP}{dP^*} = \tilde{a}\}} \tag{4.7}$$

where

$$\tilde{a} = \inf\{a \mid \int_{\{\frac{dP}{dP^*} > a\}} H dP^* \le \tilde{V}_0\} \tag{4.8}$$

and

$$\gamma = \frac{\tilde{V}_0 - \int_{\{\frac{dP}{dP^*} > \tilde{a}\}} H dP^*}{\int_{\{\frac{dP}{dP^*} = \tilde{a}\}} H dP^*} , \tag{4.9}$$

in the case that $P^*[\{\frac{dP}{dP^*} = \tilde{a}\} \cap \{H > 0\}] > 0$. *If* $P^*[\{\frac{dP}{dP^*} = \tilde{a}\} \cap \{H > 0\}] = 0$, *then* $\tilde{\varphi}_1$ *reduces to the indicator function of the success set* $\{\frac{dP}{dP^*} > \tilde{a}\}$.

Proof. The optimal test of the simple hypothesis Q against the simple alternative Q^* is described by the Neyman-Pearson lemma in terms of Q and Q^*; cf., e.g., [W (1985)]. If we rewrite it in terms of P and P^*, it takes the form (4.7). □

5. Explicit solution in the complete case

In this section we assume that the equivalent martingale measure P^* is uniquely determined, and we denote by

$$\rho^* = \frac{dP^*}{dP}$$

the corresponding Radon-Nikodym derivative. We assume that our loss function satisfies $l \in C^1(0, \infty)$, and that the derivative l' is strictly increasing with $l'(0+) = 0$ and $l'(\infty) = \infty$. Let

$$I = (l')^{-1}$$

denote the inverse function of l'.

5.1. **Structure of the modified claim.** By proposition 3.1 the solution $\widetilde{\varphi}$ of our optimization problem exists, and it is unique on $\{H > 0\}$ since l is strictly convex. On $\{H = 0\}$ we set $\widetilde{\varphi} = 1$. The following theorem provides the explicit structure of $\widetilde{\varphi}$.

Theorem 5.1. *The solution* $\widetilde{\varphi}$ *to the optimization problem (3.1), (3.2) is given by*

$$\widetilde{\varphi} = 1 - \left(\frac{I(c\rho^*)}{H} \wedge 1 \right) \quad \text{on } \{H > 0\}, \tag{5.1}$$

where the constant c *is determined by the condition*

$$E^*[\widetilde{\varphi} H] = \bar{V}_0 \; . \tag{5.2}$$

Proof. We use the method of Karlin [Ka (1959)] in order to reduce the computation of $\widetilde{\varphi}$ to an application of the Neyman-Pearson lemma.

1) For $\varphi \in \mathcal{R}$ we define

$$\varphi_\varepsilon = (1 - \varepsilon)\widetilde{\varphi} + \varepsilon \varphi \; .$$

Let F_φ the convex function defined on $[0, 1]$ via

$$F_\varphi(\varepsilon) = E[l((1 - \varphi_\varepsilon)H].$$

Applying monotone convergence separately on $\{\varphi > \widetilde{\varphi}\}$ and on $\{\varphi < \widetilde{\varphi}\}$ we see that the derivative $F'_\varphi(0+)$ exists and satisfies

$$F'_\varphi(0+) = E[l'((1-\widetilde{\varphi})H)(\widetilde{\varphi}-\varphi)H] \ .$$

The optimality of $\widetilde{\varphi}$ means that for any $\varphi \in \mathcal{R}$ the corresponding convex function F_φ on $[0,1]$ assumes its minimum in $\lambda = 0$. This is equivalent to

$$F'_\varphi(0+) \geq 0 \quad \forall \varphi \in \mathcal{R} \ ,$$

i.e., to

$$E[l'((1-\widetilde{\varphi})H)\widetilde{\varphi}H] \geq E[l'((1-\widetilde{\varphi})H)\varphi H] \quad \forall \varphi \in \mathcal{R} \ . \tag{5.3}$$

2) If we define probability measures Q, Q^* on $\{H > 0\}$ by

$$\frac{dQ}{dP} = \text{const } l'((1-\widetilde{\varphi})H)H \quad , \quad \frac{dQ^*}{dP^*} = \text{const } H$$

then (5.3) becomes the problem of testing the hypothesis Q against the alternative Q^* at the level $\alpha = \tilde{V}_0/E^*[H]$. The Neyman-Pearson lemma describes the structure of the optimal test in terms of the likelihood ratio

$$\frac{dQ}{dQ^*} = \text{const } l'((1-\widetilde{\varphi})H)\frac{dP}{dP^*}$$

where the constant c is determined from the level condition. On the set

$$\{dQ/dQ^* < c\} = \{l'((1-\widetilde{\varphi})H)/\rho^* < c\} \tag{5.4}$$

the optimal test is zero. On the set

$$\{dQ/dQ^* > c\} = \{l'((1-\widetilde{\varphi})H)/\rho^* > c\} \tag{5.5}$$

the optimal test should be equal to one. Notice however that $\widetilde{\varphi}(\omega) = 1$ implies $l'((1-\widetilde{\varphi}(\omega))H)/\rho^* = 0$ due to $l'(0) = 0$, a contradiction. Thus we have $\widetilde{\varphi} < 1$ on $\{H > 0\}$. In the ordinary Neyman-Pearson situation there is no restriction on the values on the set

$$\{dQ/dQ^* = c\} = \{l'((1-\widetilde{\varphi})H)/\rho^* = c\} \tag{5.6}$$

except compliance with the level condition. In our situation, however, we have to choose $\widetilde{\varphi}$ on this set such that

$$(1-\widetilde{\varphi})H = I(c\rho^*) \tag{5.7}$$

in order to be consistent.

3) Define

$$\varphi_c = 1 - (I(c\rho^*)/H \wedge 1) \tag{5.8}$$

on $\{H > 0\}$. Note that $I(c\rho^*)$ goes from 0 to ∞ a.s. on $\{H > 0\}$ as c goes from 0 to ∞ since $l'(0) = 0$ and $l'(\infty) = \infty$. Thus $\varphi_c H$ goes from H to 0 a.s. on $\{H > 0\}$ as c goes from 0 to ∞. Consequently $E^*[\varphi_c H]$ goes from $E^*[H]$ to 0 by dominated convergence.

