Theory and Computations for Multiple Optimization Problems with Equilibrium Constraints (MOPEC)

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Extended Mathematical Programs

- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements under resource constraints
- Problem format is old/traditional

$$\min_x f(x) \text{ s.t. } g(x) \leq 0, h(x) = 0$$

- Extended Mathematical Programs allow annotations of constraint functions to augment this format.
- Give three examples of this: complementarity problems, multi-agent competitive models and bilevel programming

The PIES Model (Hogan)

$$
\begin{array}{lll}
\min_x & c^T x & \text{cost} \\
\text{s.t.} & Ax \geq d(p) & \text{balance} \\
 & Bx = b & \text{technical constr} \\
 & x \geq 0 &
\end{array}
$$

- Issue is that p is the multiplier on the "balance" constraint of LP
- Extended Mathematical Programming (EMP) facilitates annotations of models to describe additional structure
- Can solve the problem by writing down the KKT conditions of this LP, forming an LCP and exposing p to the model
- EMP does this automatically from the annotations

Reformulation details

$$\begin{array}{lcl} 0 \le Ax - d(p) & \perp & \mu \ge 0 \\ 0 = Bx - b & \perp & \lambda \\ 0 \le -A^T\mu - B^T\lambda + c & \perp & x \ge 0 \end{array}$$

- empinfo: dualvar p balance
- replaces $\mu \equiv p$
- LCP/MCP is then solvable using PATH

$$z = \begin{bmatrix} p \\ \lambda \\ x \end{bmatrix}, \quad F(z) = \begin{bmatrix} & & A \\ & & B \\ -A^T & -B^T & \end{bmatrix} \begin{bmatrix} p \\ \lambda \\ x \end{bmatrix} + \begin{bmatrix} -d(p) \\ -b \\ c \end{bmatrix}$$

Complementarity Problems in Economics (MCP)

- p represents prices, x represents activity levels
- System model: given prices, (agent) i determines activities x_i

$$G_i(x_i, x_{-i}, p) = 0$$

  x_{-i} are the decisions of other agents.
- Walras Law: market clearing

$$0 \leq S(x,p) - D(x,p) \perp p \geq 0$$

- Key difference: optimization assumes you control the complete system
- Complementarity determines what activities run, and who produces what

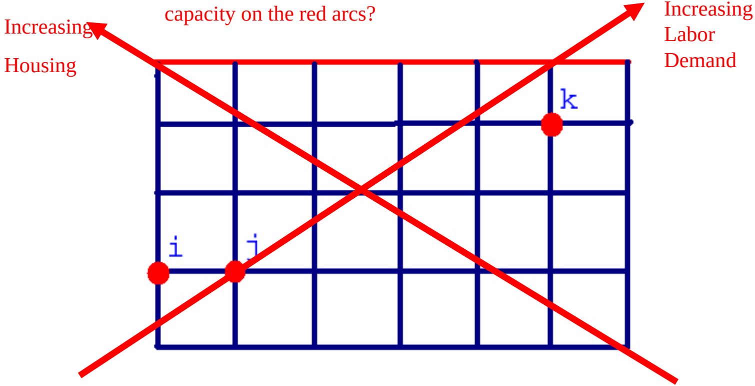
Use of complementarity

- Pricing electricity markets and options
- Video games: model contact problems
  - Friction only occurs if bodies are in contact
- Structure design
  - how springy is concrete
  - optimal sailboat rig design
- Computer/traffic networks (Wardrop)
  - The price of anarchy measures difference between "system optimal" (MPCC) and "individual optimization" (MCP)

- Complementarity facilitates modeling of competition, nonsmoothness and "switching"
- Large scale models involving complementarity now solvable
- Do you (or should you) care?

Walras meets Wardrop

What is the effect on housing prices of increasing capacity on the red arcs?



