Generating series for special cycles on unitary Shimura varieties

A Survey

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Moduli and Automorphic Forms
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The connections between

the geometry and arithmetic of
algebraic cycles on
locally symmetric varieties
$$M_\Gamma = \Gamma \backslash D$$

and

Fourier coefficients of
modular forms

is a fascinating subject with a long history.

In this lecture I will discuss old and new results in the case:

$$D = \text{ open unit ball in } \mathbb{C}^n$$

$$\Gamma = \text{arithmetic subgroup of } U(n, 1).$$

Here are the topics:

§1. A classical example

§2. Ball quotients, their special cycles and modular forms (joint work with John Millson)

§3. The arithmetic theory (joint work with M. Rapoport)

§4. The generating series for arithmetic 0-cycles

§1. A classical example:

The most familiar example of a locally symmetric variety is the quotient:

$$M_\Gamma = \Gamma\backslash D, \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R}),$$

$$D = \text{open unit ball in } \mathbb{C}$$

$$\simeq \text{upper half plane.}$$

- The complex geometry is very simple[1]:

$$j : \Gamma\backslash D^* \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C}).$$

- The arithmetic is very beautiful due to the fact that:

$$\Gamma\backslash D \quad = \quad \text{moduli space for elliptic curves.}$$

[1] A point is added at the cusp.

Because of the moduli space structure, there are certain special points on $\Gamma\backslash D$: the CM points.

- For most points $[z] \in \Gamma\backslash D$ with corresponding elliptic curve

$$E_z = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z)$$

we have

$$\mathrm{End}(E_z) = \mathbb{Z}.$$

- A point $[z] \in \Gamma\backslash D$ is a CM point if

$$\mathrm{End}(E_z) = \begin{pmatrix} \text{a ring of integers in an} \\ \text{imaginary quadratic field} \\ \mathbb{Q}(\sqrt{-t}). \end{pmatrix}$$

The curve E_z is then said to have complex multiplication.

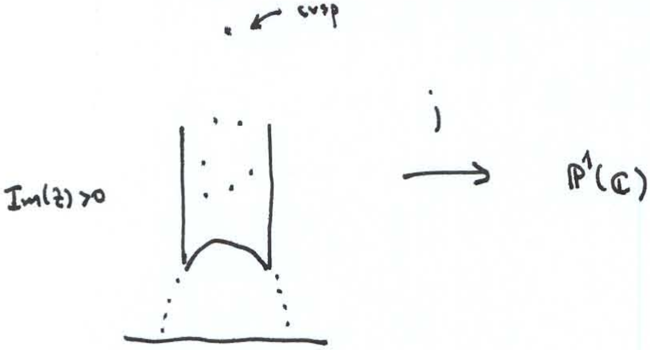
- For a positive integer t, let

$$Z(t) = \sum_{\substack{[z] \\ \mathrm{End}(E_z)=\mathbb{Z}[\sqrt{-t}]}} [z] \quad = \text{ 0-cycle in } M_\Gamma = \Gamma\backslash D^*.$$

- Its image in cohomology is simply its degree:

$$\begin{array}{ccc} H^2(M_\Gamma, \mathbb{Z}) & \stackrel{\sim}{\longrightarrow} & \mathbb{Z} \\ [Z(t)] & \mapsto & H(t) \quad = \#Z(t) \end{array}$$

where $H(t)$ is the class number of the ring $\mathbb{Z}[\sqrt{-t}]$.
This essentially goes back to Gauss.


cusp
Im(z) > 0
j
$\mathbb{P}^1(\mathbb{C})$

- A more striking fact is the following result:

Theorem (Zagier, 1975)

The generating series $(\tau = u + iv,\ q = e^{2\pi i \tau})$

$$\phi(\tau) = -\frac{1}{12} + \sum_{t>0} H(t)\, q^t + (!!!)$$

for the degrees of the 0-cycles $Z(t)$
is a (non-holomorphic) modular form of weight $\frac{3}{2}$.

To obtain a modular form, an extra bit

$$(!!!) = \sum_n \frac{1}{16\pi} v^{-\frac{1}{2}} \int_1^\infty e^{-4\pi n^2 vr}\, r^{-\frac{3}{2}}\, dr\ q^{-n^2}$$

must be added to the generating series.