Since I is continuous, $E^*[\varphi_c H]$ is continuous in c by dominated convergence. Hence we can find $c \in (0, \infty)$ such that

$$E^*[\varphi_c H] = \widetilde{V}_0 < E^*[H].$$

For this c we define

$$\widetilde{\varphi} = \varphi_c.$$

Then $\widetilde{\varphi}$ satisfies the consistency condition (5.7) on the set in (5.6). On the set in (5.4) we have

$$1 - \widetilde{\varphi} < I(c\rho^*)/H \quad ,$$

and this implies $\widetilde{\varphi} = 0$ by the definition (5.8) of $\widetilde{\varphi} = \varphi_c$. Thus $\widetilde{\varphi}$ is the optimal Neyman-Pearson test described in 2). □

Remark 5.1. *Notice that the solution $\widetilde{\varphi}$ is a function of H and ρ. If both H and ρ happen to be functions of the final stock price X_T, then $\widetilde{\varphi}$ is also a function of X_T. This will be the case in the explicit computations for the Black-Scholes model below.*

Remark 5.2. *Suppose that the objective measure P already happens to be the martingale measure P^*, i.e. $\rho^* = 1$. Then the modified claim takes the simple form*

$$\widetilde{\varphi} H = H - (I(c) \wedge H) = (H - I(c))^+. \tag{5.9}$$

If H is a call, then $\widetilde{\varphi} H$ is again a call $(X_T - \tilde{K})^+$ at the higher strike $\tilde{K} = K + I(c)$.

5.2. **Lower partial moments.** Let us consider the special case

$$l(x) = \frac{x^p}{p}$$

for some $p > 1$. Thus we want to minimize a lower partial moment of the difference $V_T - H$. As a special case of theorem 5.1 we obtain the following result:

Proposition 5.2. *The optimal hedge consists in hedging the modified claim*

$$\varphi_p H = H - c_p (\rho^*)^{\frac{1}{p-1}} \wedge H \tag{5.10}$$

where the constant c_p is determined such that

$$E^*[\varphi_p H] = \tilde{V}_0. \tag{5.11}$$

5.3. **Increasing risk-aversion.** Let us now consider the limit $p \to \infty$ corresponding to ever increasing risk-aversion with respect to large losses.

Proposition 5.3. *i) For $p \to \infty$ the modified claim $\varphi_p H$ converges to $(H - c)^+$ almost surely and in $L^1(P^*)$, where c is the unique constant that satisfies*

$$E^*[c \wedge H] = E^*[H] - \tilde{V}_0. \tag{5.12}$$

ii) If H is a call at strike K, then the limit for $p \to \infty$ is again a call at the higher strike $\tilde{K} = K + c$ which corresponds to the Black-Scholes price $\tilde{V}_0$:

$$E^*[(X_T - \tilde{K})^+] = \tilde{V}_0. \tag{5.13}$$

Proof. 1) It is straightforward to see that the constant c in (5.12) is uniquely defined. We now show that $\lim_{p\to\infty} c_p = c$. To this end consider a subsequence c_{p_n} with $\lim_n c_{p_n} = \bar{c} \in [0, \infty]$. Then

$$c_{p_n}(\rho^*)^{\frac{1}{p_n-1}} \wedge H \to \bar{c} \wedge H \quad a.s.$$

because

$$(\rho^*)^{\frac{1}{p_n-1}} \to 1 \quad a.s.$$

By dominated convergence we conclude that

$$E^*[c_{p_n}(\rho^*)^{\frac{1}{p_n-1}} \wedge H] \to E^*[\bar{c} \wedge H].$$

Thus we must have

$$E^*[\bar{c} \wedge H] = E^*[H] - \tilde{V}_0$$

due to (5.11), hence $\bar{c} = c$ since c is unique. Applying this argument to the limit inferior and to the limit superior, we see that $\liminf c_p = c = \limsup c_p$, hence $\lim c_p = c$ as claimed.

2) From 1) it follows that

$$c_p(\rho^*)^{\frac{1}{p-1}} \wedge H \to c \wedge H \quad a.s. \ ,$$

hence

$$\varphi_p H = H - c_p(\rho^*)^{\frac{1}{p-1}} \wedge H \to H - c \wedge H = (H - c)^+$$

both almost surely and in $L^1(P^*)$ (by dominated convergence since $H \in L^1(P^*)$ due to our assumption (2.3)). □

5.4. **Risk-taking and quantile hedging.** In this section we assume that our investor, instead of being a standard risk-averse agent, is in fact inclined to take risk. In our setting this corresponds to a concave rather than a convex loss function. Hence let $k : [0, \infty) \to [0, \infty)$ be increasing and strictly concave with $k(0) = 0$.

Our basic optimization problem is still the same:

$$E[k((H - V_T)^+)] = \min \tag{5.14}$$

under the constraint

$$V_0 \le \tilde{V}_0 \tag{5.15}$$

As before this is equivalent to finding $\varphi \in \mathcal{R}$ such that

$$E[k((1 - \varphi)H)] = \min \tag{5.16}$$

under the constraint

$$E^*[\varphi H] \leq \tilde{V}_0 \tag{5.17}$$

It is straightforward to see that a solution $\widetilde{\varphi}$ exists and that $\widetilde{\varphi}$ is extremal on the set $\{H > 0\}$.

From the concavity of k and from $k(0) = 0$ the following inequality is immediate:

$$E[k((1-\varphi)H)] \geq E[k(H)] - E[\varphi k(H)]. \tag{5.18}$$

But minimizing the lower bound in (5.18) can be done by a direct application of Neyman-Pearson because it is equivalent to maximizing $E_Q[\varphi]$ under the constraint $E_{Q^*}[\varphi] \leq \tilde{V}_0/E^*[H]$ where the measures Q, Q^* are defined by $dQ = \text{const } k(H)\ dP$ and $dQ^* = \text{const } H\ dP^*$.

To avoid technicalities we only sketch the case where no randomization is needed; for details and proofs see [L (1998)]. In this case the optimal Neyman-Pearson test

$$\widetilde{\varphi} = \mathbf{1}_{\{k(H) > aH\rho^*\}} \tag{5.19}$$

is simply an indicator function where the constant a is determined by the constraint. Hence $\widetilde{\varphi}$ minimizes the lower bound in (5.18). But for this $\widetilde{\varphi}$ (5.18) holds in fact as an equality. Thus $\widetilde{\varphi}$ must be the optimal solution to our optimization problem (5.16), (5.17).