Features

- We buy a house to "optimize" some measure
  - Price driven by market
  - We compete against each other
- Driver's choose routes to "optimize" travel time
  - Choices affect congestion
  - Your choice affects me!
- Production processes are "optimized"
- But the road designer does not control any of these!
- Other models allow for parking policy design for street and ramp options

Simplified AGE model

$$(P): \min_{y \geq 0} \quad c^T y \qquad \text{s.t.} \quad Ay \geq d \quad (\perp p \geq 0)$$

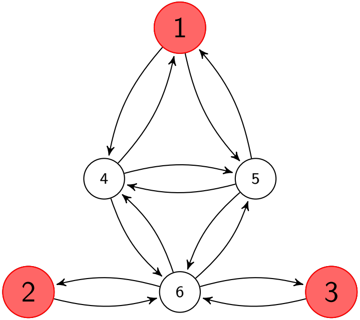
$$(C): \max_{d \geq 0} \quad u(d) \qquad \text{s.t.} \quad p^T d \leq I$$

- In equilibrium, the optimal demand d from (C) will be the demand in (P), and the sales price p in (C) will be the marginal price on production from (P)
- Complementarity conditions of (P) and (C) have both primal and dual variables
- Optimization models linked by variables and multipliers
- Equilibrium problem solvable as a complementarity problem
- Can add "other features" such as taxation, transportation, tolls.

Model building with EMP

- Take one system of (nonlinear) equations and annotate them to:
  - form a simple nonlinear program (no annotations)
  - form a complementarity problem from an embedded optimization problem (nlp with side constraints outside of optimizers control)
  - form an equilibrium model consisting of optimality conditions of several nlp's along with equilibrium constraints (MOPEC)
  - form a bilevel program (an optimization problem with optimization problems as constraints)
  - Can assign multipliers (prices) from one sub-model as variables in another model
  - Can reformulate nonsmooth models using duality
  - Can introduce random variables into a model
- The annotations essentially detail who controls which equations and variables

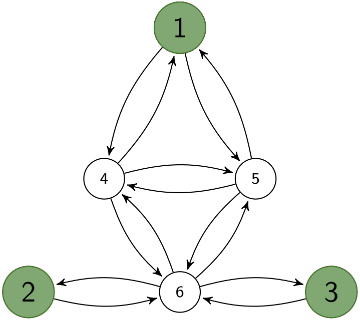
Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$

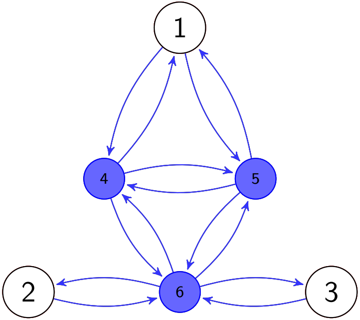
Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$

Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$
Transport: T_{ij}
Unit transport cost: $c_{ij}(T_{ij}) = ..$

One large system of equations and inequalities to describe this (GAMS).
(Assume for ease that $S_i = D_i = 0$ for $i \in n \setminus L$)

Nonlinear Program Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Full knowledge of transportation system

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L} \theta_l(D_l)D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij}$$

$$\text{s.t.} \quad S_l + \sum_i T_{il} = D_l + \sum_j T_{lj}, \quad \forall l \in n$$

EMP = NLP

2 agents: NLP + VI Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Price-taker in transportation system

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L}\theta_l(D_l)D_l - \sum_{l\in L}\Psi_l(S_l) - \sum_{i,j} \overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}} T_{ij} \tag{1}$$

$$\text{s.t.} \quad S_l + \sum_i T_{il} = D_l + \sum_j T_{lj}, \quad \forall l \in n$$

$$p_{ij} = c_{ij}(T_{ij}) \tag{2}$$

empinfo: vi tcDef tc

2 agents: NLP + VI Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
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$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L}\theta_l(D_l)D_l - \sum_{l\in L}\Psi_l(S_l) - \sum_{i,j}\overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}}T_{ij} \qquad (1)$$

$$\text{s.t.} \quad S_l + \sum_i T_{il} = D_l + \sum_j T_{lj}, \quad \forall l \in n$$

$$p_{ij} = c_{ij}(T_{ij}) \qquad (2)$$

$$\text{EMP} = \text{MOPEC} \implies \text{MCP}$$

EMP(iii): MOPEC

- Model has the format:

$$\text{Agent o:} \quad \min_x \; f(x,y) \quad \text{s.t.} \; g(x,y) \le 0 \quad (\perp \lambda \ge 0)$$

$$\text{Agent v:} \quad H(x,y,\lambda) = 0 \quad (\perp y \text{ free})$$

- Difficult to implement correctly (multiple optimization models)
- Can do automatically - simply annotate equations

  empinfo: equilibrium

  min f x defg

  vi H y dualvar λ defg
- EMP tool automatically creates an MCP

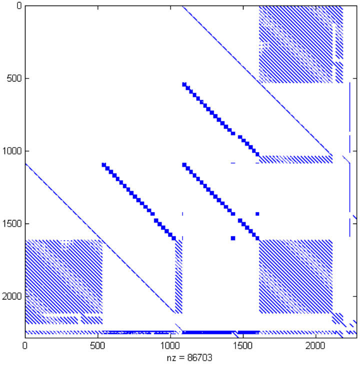
$$\nabla_x f(x,y) + \lambda^T \nabla g(x,y) = 0$$

$$0 \le -g(x,y) \perp \lambda \ge 0$$

$$H(x,y,\lambda) = 0$$

World Bank Project (Uruguay Round)

- 24 regions, 22 commodities
  - Nonlinear complementarity problem
  - Size: 2200 x 2200
- Short term gains $53 billion p.a.
  - Much smaller than previous literature
- Long term gains $188 billion p.a.
  - Number of less developed countries loose in short term
- Unpopular conclusions - forced concessions by World Bank
- Region/commodity structure not apparent to solver



Classic SPE Model (NLP + VI agents)

- One producer controlling all regions
- Price-taker in demand system
- Price-taker in transportation system

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L} \overset{\pi_l}{\cancel{\theta_l(D_l)}} D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} \overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}} T_{ij} \qquad (1)$$

$$\text{s.t.} \quad S_l + \sum_i T_{il} = D_l + \sum_j T_{lj}, \quad \forall l \in n$$

$$p_{ij} = c_{ij}(T_{ij}) \qquad (2)$$

$$\pi_l = \theta_l(D_l) \qquad (3)$$

empinfo: vi tcDef tc
vi pricedef price

Classic SPE Model (NLP + VI agents)

- One producer controlling all regions
- Price-taker in demand system
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$$\max_{(D,S,T)\in\mathcal{F}} \sum_{l\in L} \overset{\pi_l}{\cancel{\theta_l(D_l)}} D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} \overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}} T_{ij} \tag{1}$$

$$\text{s.t.} \quad S_l + \sum_i T_{il} = D_l + \sum_j T_{lj}, \quad \forall l \in n$$

$$p_{ij} = c_{ij}(T_{ij}) \tag{2}$$

$$\pi_l = \theta_l(D_l) \tag{3}$$

EMP = MOPEC $\implies$ MCP

Cournot-Nash equilibrium (multiple agents)

Assumes that each agent (producer):

- Treats other agent decisions as fixed
- Is a price-taker in transport and demand

EMP info file

```
equilibrium
max obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price
```

EMP = MOPEC $\Longrightarrow$ MCP

Nash Equilibria

- Nash Games: x^* is a Nash Equilibrium if

$$x_i^* \in \arg\min_{x_i \in X_i} \ell_i(x_i, x_{-i}^*, q), \forall i \in \mathcal{I}$$

  x_{-i} are the decisions of other players.

- Quantities q given exogenously, or via complementarity:

$$0 \leq H(x, q) \perp q \geq 0$$

- empinfo: equilibrium
  min loss(i) x(i) cons(i)
  vi H q
- Applications: Discrete-Time Finite-State Stochastic Games. Specifically, the Ericson & Pakes (1995) model of dynamic competition in an oligopolistic industry.