Some background about modular forms:

- Recall that a modular form f of weight k is a holomorphic function of $\tau = u + iv \in \mathfrak{H}$, $v > 0$, such that

$$f((a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^k f(\tau)$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' = \begin{matrix} \text{a subgroup of finite} \\ \text{index in } \mathrm{SL}_2(\mathbb{Z}). \end{matrix}$$

- Due to the invariance under a suitable power of the translation $\tau \mapsto \tau + 1$, a modular form has a Fourier series

$$f(\tau) = \sum_n a(n)\, q^n, \qquad q = e^{2\pi i \tau}.$$

Some philosophy:

- The Fourier coefficients $a(n)$ of a modular form satisfy many 'mysterious' relations and seem to have some 'unreasonable effectiveness' as models of generating series arising in many areas of mathematics.
- The statement that some generating series

$$\sum_n a(n)\, q^n$$

is a modular form is a very strong assertion concerning the 'coherence' of the $a(n)$'s.
- We will be interested in more exotic modular forms (possibly non-holomorphic) whose coefficients are cohomology classes, or cycle classes, for more general arithmetic quotients $M_\Gamma = \Gamma \backslash D$.

§2. Ball Quotients:

These arithmetic quotients are constructed as follows:

- Fix an imaginary quadratic field $\Bbbk = \mathbb{Q}(\sqrt{\Delta})$, $\Delta \in \mathbb{Z}_{<0}$ with ring of integers $O_\Bbbk$, e.g. $\Bbbk = \mathbb{Q}(i)$, $O_\Bbbk = \mathbb{Z}[i]$, and
- a $\Bbbk$-vector space with hermitian form of signature $(n, 1)$:

$$V = \Bbbk^{n+1} \quad \subset \quad \mathbb{C}^{n+1}$$

e.g. $$(x, y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n - x_{n+1}\bar{y}_{n+1}$$

- Let

$$D = \text{space of negative lines in } \mathbb{C}^{n+1}$$
$$\simeq \text{open unit ball in } \mathbb{C}^n.$$

$$G = U(V) = \text{isometry group of } (\ ,\ ) \text{ in } \mathrm{GL}_{n+1}(\Bbbk).$$

$$G(\mathbb{R}) \simeq U(n, 1), \qquad D \simeq U(n, 1)/(U(n) \times U(1))$$

- The arithmetic subgroup Γ of G arises as follows:

$$L = (O_k)^{n+1} \quad \subset V, \quad \text{an } O_k\text{-lattice}$$

$$\Gamma = \text{the stabilizer of } L \text{ in } G.$$

$$\text{or a subgroup of finite index}$$

$$= \text{a discrete subgroup of } U(n,1),$$

- and the ball quotient is:

$$M_\Gamma = \Gamma \backslash D.$$

- By general results (Bailey-Borel, Shimura, Deligne and others)

$$j : \Gamma\backslash D = M_\Gamma \quad \xrightarrow{\sim} \quad X_\Gamma(\mathbb{C}) \ \subset \mathbb{P}^N(\mathbb{C}),$$

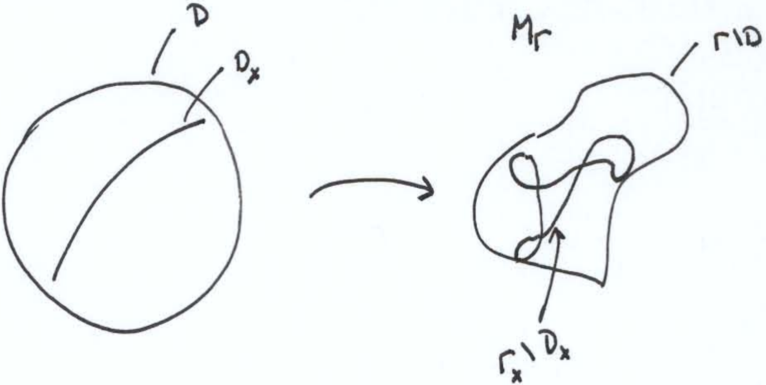
where X_Γ is a quasi-projective variety of dimension n defined over k.