Now let us consider lower partial moments again, i.e. $k(x) = x^p$ for some $0 < p < 1$. Then the set $\{k(H) > a_p H\rho^*\} \cup \{H = 0\}$ takes the form $\{1 > a_p H^{1-p}\rho^*\}$, and the optimal solution is given by

$$\varphi_p = \mathbf{1}_{\{1 > a_p H^{1-p}\rho^*\}}\ , \tag{5.20}$$

if we assume for simplicity that no randomization is needed, i.e. $E^*[\varphi_p H] = \tilde{V}_0$.

Let us also assume that there is unique constant $\tilde{a}$ such that $E^*[\varphi_0 H] = \tilde{V}_0$ where φ_0 denotes the indicator function of the set $\{1 > \tilde{a}H\rho^*\}$. The following proposition shows that quantile hedging, as introduced in [FL (1998)], appears as the limiting strategy with minimal shortfall risk as p decreases to zero.

Proposition 5.4. *For $p \to 0$ the solution φ_p in (5.20) converges to the solution φ_0 in the case of quantile hedging, both almost surely and in $L^1(P^*)$.*

Proof. Let $a_{p_n} \to a^*$ be a convergent subsequence. From $H^{1-p_n} \to H$ it follows that

$$\mathbf{1}_{\{1 > a_{p_n} H^{1-p_n}\rho^*\}} \to \mathbf{1}_{\{1 > a^* H\rho^*\}} \quad a.s.$$

Due to assumption (2.3) we can apply the dominated convergence theorem to conclude that

$$E^*[H\mathbf{1}_{\{1 > a_{p_n} H^{1-p_n}\rho^*\}}] \to E^*[H\mathbf{1}_{\{1 > a^* H\rho^*\}}] \quad ,$$

and this implies

$$E^*[H\mathbf{1}_{\{1 > a^* H\rho^*\}}] = \tilde{V}_0.$$

From the uniqueness of $\tilde{a}$ it follows that $a^* = \tilde{a}$. Applying this argument to the limit superior and inferior respectively we see that $\lim a_p = \tilde{a}$. Consequently

$$\varphi_p = \mathbf{1}_{\{1 > a_p H^{1-p} \rho^*\}} \to \mathbf{1}_{\{1 > \tilde{a} H \rho^*\}} = \varphi_0$$

both a.s. and in $L^1(P^*)$. □

In an analogous manner one can establish the following proposition

Proposition 5.5. *For $p \nearrow 1$ the solution φ_p in (5.20) converges to the solution $\widetilde{\varphi}_1$ in the linear case (cf. proposition 4.1), both almost surely and in $L^1(P^*)$.*

6. Computations in the Black-Scholes model

In the standard Black-Scholes model with constant volatility $\sigma > 0$ the underlying discounted price process is given by a geometric Brownian motion

$$dX_t = X_t(\sigma dW_t + m dt)$$

with initial value $X_0 = x_0$, where W is a Wiener process under P and m is a constant. We assume that $m > 0$. The unique equivalent martingale measure P^* is given by

$$\frac{dP^*}{dP} = \rho^* = \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) = \text{const } X_T^{-\alpha}$$

where we set

$$\alpha = \frac{m}{\sigma^2} \ .$$

The process W^* defined by

$$W_t^* = W_t + \frac{m}{\sigma} t$$

is a Brownian motion under P^*.

A European call $H = (X_T - K)^+$ can be hedged perfectly if we provide the initial capital

$$H_0 = E^*[H] = x_0 \Phi(d_+) - K\Phi(d_-) \ ,$$

where

$$d_\pm(x_0, K) = \frac{\ln x_0 - \ln K}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$$

and Φ denotes the distribution function of the standard normal distribution. Suppose we want to use only an initial capital $\widetilde{V}_0$ which is smaller than the Black-Scholes price H_0. Under this constraint we want to minimize the shortfall risk $E[l((H - V_T)^+)]$ where l is a given loss function satisfying the assumptions of section 2. We know that the optimal strategy consists in hedging the modified option $\widetilde{\varphi} H$ where $\widetilde{\varphi}$ solves the optimization problem defined by (3.1), (3.2). We will work out the solution explicitly in the case of lower partial moments.

6.1. **Lower partial moments.** We know from (5.1) that the modified claim is given by

$$\widetilde{\varphi}_p H = (X_T - K)^+ - [c^{\frac{1}{p-1}} X_T^{\frac{-\alpha}{p-1}} \wedge (X_T - K)^+] \ . \tag{6.1}$$

Since $c^{\frac{1}{p-1}} x^{\frac{-\alpha}{p-1}}$ is convex and decreasing, there is at most one point of intersection L with $(x-K)^+$. So we get

$$\begin{aligned} \widetilde{\varphi}_p H &= \left[X_T - K - L^{\frac{\alpha}{p-1}}(L-K) X_T^{\frac{-\alpha}{p-1}} \right] \mathbf{1}_{\{X_T \geq L\}} \\ &= f_p(X_T) \end{aligned} \tag{6.2}$$

where

$$f_p(x) = \left[x - K - L^{\frac{\alpha}{p-1}}(L-K) x^{\frac{-\alpha}{p-1}} \right] \mathbf{1}_{\{x \geq L\}} \ . \tag{6.3}$$

Let $\tau = T - t$. Consider the conditional expectation

$$\begin{aligned} V_t &= E^*[\widetilde{\varphi}_p H \mid \mathcal{F}_t] \\ &= E^*[f_p(X_t \exp[\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)]) \mid \mathcal{F}_t] \\ &= F_p(t, X_t) \end{aligned} \tag{6.4}$$

where the function F_p is given by

$$\begin{aligned} F_p(t,x) &= \int_{-\infty}^{\infty} f_p(x \exp[\sigma\sqrt{\tau} y - \frac{1}{2}\sigma^2\tau]) \exp(-\frac{1}{2}y^2) \frac{dy}{\sqrt{2\pi}} \\ &= \int_{-d_-(x,L)}^{\infty} (x \exp[\sigma\sqrt{\tau} y - \frac{1}{2}\sigma^2\tau] - K) \exp(-\frac{1}{2}y^2) \frac{dy}{\sqrt{2\pi}} - (\frac{L}{x})^{\frac{\alpha}{p-1}} (L-K) \times \\ &\quad \times \int_{-d_-(x,L)}^{\infty} \exp[-\frac{\alpha}{p-1}(\sigma\sqrt{\tau} y + \frac{1}{2}\sigma^2\tau)] \exp(-\frac{1}{2}y^2) \frac{dy}{\sqrt{2\pi}} \\ &= x\Phi\left(d_+(x,L)\right) - K\Phi\left(d_-(x,L)\right) - \frac{L^{\frac{\alpha}{p-1}}(L-K)}{x^{\frac{\alpha}{p-1}}} \times \\ &\quad \times \exp[\frac{1}{2}\sigma^2\tau \frac{\alpha}{p-1}(\frac{\alpha}{p-1}+1)] \Phi\left(d_-(x,L) - \frac{\alpha\sigma\sqrt{\tau}}{p-1}\right) . \end{aligned} \tag{6.5}$$