How to combine: Nash Equilibria

- Non-cooperative game: collection of players $a \in \mathcal{A}$ whose individual objectives depend not only on the selection of their own strategy $x_a \in C_a = \mathrm{dom} f_a(\cdot, x_{-a})$ but also on the strategies selected by the other players $x_{-a} = \{x_a : o \in \mathcal{A} \setminus \{a\}\}$.
- Nash Equilibrium Point:

$$\bar{x}_{\mathcal{A}} = (\bar{x}_a, a \in \mathcal{A}) : \forall a \in \mathcal{A} : \bar{x}_a \in \mathrm{argmin}_{x_a \in C_a} f_a(x_a, \bar{x}_{-a}).$$

1. for all $x \in \mathcal{A}$, $f_a(\cdot, x_{-a})$ is convex
2. $C = \prod_{a \in \mathcal{A}} C_a$ and for all $a \in \mathcal{A}$, C_a is closed convex.

VI reformulation

Define

$$G : \mathbf{R}^N \mapsto \mathbf{R}^N \text{ by } G_a(x_{\mathcal{A}}) = \partial_a f_a(x_a, x_{-a}), a \in \mathcal{A}$$

where ∂_a denotes the subgradient with respect to x_a. Generally, the mapping G is set-valued.

Theorem

Suppose the objectives satisfy (1) and (2), then every solution of the variational inequality

$$x_{\mathcal{A}} \in C \text{ such that } -G(x_{\mathcal{A}}) \in N_C(x_{\mathcal{A}})$$

is a Nash equilibrium point for the game.
Moreover, if C is compact and G is continuous, then the variational inequality has at least one solution that is then also a Nash equilibrium point.

Key point: models generated correctly solve quickly

Here S is mesh spacing parameter

S	Var	rows	non-zero	dense(%)	Steps	RT (m:s)
20	2400	2568	31536	0.48	5	0 : 03
50	15000	15408	195816	0.08	5	0 : 19
100	60000	60808	781616	0.02	5	1 : 16
200	240000	241608	3123216	0.01	5	5 : 12

Convergence for $S = 200$ (with new basis extensions in PATH)

Iteration	Residual
0	1.56(+4)
1	1.06(+1)
2	1.34
3	2.04(−2)
4	1.74(−5)
5	2.97(−11)

General Equilibrium models

$$(C): \max_{x_k \in X_k} U_k(x_k) \text{ s.t. } p^T x_k \leq i_k(y,p)$$

$$(I): i_k(y,p) = p^T \omega_k + \sum_j \alpha_{kj} p^T g_j(y_j)$$

$$(P): \max_{y_j \in Y_j} p^T g_j(y_j)$$

$$(M): \max_{p \geq 0} p^T \left(\sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \right) \text{ s.t. } \sum_l p_l = 1$$

General Equilibrium models

$$(C): \max_{x_k \in X_k} U_k(x_k) \text{ s.t. } p^T x_k \leq i_k(y, p)$$

$$(I): i_k(y, p) = p^T \omega_k + \sum_j \alpha_{kj} p^T g_j(y_j)$$

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$$(M): \max_{p \geq 0} p^T \left(\sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \right) \text{ s.t. } \sum_l p_l = 1$$

- KKT(C) + (I) + KKT(P) + KKT(M) form an MCP
- Loose structure for solution
- EMP facilitates modeling as (C) + (I) + (P) + (M) and either forms MCP automatically, or allows different solution method that exploits underlying structure

Negishi Weights

Can reformulate (GE) as an embedded problem (Ermoliev et al):