- Special algebraic cycles in X_Γ can be defined as follows:

$$\begin{aligned} x \in V, &\qquad \text{a vector with } (x,x) > 0, \\ D_x &= \{\, z \in D \mid x \perp z \,\} \\ &= \text{negative lines perpendicular to } x \\ &= \text{ a complex } (n-1) \text{ ball in } D. \\ \Gamma_x &= \text{stablizer of } x \text{ in } \Gamma \\ Z(x) &= \Gamma_x \backslash D_x \longrightarrow \Gamma \backslash D \\ &= \text{ a divisor in } X_\Gamma. \text{ (dep. only on the } \Gamma \text{ orbit of } x) \end{aligned}$$

- For $t > 0$, there is a finite sum of such divisors:

$$Z(t) = \sum_{\substack{x \in L \\ (x,x)=t \\ \text{mod } \Gamma}} Z(x).$$


D
D_x
M_Γ
Γ\D
Γ_x\D_x

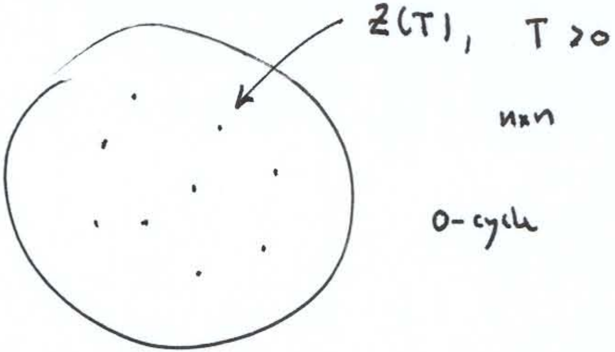
- A little more generally, there are special algebraic cycles of any codimension r, where $1 \le r \le n$:

$$\begin{aligned} \mathbf{x} &= [x_1, \dots, x_r] \in V^r, \quad \text{an } r\text{-tuple with} \\ (\mathbf{x}, \mathbf{x}) &= ((x_i, x_j)) > 0 \\ D_{\mathbf{x}} &= \{\, z \in D \mid z \perp \mathbf{x} \,\} \simeq (n-r)\text{-ball in } D \\ Z(\mathbf{x}) &= \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x}} \longrightarrow \Gamma \backslash D \\ &= \text{algebraic cycle in } X_\Gamma \text{ of codimension } r. \end{aligned}$$

- Again we can define certain composite cycles:

$$T = {}^t\bar{T} > 0, \quad \text{a positive definite hermitian in } M_r(O_{\Bbbk})$$

$$Z(T) = \sum_{\substack{\mathbf{x} \in L^r \\ (\mathbf{x}, \mathbf{x}) = T \\ \text{mod } \Gamma}} Z(\mathbf{x}).$$


$Z(T)$, $T > 0$
$n \times n$
0-cycle

With this big supply of algebraic cycles in X_Γ we can construct generating functions.

For $T \in \mathrm{Herm}_r(O_k)_{>0}$, there are cohomology class

$$[Z(T)] \in H^{2r}(X_\Gamma, \mathbb{C}).$$

(We also need classes for singular T's.)

Theorem

(K.-Millson). For any r with $1 \leq r \leq n$, the generating series

$$\phi_r(\tau) = \sum_{T \geq 0} [Z(T)]\, q^T, \qquad q^T = e^{2\pi i \mathrm{tr}(T\tau)},$$

is a hermitian modular form of weight $n+1$ with values in $H^{2r}(X_\Gamma)$.

- A hermitian modular form is a holomorphic function ϕ on the space

$$\mathfrak{H}_r = \{\tau \in M_r(\mathbb{C}) \mid v = \frac{1}{2i}(\tau - {}^t\bar{\tau}) \in \mathrm{Herm}_r(\mathbb{C})_{>0}\},$$

such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subset \mathrm{U}(r,r)$,

$$\phi((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^{n+1}\, \phi(\tau).$$

This theorem is proved by constructing a (non-holomorphic) theta series

$$\theta_r(\tau, L) = \sum_{\mathbf{x} \in L^r} \varphi_{KM}(\tau, \mathbf{x}) \in A^{(r,r)}(X_\Gamma)$$

valued in the deRham complex of $X_\Gamma \simeq \Gamma \backslash D$.

This is a (non-holomorphic) modular form.