The constant L is determined by the equation

$$\widetilde{V}_0 = E^*[\widetilde{\varphi}_p H] = F_p(0, x_0) \ . \tag{6.6}$$

The strategy is obtained from this by differentiation:

$$\begin{aligned} \xi_p(t,x) &= \frac{\partial}{\partial x} F_p(t,x) = \Phi\left(\frac{\ln x - \ln L}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right) + \frac{\alpha}{p-1} \frac{L^{\frac{\alpha}{p-1}}(L-K)}{x^{\frac{\alpha}{p-1}+1}} \times \\ &\quad \times \exp[\frac{1}{2}\sigma^2\tau \frac{\alpha}{p-1}(\frac{\alpha}{p-1}+1)] \Phi\left(\frac{\ln x - \ln L}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}(\frac{\alpha}{p-1} + \frac{1}{2})\right) \ . \end{aligned} \tag{6.7}$$

Remark 6.1. *As shown in section 5.3 the limit for $p \to \infty$ is again a call option with a strike $\widetilde{K}$ corresponding to the Black-Scholes price $\widetilde{V}_0$:*

$$\widetilde{\varphi}_\infty H = (X_T - \widetilde{K})^+ \tag{6.8}$$

where $\widetilde{K}$ is such that

$$\begin{aligned}\widetilde{V}_0 &= E^*[(X_T - \widetilde{K})^+] \\ &= x_0 \Phi\left(\frac{\ln x_0 - \ln \widetilde{K}}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right) - \widetilde{K}\Phi\left(\frac{\ln x_0 - \ln \widetilde{K}}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}\right) .\end{aligned} \tag{6.9}$$

6.2. **The linear case.** In the case $l(x) = x$ proposition 4.1 shows that we have to knock out the option outside the success set

$$A = \{dP/dP^* > \widetilde{a}\} = \{X_T^\alpha > c'\} = \{X_T > c\} , \tag{6.10}$$

where the constant c is determined by the equation

$$\begin{aligned}\widetilde{V}_0 &= E^*[H\mathbf{1}_A] \\ &= x_0 \Phi\left(\frac{\ln x_0 - \ln c}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - K\Phi\left(\frac{\ln x_0 - \ln c}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) .\end{aligned} \tag{6.11}$$

Thus we have to hedge the modified claim

$$\widetilde{\varphi}_1 H = H\mathbf{1}_{\{X_T > c\}} = (X_T - c)^+ + (c - K)\mathbf{1}_{\{X_T > c\}} . \tag{6.12}$$

The conditional expectations are of the form

$$E^*[\widetilde{\varphi}_1 H \mid \mathcal{F}_t] = F_1(t, X_t) \tag{6.13}$$

where

$$F_1(t,x) = x\Phi\left(\frac{\ln x - \ln c}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right) - K\Phi\left(\frac{\ln x - \ln c}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}\right) . \tag{6.14}$$

By differentiation we obtain the hedging strategy:

$$\xi_1(t, X_t) = \Phi\left(\frac{\ln X_t - \ln c}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right) + \frac{c - K}{\sqrt{2\pi}c\sigma\sqrt{\tau}\exp(\frac{1}{8}\sigma^2\tau)}\left(\frac{c}{X_t}\right)^{\frac{\ln X_t - \ln c}{2\sigma^2(T-T)} + \frac{1}{2}} . \tag{6.15}$$

6.3. **The case of risk-taking.** For $p < 1$ the optimal hedge consists in replicating the knock-out option $H\mathbf{1}_{A_p}$ where

$$A_p = \{X_T^\alpha \geq a_p((X_T - K)^+)^{1-p}\} ,$$

see (5.20). Let us illustrate this result in the case $\alpha < 1$. For $p = 1$ the call is hedged for values in $[0, K] \cup [c, \infty)$, see (6.10). As soon as p drops below 1 both thresholds move up, i.e. the success set takes the form

$$A_p = \{X_T \in [0, b_p] \cup [c_p, \infty)\} \tag{6.16}$$

with $K < b_p < c_p < \infty$. As p decreases to the level $1-\alpha$, the upper threshold c_p goes to ∞ and the lower threshold b_p increases to the value b which is determined by the capital constraint

$$E^*[(X_T - K)^+ \mathbf{1}_{\{X_T \leq b\}}] = \widetilde{V}_0 \ . \tag{6.17}$$

The resulting knock-out option $\widetilde{H} = (X_T - K)^+ \mathbf{1}_{\{X_T \leq b\}}$ is the optimal hedge for any value $p \in (0, 1-\alpha]$, and it is exactly the knock-out option which appears in the case of quantile hedging.

In the case $\alpha \geq 1$ the optimal hedge in the quantile case is of the form

$$(X_T - K)^+ \mathbf{1}_{\{X_T \leq b\} \cup \{X_T \geq c\}}.$$

In accordance with proposition 5.4, the thresholds b and c appear as decreasing limits of the corresponding thresholds b_p and c_p as p goes to zero.

6.4. **Illustrations.** In the following we illustrate the modified claims and strategies determined above and compare them to the Black-Scholes (perfect) hedging strategy. The parameters are chosen as

$$\begin{array}{lll} K = 110 & T = 0.25 & x_0 = 100 \\ m = 0.08 & \sigma = 0.3 & t = 0.1 \end{array}$$

The resulting Black-Scholes price is $H_0 = 2.5$, but we choose to provide only an amount $\widetilde{V}_0 = 1$ of initial capital.

[Insert Graph 1 here]

The dashed line shows the original call. The thick line shows the modified option for $p = 1$, the medium line that for $p = 1.1$, the thin line that for $p = 1.5$ and the dotted line that for $p = 5$. Notice that already for $p = 5$ the modified option is close to a call at a higher strike. This corresponds to the limit for $p \to \infty$ as described in the remark above. Notice also that in the case of risk-aversion ($p > 1$) the modified options are continuous as opposed to the case of the knock-in option for $p = 1$ and the case of the knock-out option of quantile hedging (see [FL (1998)]). In particular undesirable features arising in the hedging of knock-in/out options (see e.g. [W (1998)]) are avoided.