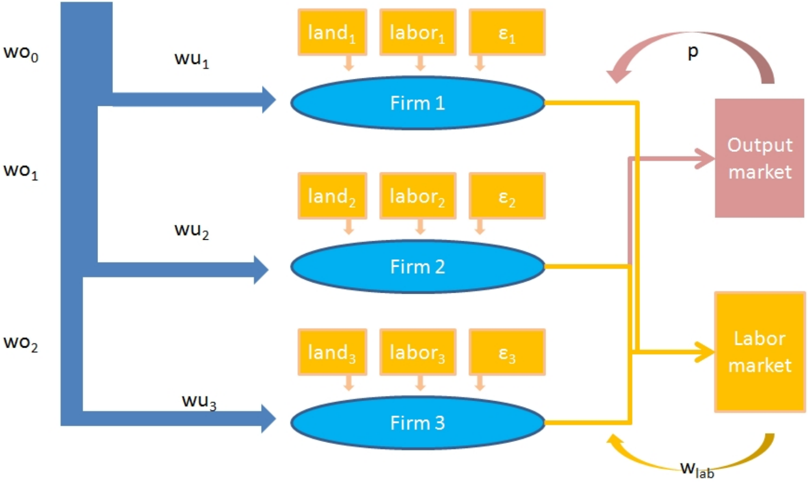
$$\max_{x \in X, y \in Y} \quad \sum_k \frac{t_k}{\beta_k} \log U_k(x_k)$$

$$\text{s.t.} \quad \sum_k x_k \leq \sum_k \omega_k + \sum_j g_j(y_j)$$

$t_k = i_k(y, p)$ where p is multiplier on NLP constraint

- KKT(NLP) + t_k definitional constraint form an alternative MCP
- this MCP often solves faster than the original MCP from Nash game

Water rights pricing (Britz/F./Kuhn)



The model AO (cooperative firms)

$$\max_{q_i, x_i, wo_i \geq 0} \sum_i \left(q_i \cdot p - \sum_{f \in \{int, lab\}} x_{i,f} \cdot w_f \right)$$

$$\begin{aligned} \text{s.t.} \quad & q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}} \\ & x_{i,land} \leq e_{i,land} \\ & wo_{i-1} = x_{i,wat} + wo_i \end{aligned}$$

The model AO (this is not an MPEC)

$$\max_{q_i, x_i, wo_i \geq 0} \sum_i \left(q_i \cdot p - \sum_{f \in \{int, lab\}} x_{i,f} \cdot w_f \right)$$

$$\text{s.t.} \quad q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}}$$

$$x_{i,land} \leq e_{i,land}$$

$$wo_{i-1} = x_{i,wat} + wo_i$$

$$0 \leq p \perp \sum_i q_i - d(p) \geq 0$$

$$0 \leq w_{lab} \perp \sum_i e_{i,lab} - \sum_i x_{i,lab} \geq 0$$

(M)OPEC

$$\max_{x} \theta(x, p) \text{ s.t. } g(x, p) \leq 0$$

$$0 \leq p \perp h(x, p) \geq 0$$

```
equilibrium
max theta x g
vi h p
```

- Solved concurrently (in a Nash manner)

(M)OPEC

$$\max_{x} \theta(x, p) \text{ s.t. } g(x, p) \leq 0$$

$$0 = h(x, p)$$

```
equilibrium
max theta x g
vi h p
```

- Solved concurrently (in a Nash manner)

(M)OPEC

$$\max_{x} \theta(x,p) \text{ s.t. } g(x,p) \leq 0$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$0 \leq p \perp h(x,p) \geq 0$$

$$x \perp -\nabla_x \theta(x,p) + \lambda^T \nabla_x g(x,p)$$
$$0 \leq \lambda \perp -g(x,p) \geq 0$$
$$0 \leq p \perp h(x,p) \geq 0$$

```
equilibrium
max theta x g
vi h p
```