Its image in cohomology is the generating series!

$$\phi_r(\tau) = [\theta(\tau, L)] \in H^{2r}(X_\Gamma).$$

§3. The arithmetic theory (joint work with M. Rapoport)

- There is deeper arithmetic theory of special algebraic cycles on ball quotients due to the fact that

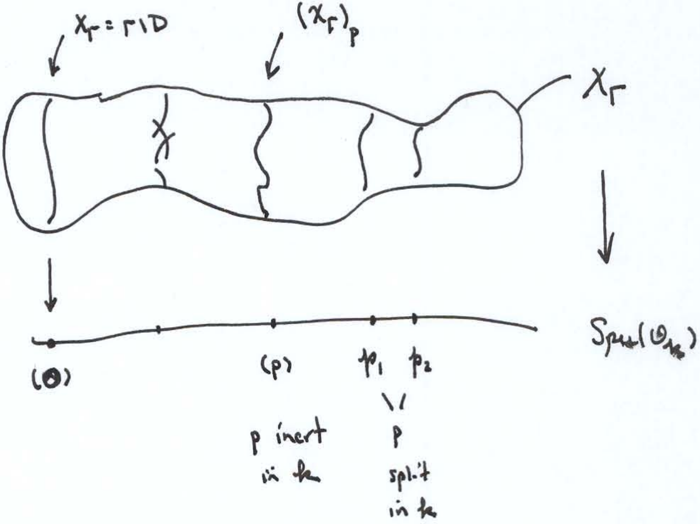
$$X_\Gamma = \begin{pmatrix} \text{a moduli space for polarized} \\ \text{abelian varieties} \\ O_k\text{-action of signature } (n,1) \end{pmatrix}$$

i.e., certain algebraic complex tori of dimension $n+1$.

- This definition of X_Γ can be extended to give a moduli scheme $\mathcal{X}_\Gamma$:

$$\begin{array}{ccccc} X_\Gamma & \longrightarrow & \mathcal{X}_\Gamma & \longleftarrow & (\mathcal{X}_\Gamma)_p \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\,(\mathbb{C}) & \longrightarrow & \mathrm{Spec}\,(O_k) & \longleftarrow & \mathrm{Spec}\,(\mathbb{F}_{p^2}). \end{array}$$

$\mathcal{X} = \mathcal{X}_\Gamma$ has dimension $n+1$.


$X_\Gamma = \Gamma\backslash D$
$(X_\Gamma)_p$
X_Γ
$Spec(O_k)$
(0)
(p)
$\mathfrak{p}_1$ $\mathfrak{p}_2$
p inert
in k
p
split
in k

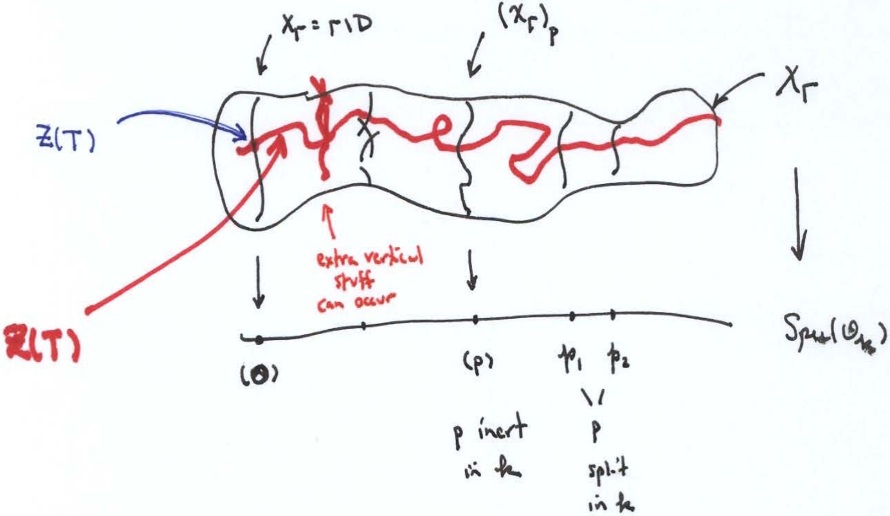
- The special cycles $Z(T)$ in X_Γ can be described in terms of the moduli of abelian varieties:
- Given (A, ι, λ) in X_Γ, and (E, ι, λ_E), a CM elliptic curve

$$\mathrm{Hom}_{O_k}(E, A)$$

has an O_k-valued hermitian form h.
- Then

$$Z(T) = \left\{ \begin{array}{c} \text{locus in } X_\Gamma \text{ of } (A, E, \mathbf{x}) \\ \mathbf{x} = [x_1, \dots, x_r] \\ h(\mathbf{x}, \mathbf{x}) = (h(x_i, x_j)) = T \end{array} \right\}.$$

- This allows us to generalize the definition to give special cycles $\mathcal{Z}(T)$ in $\mathcal{X} = \mathcal{X}_\Gamma$, extending the $Z(T)$'s in the generic fiber.