[Insert Graph 2 here]

The dashed line shows the Black-Scholes strategy. The thick line shows the efficient strategy for $p = 1$, the medium line for $p = 1.1$, the thin line for $p = 1.5$, and the dotted line for $p = 5$.

The differences in the attitudes towards shortfall risk are reflected in the shape of the modified claims and the resulting hedging strategies. See [F (1977)] for a thorough discussion of the microeconomic aspects of such a risk analysis. The efficient hedging strategies above allow the investor to interpolate in a systematic way between the extremes of a perfect hedge (no chance of making a profit) and no hedge (full risk of shortfall, full chance of a profit) depending on the investor's appetite for risk.

7. Convex duality methods

Convex duality is a well established tool in mathematical finance, see e.g. [K (1997)]. In this section we briefly show how convex duality methods may be used as an alternative to the reduction to the Neyman Pearson lemma in solving our problem. Additionally this approach reveals some qualitative features of the value function, i.e. the shortfall risk as a function of the initial capital employed. This value function describes the efficient frontier in our problem of balancing cost and shortfall risk. We only sketch the basic steps in this approach; for detailed proofs see [L (1998)].

For the case of minimizing the expected shortfall in a diffusion model the convex duality approach has been worked out in [CK (1998)] and in [C (1998)] under additional constraints on the strategies. In a discrete time setting the case of lower partial moments is treated in [P (1998)]. In our approach we combine the basic duality result from [KS (1997)] with the technique of considering state-dependent Legendre transforms which is used in [CK (1998)], [C (1998)] and[P (1998)].

7.1. **State-dependent utility.** In the initial formulation of our optimization problem we want to minimize the shortfall risk given an amount of initial capital which we now call z. In other words we are looking for a random variable Z (corresponding to the terminal wealth resulting from an admissible strategy) such that

$$E[l((H-Z)^+)] = \min \tag{7.1}$$

under the capital constraint

$$\sup_{P^*\in\mathcal{P}} E^*[Z] \leq z\ . \tag{7.2}$$

Let us reformulate the problem in terms of maximizing expected utility. To this end we introduce the state-dependent utility function

$$U_l(z,\omega) = l(H(\omega)) - l((H(\omega)-z)^+) \tag{7.3}$$

Then the problem (7.1), (7.2) is equivalent to

$$E[U_l(Z(\omega),\omega)] = \max \tag{7.4}$$

under the constraint

$$\sup_{P^*\in\mathcal{P}} E^*[Z] \leq z\ . \tag{7.5}$$

For later purposes we generalize the problem as follows. Let us consider a general state-dependent utility function $U(z,\omega)$ which is non-decreasing and concave in z, strictly concave on $[0,H(\omega)]$ and satisfies $U(.,\omega)\in C^1(0,H(\omega))$. We also assume that

$$-\infty < E[U(0,.)] \quad \text{and} \quad E[U(H,.)] < \infty. \tag{7.6}$$

As in the usual optimization of expected utility the inverse of marginal utility plays a central role. In our case the inverse

$$I(y,\omega) = \inf\{z\in[0,H(\omega)] \mid U'(z,\omega) < y\}$$

is state dependent. We use the convention that $I(y,\omega)=\infty$ if $U'(H(\omega),\omega)\geq y$. We need the stochastic conjugate

$$V(y,\omega) = \max_{0\leq z\leq H(\omega)}(U(z,\omega)-zx) \tag{7.7}$$
$$= U(I(y,\omega)\wedge H(\omega),\omega)-y(I(y,\omega)\wedge H(\omega))$$

of U. The function $V(.,\omega)$ is non-increasing and convex in y, strictly convex on $[U'(H(\omega),\omega),U'(0,\omega)]$, and differentiable with derivative

$$V'(y,\omega)=-(I(y,\omega)\wedge H(\omega)). \tag{7.8}$$

7.2. **Complete case.** In this section we consider the complete case where the equivalent martingale measure is unique. In this context we define the value function

$$u(z)=\sup\{E[U(Z,.)]\mid 0\leq Z\leq H \quad \text{and } E^*[Z]\leq z\}\ . \tag{7.9}$$

We can then express the solution in terms of I via convex duality methods. In particular we recover the solution from section 5 for $U=U_l$. In addition we obtain qualitative properties of the value function u. These properties will be used in our discussion of a volatility jump in section 8.

Theorem 7.1. *1) For each $z\leq E^*[H]$ there is a unique solution $\widetilde{Z}$ such that $u(z)=E[U(\widetilde{Z},.)]$. It takes the form*

$$\widetilde{Z}(\omega)=I(y(z)\rho^*(\omega),\omega)\wedge H(\omega) \tag{7.10}$$

where $y(z)$ is the solution of

$$E^*[I(y(z)\rho^*(\omega),\omega)\wedge H(\omega)]=z \quad . \tag{7.11}$$

2) The conjugate function

$$v(y)=\max_{z\geq 0}(u(z)-zy) \tag{7.12}$$

of u is given by

$$v(y)=E[V(y\rho^*(\omega),\omega] \quad , \tag{7.13}$$

where

$$V(y,\omega)=U(I(y,\omega)\wedge H(\omega))-y(I(y,\omega)\wedge H(\omega))\ . \tag{7.14}$$

Conversely we have

$$u(z)=\min_{y\geq 0}(v(y)+yz). \tag{7.15}$$

3) The value function u is strictly increasing and strictly concave on $[0,E^*[H]]$, and belongs to $\mathcal{C}^1(0,E^*[H])$. Its (right) derivative on $[0,E^*[H])$ is given by

$$u'(z+)=y(z) \tag{7.16}$$

4) In particular we have

$$u'(0+) = +\infty \quad , \tag{7.17}$$

if $\infty = y_1 = \inf\{y \geq 0 \mid v(y) = E[U(0,.)]\}$, *and this is the case if*

$$\text{ess.sup}\, \frac{U'(0,.)}{\rho^*} \mathbf{1}_{\{H>0\}} = \infty. \tag{7.18}$$

The proof is similar to maximization of expected utility except that, as in [CK (1998)], [C (1998)], [P (1998)], one has to use stochastic instead of deterministic conjugates; for a detailed proof see [L (1998)].