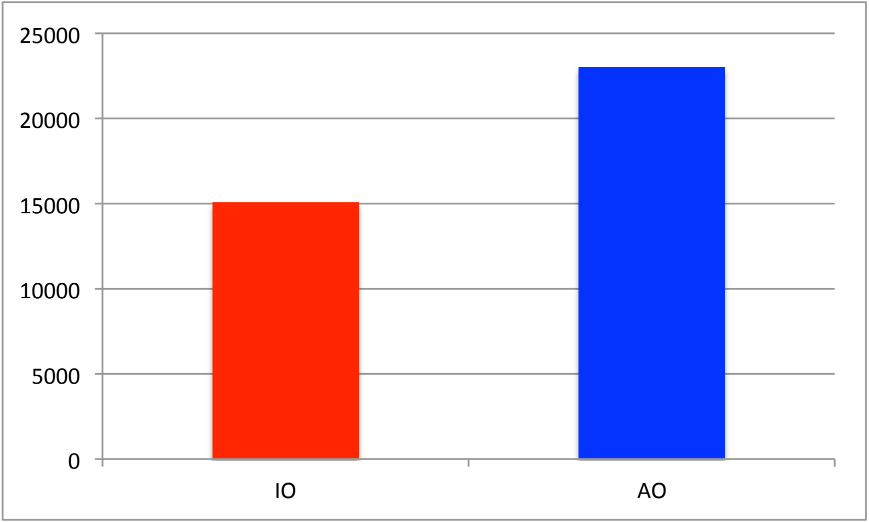
- Solved concurrently (in a Nash manner)
- Requires global solutions of agents problems (or theory to guarantee KKT are equivalent)
- Theory of existence, uniqueness and stability based in variational analysis

The model IO (independent firms)

$$\begin{aligned} \max_{q_i, x_i, wo_i \geq 0} \quad & \left(q_i \cdot p - \sum_f x_{i,f} \cdot w_f \right) \\ \text{s.t.} \quad & q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}} \\ & x_{i,land} \leq e_{i,land} \\ & wo_{i-1} = x_{i,wat} + wo_i \end{aligned}$$

$$\begin{aligned} 0 \leq \quad p \quad & \perp \quad \sum_i q_i - d(p) \geq 0 \\ 0 \leq \quad w_{lab} \quad & \perp \quad \sum_i e_{i,lab} - \sum_i x_{i,lab} \geq 0 \end{aligned}$$

IO vs AO (price of anarchy)



The model IO + trade (mechanism design)

$$\max_{q_i, x_i, wo_i, wr_i^b, wr_i^s \geq 0} \left(q_i \cdot p - \sum_f x_{i,f} \cdot w_f - wr_i^b \cdot (w_{wr} + \tau) + wr_i^s \cdot w_{wr} \right)$$

s.t.

$$q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}}$$

$$x_{i,land} \leq e_{i,land}$$

$$wo_{i-1} = x_{i,wat} + wo_i$$

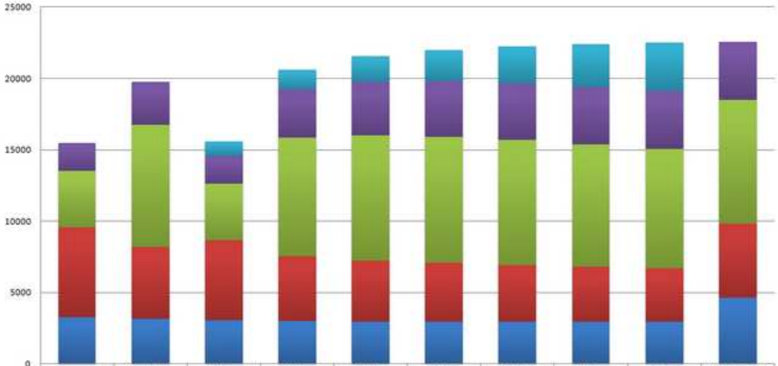
$$wr_i + wr_i^b \geq x_{i,wat} + wr_i^s$$

$$0 \leq p \perp \sum_i q_i - d(p) \geq 0$$

$$0 \leq w_{lab} \perp \sum_i e_{i,lab} - \sum_i x_{i,lab} \geq 0$$

$$0 \leq w_{wr} \perp \sum_i wr_i^s - \sum_i wr_i^b \geq 0$$

Different Management Strategies



	No institution	Equitable rights	Water charges 10	Water charges 14	Water charges 18	Water charges 22	Water charges 26	Water charges 30	Water charges 34	Water right trade
Water charge revenue	0	0	1000	1400	1800	2200	2600	3000	3400	0
Consumer surplus	1943	3040	1969	3398	3705	3865	3959	4018	4054	4099
Profit Firm 3	3983	8513	3979	8279	8776	8869	8784	8601	8356	8642
Profit Firm 2	6306	5033	5580	4557	4281	4097	3960	3853	3766	5217
Profit Firm 1	3255	3176	3068	2995	2972	2957	2947	2940	2936	4617