$\mathcal{X}_\Gamma = \Gamma \backslash D$
$(\mathcal{X}_\Gamma)_p$
$\mathcal{X}_\Gamma$
$Z(T)$
$\mathcal{Z}(T)$
extra vertical stuff can occur
$\text{Spec}(O_k)$
(0)
(p)
p_1
p_2
p inert in k
p split in k

- We would again like to form generating functions.
- The role of the cohomology groups $H^{2r}(X_\Gamma)$ is now played by the arithmetic Chow groups $\widehat{\mathrm{CH}}^r(\mathcal{X}_\Gamma)$, $0 \le r \le n+1$.
- To obtain classes

$$\widehat{\mathcal{Z}}(T) \in \widehat{\mathrm{CH}}^r(\mathcal{X}_\Gamma)$$

from the $\mathcal{Z}(T)$'s, we still need to define Green currents...
- The goal is to define

$$\widehat{\phi}_r(\tau) = \sum_T \widehat{\mathcal{Z}}(T)\, q^T$$

and to show that it is a (non-holomorphic) hermitian modular form with values in $\widehat{\mathrm{CH}}^r(\mathcal{X}_\Gamma)$....
- This is work in progress!

§4. Generating series for arithmetic 0-cycles

For $T \in \mathrm{Herm}_{n+1}(O_k) > 0$,

$$\text{expected dim}\, \mathcal{Z}(T) = 0.$$

This is usually false!

Theorem 1. (K.-Rapoport) (i) $\mathcal{Z}(T)_{\mathbb{Q}} = \emptyset$.

(ii) Either $\mathcal{Z}(T)$ is empty or

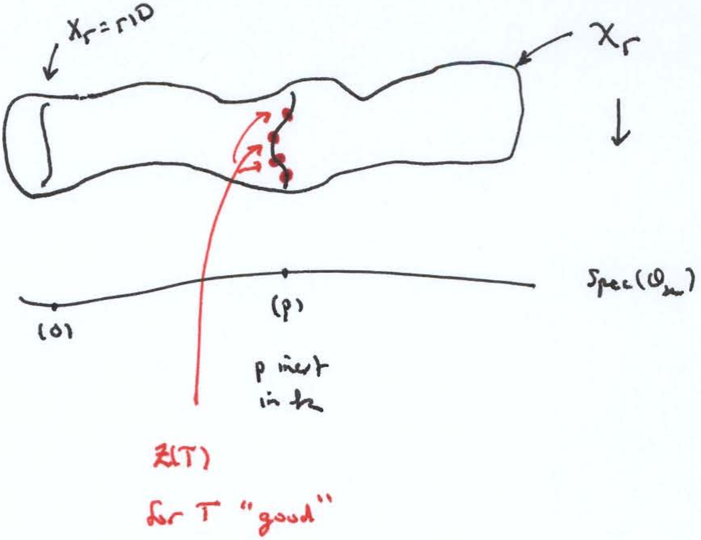
$$\text{supp}\, \mathcal{Z}(T) \subset \mathcal{X}_p^{\mathrm{ss}}$$

for a unique prime p, non-split in k.

(iii) Suppose that p is inert in k. Then $\mathcal{Z}(T)$ is a 0-cycle if and only if

$$T \simeq \mathrm{diag}(1_{n-1}, p^a, p^b) \in \mathrm{Herm}_{n+1}((O_k)_p), \quad 0 \le a \le b.$$

Definition: In case (iii), we call T 'good'.


X_r
Spec(O_k)
(0)
(p)
p inert
in k
Z(T)
for T "good"

Theorem 2. (K.-R.)
When $T \in \mathrm{Herm}_{n+1}(O_k) > 0$ is good the 0-cycle $\mathcal{Z}(T)$ has arithmetic degree

$$\widehat{\deg}\, \mathcal{Z}(T) = \text{ \# of points } \cdot \mu_p(T).$$

$$\mu_p(T) = \text{local multiplicity}.$$

Moreover,

$$\mu_p(T) = \frac{1}{2} \sum_{\ell=0}^{a} p^{\ell}\,(a+b+1-2\ell).$$

Using this formula, and counting points in the support of $\mathcal{Z}(T)$, we obtain a partial modularity result:

Theorem 3. (K.-R.). The partial generating series

$$\widehat{\phi}^{\,\text{partial}}_{n+1}(\tau) = \sum_{\substack{T \in \mathrm{Herm}_n(O_{\boldsymbol{k}})_{>0} \\ T \text{ good}}} \widehat{\deg}\, \mathcal{Z}(T)\, q^T,$$

is part of the q-expansion of a (non-holomorphic) hermitian modular form of weight $n+1$.