Remark 7.1. *In the Black-Scholes model the sufficient condition (7.18) is always satisfied if the drift* m *is positive.*

7.3. **Incomplete case.** The incomplete case involves a fundamental duality which is shown in full generality in [KS (1997)]. Let

$$\mathcal{Z}(z) = \{Z = (Z_t) \geq 0 \mid Z_t = z + \int_0^t \xi dX \quad , \quad \xi \text{ admissible } \}$$

denote the class of possible value processes for admissible strategies starting with initial capital z. Let

$$\mathcal{C}(z) = \{g \in L^0 \mid 0 \leq g \leq Z_T \text{ for some } z \in \mathcal{Z}(z)\}$$

denote the set of claims attainable with initial capital z. Consider the set of supermartingale densities

$$\mathcal{Y}(y) = \{Y \geq 0 \mid Y_0 = y \text{ and } ZY \text{ supermartingale } \forall Z \in \mathcal{Z}(1)\}.$$

and define

$$\mathcal{D}(y) = \{h \in L^0 \mid 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\} \ .$$

Notice that $\mathcal{C}(z) = z\mathcal{C}(1)$ and $\mathcal{D}(y) = y\mathcal{D}(1)$. For brevity let us denote $\mathcal{C}(1)$ simply by $\mathcal{C}$ and $\mathcal{D}(1)$ by $\mathcal{D}$. We quote the following result :

Theorem 7.2. *(Proposition 3.1 in* [KS (1997)]*)*
(i) $\mathcal{C}$,$\mathcal{D}$ *are convex and solid subsets of* L^0_+ *which are closed in the topology of convergence in measure*
(ii) $\mathcal{C}$ *and* $\mathcal{D}$ *stand in bi-polar relation, i.e.*

$$g \in \mathcal{C} \Leftrightarrow E[gh] \leq 1 \quad \forall h \in \mathcal{D} \tag{7.19}$$

$$h \in \mathcal{D} \Leftrightarrow E[gh] \leq 1 \quad \forall g \in \mathcal{C} \tag{7.20}$$

The value function for our problem is defined as

$$u(z) = \sup_{g \in \mathcal{C}(z)} E[U(g \wedge H, .)] \ , \quad z \in [0, \infty). \tag{7.21}$$

It is straightforward to see that u is non-decreasing and concave. With the help of the stochastic conjugate V we can then set up the dual problem

$$\begin{aligned} v(y) &= \inf_{h \in \mathcal{D}(y)} E[V(h(\omega), \omega)] \\ &= \inf_{h \in \mathcal{D}} E[V(yh(\omega), \omega)]. \end{aligned} \tag{7.22}$$

It is again straightforward to see that v is non-increasing and convex.

The full duality picture is described by the following theorem.

Theorem 7.3. *1) For every $y > 0$ there is a solution $\widetilde{h}(y) \in \mathcal{D}$ to the dual problem*

$$v(y) = \inf_{h \in \mathcal{D}} E[V(yh(\omega), \omega)]. \tag{7.23}$$

For any two solutions $\widetilde{f}, \widetilde{h}$

$$U'(H,.) \vee \widetilde{f} \wedge U'(0,.) = U'(H,.) \vee \widetilde{h} \wedge U'(0,.) \quad P-a.s. \text{ on } \{H > 0\}. \tag{7.24}$$

2) $v \in C^1(0, \infty)$ and

$$\begin{aligned} v'(y) &= E[\widetilde{h}(y) V'(y\widetilde{h}(y))] \\ &= -E[\widetilde{h}(y)(I(y\widetilde{h}(y)) \wedge H)] \end{aligned} \tag{7.25}$$

3) For $0 < z < U_0$ there is $y(z) \geq 0$ such that

$$v'(y(z)) = -z. \tag{7.26}$$

and

$$\tilde{g}(z) = I(y(z)\widetilde{h}(y(z))) \wedge H \tag{7.27}$$

is the unique solution ($\leq H$) to the optimization problem

$$u(z) = \sup_{g \in \mathcal{C}(z)} E[U(g \wedge H, .)]. \tag{7.28}$$

4) u and v are conjugates, i.e.

$$u(z) = \min_{y \geq 0} (v(y) + zy) \tag{7.29}$$

$$v(y) = \max_{z \geq 0} (u(z) - zy) \tag{7.30}$$

5) u is strictly increasing and strictly concave on $[0, U_0]$.

6) Assume that $U'(H,.) = 0$ on $\{H > 0\}$. Then v is strictly convex on $(0, y_1)$ where

$$y_1 = \inf\{y \geq 0 \mid v(y) = E[U(0,.)]\} \tag{7.31}$$

and u is in $C^1(0, U_0)$ with

$$u'(z) = y(z). \tag{7.32}$$

If in addition $y_1 = \infty$, then

$$u'(0+) = +\infty. \tag{7.33}$$

Remark 7.2. *Since we know that $U'(H,.) \leq U'(0,.)$, we have*

$$(U'(H,.) \vee g) \wedge U'(0,.) = U'(H,.) \vee (g \wedge U'(0,.)) \tag{7.34}$$

for any function g. Thus there is no ambiguity in the expression $U'(H,.) \vee \widetilde{f} \wedge U'(0,.)$ in (7.24).

Remark 7.3. *At first glance the formulae (7.25) for v' and (7.27) for $\widetilde{g}$ might appear to depend on the choice of the solution $\widetilde{h}$. This is of course not so: For any two solutions f, h one has*

$$E[f(I(yf) \wedge H)] = E[h(I(yh) \wedge H)] \tag{7.35}$$

and

$$I(yf) \wedge H = I(yh) \wedge H \quad P - a.s. \tag{7.36}$$

Remark 7.4. *The condition $U'(H,.) = 0$ in 6) is always satisfied in our original problem where $U = U_l$ for a convex loss function with $l'(0) = 0$.*

There is a difference in the assumptions of the above optimization problem in comparison to [KS (1997)]. Conditions on the asymptotic elasticity are used in [KS (1997)] to control the behavior for large values of the utility. In our context any overshooting the goal H does no good. Consequently the conditions of finiteness of the value function and on the asymptotic elasticity are replaced by our conditions on the integrability of $U(H,.)$ and $U(0,.)$.

As in [KS (1997)] the proof of the above theorem uses the fundamental duality result theorem 7.2. However the rest of our proof is more direct. It is closer in spirit to the proofs in [P (1998)], [CK (1998)] and [C (1998)] - except that we work in a general semimartingale setting with a general state-dependent utility function. The details are given in [L (1998)].