More precisely, there is an (incoherent) Eisenstein series:

$$\mathcal{E}(\tau, s) = C(s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \det(c\tau + d)^{-n-1} \frac{\det v(\tau)^{\frac{s}{2}}}{|\det(c\tau + d)|^s} \Phi(\gamma, s),$$

convergent for $s \in \mathbb{C}$, $\mathrm{Re}(s) > n+1$, such that

$$\mathcal{E}(\tau, 0) = 0,$$

and

$$\mathcal{E}'(\tau, 0) = \widehat{\phi}^{\,\text{partial}}_{n+1}(\tau) + \sum_{T \text{ other}} a(T, v(\tau))\, q^T.$$

Via p-adic uniformization of $\mathcal{X}_p^{\mathrm{ss}}$, we have

Conjecture: For any $T \in \mathrm{Herm}_{n+1}(O_k)_{>0}$, the virtual 0-cycle $\mathcal{Z}(T)$ has arithmetic degree

$$\widehat{\deg}\, \mathcal{Z}(T) = \text{ \# of "components" } \cdot \mu_p(T).$$

with local multiplicity

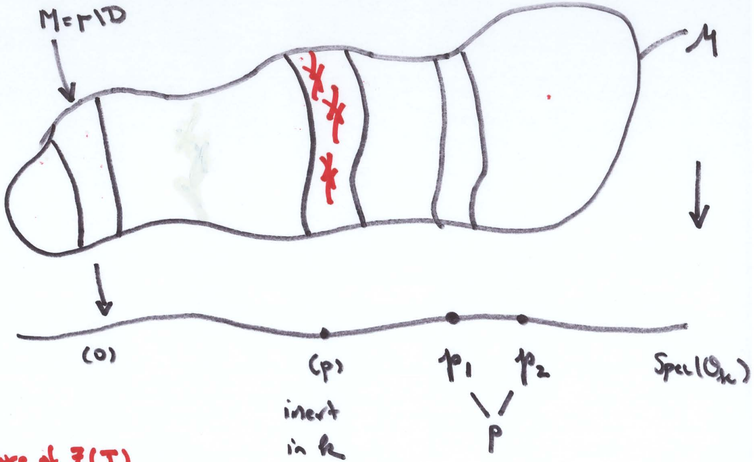
$$\mu_p(T) := \chi(\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_{n+1})}),$$

where $\mathcal{Z}(x_i)$ are certain special cycles on the Rapoport-Zink space $\mathcal{N} = \mathcal{N}(n, 1)$.

Moreover,

$$\mu_p(T) = c_p\, \alpha'_p(S, T)$$

for the derivative of a representation density for hermitian forms and an explicit constant c_p.



Picture of $\mathcal{Z}(T)$

$n=3$

support in a unique $\mathcal{M}_p$.

Theorem.(Terstiege (2010)) For $T \in \mathrm{Herm}_3(O_k)_{>0}$, with

$$T \sim \mathrm{diag}(p^a, p^b, p^c), \qquad 0 \leq a \leq b \leq c,$$

the multiplicity of each connected component is

$$\mu_p(T) = \chi(\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_2)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_3)})$$

$$= -\frac{1}{2} \sum_{k=0}^{a} \sum_{\ell=0}^{a+b-2k} (-1)^k \left((k+\ell)\, p^{2k+\ell} - (k+\ell+c+1)\, p^{a+b-\ell}\right).$$

Moreover $\mu_p(T) = c_p\, \alpha'_p(S, T)$, as predicted.

Terstiege also does the counting so that:

Corollary (T). For the incoherent Eisenstein series $\mathcal{E}(\tau, s)$ on $U(3,3)$,

$$\mathcal{E}'(\tau, 0) = \widehat{\phi}_3^{\text{partial}}(\tau) + \sum_{T \text{ other}} a(T, v(\tau))\, q^T,$$

where all positive definite coefficients coincide!

- These results provide evidence for the existence of the hermitian modular forms $\widehat{\phi}_r(\tau)$'s.
- There are many fascinating open problems in this area and a lot of work remains to be done.

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