8. Hedging of a volatility jump

As in [FL (1998)] we consider a geometric Brownian motion with positive drift m where the volatility has a constant value $\sigma > 0$ up to time t_0 and then jumps to a new constant value η according to some distribution μ on $(0, \infty)$.

We use an explicit model $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ of the following form. Put $\bar{\Omega} = C[0,T] \times (0, \infty)$ and for $\bar{\omega} = (\omega, \eta)$ define $X_t(\omega) = \bar{X}_t(\bar{\omega}) = \omega(t)$. We fix a time $t_0 \in (0,T)$ and an initial value $x_0 > 0$. For each $\eta > 0$ we define a time-dependent volatility by $\sigma_t(\eta) = \sigma$ for $t < t_0$ and

$\sigma_t(\eta) = \eta$ for $t \geq t_0$. Let P_η denote the unique probability measure on $\Omega = C[0,T]$ such that the process (X_t) satisfies the stochastic differential equation

$$dX_t = X_t(mdt + \sigma_t(\eta)dW_t^\eta) \quad , \quad X_0 = x_0$$

under P_η where W^η is a Wiener process under P_η. The measure $\bar{P}$ on $\bar{\Omega}$ is defined by $\bar{P}(d\omega, d\eta) = \mu(d\eta)P_\eta(d\omega)$. We denote by $\bar{\mathcal{F}}$ the completion of the natural product σ-field on $\bar{\Omega}$ under $\bar{P}$ and by $(\bar{\mathcal{F}}_t)_{t\in[0,T]}$ the right-continuous complete filtration on $\bar{\Omega}$ generated by the processes $(\bar{X}_t)$ and (σ_t). The projection of $\bar{P}$ on Ω is denoted by P and $(\mathcal{F}_t)$ is the right-continuous complete filtration on Ω generated by (X_t).

Consider a European option of the form $H = h(X_T)$ where h is some non-negative continuous function such that our integrability assumptions (2.3) and (2.4) are satisfied. At time t_0 the value $X_{t_0} = x$ is observed and the new volatility η is revealed. From this time on the option can be replicated perfectly using the standard Black-Scholes hedging strategy in the complete model P_η. Let P_η^* denote the unique equivalent martingale measure of this model. The required initial capital is given by

$$v^\eta(x) = E_\eta^*[h(X_T) \mid X_{t_0} = x] \ .$$

We set

$$v(x) = \sup_\eta v^\eta(x) \quad , \tag{8.1}$$

where the supremum is taken over the support of μ.

For $t < t_0$ the value η is still unknown. As in [FL (1998)] the capital required for superhedging at time t is given by

$$U_t = E^*[v(X_{t_0}) \mid \mathcal{F}_t] \ , \tag{8.2}$$

where P^* denotes the common projection of all equivalent martingale measures $\bar{P}^* \approx \bar{P}$ on $(\Omega, \mathcal{F}_{t_0})$.

Remark 8.1. *If H is a call and μ has unbounded support, then we get $v(x) = x$, hence $U_t = X_t$ for $t < t_0$ and in particular $U_0 = x_0$. In this case the superhedging strategy is reduced to the following trivial procedure: Buy one unit of the underlying asset at time 0 and hold it up to time t_0. At that time the value η is revealed. Pay out the refund $C_{t_0} = X_{t_0} - v^\eta(X_{t_0})$ and use the remaining capital $v^\eta(X_{t_0})$ to implement a perfect hedge of the option.*

Let us now fix a loss function l. Our aim is to minimize the shortfall risk under the constraint that the initial capital is not larger than some fixed amount $\widetilde{V}_0$ such that

$$0 < \widetilde{V}_0 < U_0 \ . \tag{8.3}$$

At time t_0 let $\beta^\eta(z,x)$ denote the minimal shortfall risk that can be achieved given the present state $x = X_{t_0}$ and some capital $z \geq 0$ if the new volatility is given by η, i.e.

$$\beta^\eta(z,x) = \min\{E_\eta[l((1-\varphi)H) \mid X_{t_0} = x] \mid E_\eta^*[\varphi H \mid X_{t_0} = x] \leq z \ , \ \varphi \in \mathcal{R}\}. \tag{8.4}$$

Further let $u^\eta(z,x)$ denote the value function of the corresponding complete problem. In analogy to (7.3) it is related to $\beta^\eta(z,x)$ via

$$u^\eta(z,x) = E_\eta[l(H) \mid X_{t_0} = x] - \beta^\eta(z,x). \tag{8.5}$$

We know from theorem 7.1 that $u^\eta(.,x)$ is non-decreasing and concave, strictly increasing and strictly concave on the interval $[0, E^*_\eta[H \mid X_{t_0} = x]]$. Furthermore it belongs to $\mathcal{C}^1(0, E^*_\eta[H \mid X_{t_0} = x])$ and its (right) derivative equals

$$\frac{\partial}{\partial z} u^\eta(z,x) = y^\eta(z,x) \ ,$$

where $y^\eta(z,x)$ is the solution of

$$E^*_\eta[I(y^\eta(z,x)\rho^*) \wedge H \mid X_{t_0} = x] = z \ .$$

Now we assume that the functions β and u defined by

$$\beta(z,x) = \int \beta^\eta(z,x)\mu(d\eta)$$

and

$$u(z,x) = \int u^\eta(z,x)\mu(d\eta)$$

are finite for all z,x. In the case $l(x) = x^p$ these conditions are satisfied if

$$\int \exp[\frac{p}{2}(p+1)\eta^2 T]\mu(d\eta) < \infty \ . \tag{8.6}$$

The function u has again the relevant properties of a state-dependent utility function:

Lemma 8.1. *The function $u(.,x)$ is non-decreasing, strictly increasing on $[0, v(x))$, concave and strictly concave on $[0, v(x))$. Furthermore it belongs to $\mathcal{C}^1(0, v(x))$ with derivative*

$$\frac{\partial}{\partial z} u(z,x) = \int \frac{\partial}{\partial z} u^\eta(z,x)\mu(d\eta). \tag{8.7}$$

Proof. 1) It is straightforward to see that u is non-decreasing and concave. Now let $0 \le z < z' < v(x)$. There exists a set $A \subset (0,\infty)$ with $\mu(A) > 0$ and

$$E^*_\eta[H \mid X_{t_0} = x] \ge z' \quad \forall \eta \in A.$$

Otherwise we would have $E^*_\eta[H \mid X_{t_0} = x] < z'$ for μ-almost all η in contradiction to $z' < v(x)$. Consequently we have for all $\eta \in A$

$$u^\eta(z,x) < u^\eta(z',x)$$

and

$$u^\eta(\lambda z + (1-\lambda)z', x) > \lambda u^\eta(z,x) + (1-\lambda)u^\eta(z',x).$$

As $\mu(A) > 0$, this implies

$$u(z,x) < u(z',x)$$

and

$$u(\lambda z + (1-\lambda)z', x) > \lambda u(z,x) + (1-\lambda)u(z',x).$$

2) We have

$$u^\eta(z',x) = \int_0^{z'} \frac{\partial}{\partial z} u^\eta(z,x)dz$$

by the properties of u^η and the fact that $u^\eta(0,x) = 0$. Because

$$\frac{\partial}{\partial z} u^\eta(z,x) = y^\eta(z,x) \geq 0,$$

we can apply Fubini's theorem to conclude that

$$\begin{aligned} u(z',x) &= \int u^\eta(z',x)\mu(d\eta) \\ &= \int (\int_0^{z'} \frac{\partial}{\partial z} u^\eta(z,x)dz)\ \mu(d\eta) \\ &= \int_0^{z'} (\int \frac{\partial}{\partial z} u^\eta(z,x)\mu(d\eta))\ dz. \end{aligned}$$

This shows that $u(.,x)$ is differentiable on $(0, v(x))$ with derivative

$$\frac{\partial}{\partial z} u(z,x) = \int \frac{\partial}{\partial z} u^\eta(z,x)\mu(d\eta).$$

Since u is a concave function which is differentiable on the convex, open set $(0, v(x))$, it is also continuously differentiable on this set. $\square$

Let ν resp. ν^* denote the distribution of X_{t_0} under P resp. P^*. We define the inverse $J(.,x)$ of $\frac{\partial}{\partial z} u(.,x)$ by

$$J(y,x) = \inf\{z \in [0, v(x)] \mid \frac{\partial}{\partial z} u(z,x) = \int \frac{\partial}{\partial z} u^\eta(z,x)\mu(d\eta) < y\}$$

with the convention that $J(y,x) = \infty$ if $\frac{\partial}{\partial z} u(v(x),x) \geq y$. We can now apply again theorem 7.1 to obtain the following proposition.

Proposition 8.2. *There exists a unique function $\widetilde{f}$ such that*

$$\int u(\widetilde{f}(x),x)\nu(dx) = \sup_f \int u(f(x),x)\nu(dx) \tag{8.8}$$

where the supremum is taken over all measurable functions $f \geq 0$ on $(0,\infty)$ with

$$\int f d\nu^* \leq \widetilde{V}_0. \tag{8.9}$$

The solution is of the form

$$\widetilde{f}(x) = J(c(\widetilde{V}_0)\frac{d\nu^*}{d\nu}(x)) \wedge v(x) \ , \tag{8.10}$$

where $c(\widetilde{V}_0)$ is the solution of

$$\int J(c(\widetilde{V}_0)\ \frac{d\nu^*}{d\nu}(x)) \wedge v(x)\ \nu^*(dx) = \widetilde{V}_0 \ . \tag{8.11}$$

Proof. It is easy to see that a function which maximizes the integral in (8.8) must belong to the class C of all measurable functions $f \geq 0$ with $\int f d\nu^* = \widetilde{V}_0$ and $0 \leq f \leq v$. But then we are precisely in the situation of theorem 7.1 if we take v as the contingent claim, $u(z,x)$ as the state-dependent utility function, and consider $(0,\infty)$ with ν resp. ν^* as the basic probability space. □

Theorem 8.3. *The following strategy is optimal:*
i) Up to time t_0 use the strategy which replicates the contingent claim $\widetilde{f}(X_{t_0})$ where $\widetilde{f}$ is the solution to the optimization problem (8.8), (8.9).
ii) From time t_0 on use the strategy which minimizes the shortfall risk under the new volatility η given the initial capital $\widetilde{f}(X_{t_0})$ (see section 5).

Proof. Consider any admissible strategy (V_0, ξ) with $V_0 \leq \widetilde{V}_0$ and denote by φ the corresponding succes ratio. The resulting value

$$V_t = V_0 + \int_0^t \xi_s dX_s$$

will be viewed as a random variable on $(\Omega, \mathcal{F}_{t_0})$ for any $t \leq t_0$. We have

$$V_0 \leq \widetilde{V}_0 \ ,$$

and the conditional shortfall risk satisfies

$$\bar{E}[l((\bar{H} - V_T)^+) \mid \mathcal{F}_{t_0}] = \bar{E}[l((1-\varphi)\bar{H}) \mid \mathcal{F}_{t_0}] \geq \beta(V_{t_0}, X_{t_0}) \ .$$

This implies

$$\bar{E}[l((1-\varphi)\bar{H})] \geq E[\beta(V_{t_0}, X_{t_0})] \ . \tag{8.12}$$

Let g be a measurable function such that

$$g(X_{t_0}) = E[V_{t_0} \mid X_{t_0}] \quad P-a.s.$$

Since $\beta(.,x)$ is convex, (8.12) implies

$$\bar{E}[l((1-\varphi)\bar{H})] \geq E[E[\beta(V_{t_0}, X_{t_0}) \mid X_{t_0}]] \geq E[\beta(g(X_{t_0}), X_{t_0})] \tag{8.13}$$

via Jensen's inequality for conditional expectations. Since

$$E[V_{t_0} \mid X_{t_0}] = E^*[V_{t_0} \mid X_{t_0}] \ ,$$

we have

$$E^*[g(X_{t_0})] = E^*[V_{t_0}] \leq V_0 \ .$$

The optimality of $\widetilde{f}$ with respect to (8.8) and the relation (8.5) now imply

$$E[\beta(\widetilde{f}(X_{t_0}), X_{t_0})] \leq E[\beta(g(X_{t_0}), X_{t_0})] \ .$$

Thus

$$\bar{E}[l((\bar{H} - V_T)^+)] \geq E[\beta(\widetilde{f}(X_{t_0}), X_{t_0})] \ ,$$

and this shows that $E[\beta(\widetilde{f}(X_{t_0}), X_{t_0})]$ is a lower bound for the shortfall risk if the initial cost is bounded by $\widetilde{V}_0$. But this bound is actually achieved if we use the strategy described in the theorem. □

Remark 8.2. *It is straightforward to iterate backwards the arguments of this section and to treat the case where volatility jumps at finitely many time points. This is based on theorem 7.1 which shows that the value function in each step inherits the regularity properties of the respective utility function.*

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