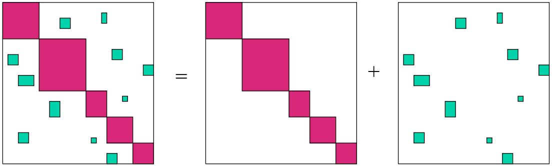
MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, p) \text{ s.t. } g_i(x_i, x_{-i}, p) \leq 0, \forall i$$

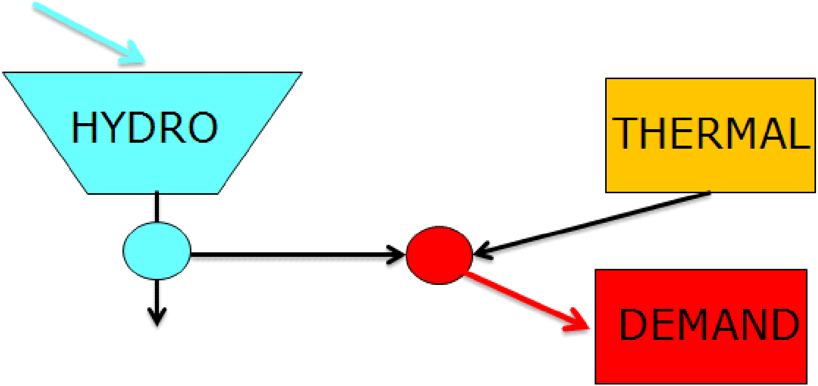
$$p \text{ solves } \mathrm{VI}(h(x, \cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h p cons
```

- Reformulate optimization problem as first order conditions (complementarity)
- Use nonsmooth Newton methods to solve complementarity problem
- Precondition using "individual optimization" with fixed externalities



Hydro-Thermal System (Philpott/F./Wets)



Simple electricity "system optimization" problem

$$\text{SO:} \quad \max_{d_k, u_i, v_j, x_i \geq 0} \quad \sum_{k \in \mathcal{K}} W_k(d_k) - \sum_{j \in \mathcal{T}} C_j(v_j) + \sum_{i \in \mathcal{H}} V_i(x_i)$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{H}} U_i(u_i) + \sum_{j \in \mathcal{T}} v_j \geq \sum_{k \in \mathcal{K}} d_k,$$

$$x_i = x_i^0 - u_i + h_i^1, \quad i \in \mathcal{H}$$

- u_i water release of hydro reservoir $i \in \mathcal{H}$
- v_j thermal generation of plant $j \in \mathcal{T}$
- x_i water level in reservoir $i \in \mathcal{H}$
- prod fn U_i (strictly concave) converts water release to energy
- $C_j(v_j)$ denote the cost of generation by thermal plant
- $V_i(x_i)$ future value of terminating with storage x (assumed separable)
- $W_k(d_k)$ utility of consumption d_k

SO equivalent to CE

$$\text{Consumers } k \in \mathcal{K} \text{ solve CP}(k): \max_{d_k \geq 0} \quad W_k(d_k) - p^T d_k$$

$$\text{Thermal plants } j \in \mathcal{T} \text{ solve TP}(j): \max_{v_j \geq 0} \quad p^T v_j - C_j(v_j)$$

$$\text{Hydro plants } i \in \mathcal{H} \text{ solve HP}(i): \max_{u_i, x_i \geq 0} \quad p^T U_i(u_i) + V_i(x_i)$$

$$\text{s.t.} \quad x_i = x_i^0 - u_i + h_i^1$$

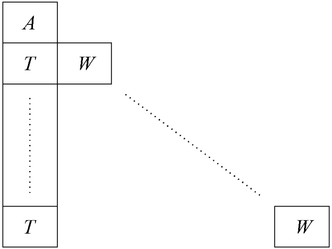
Perfectly competitive (Walrasian) equilibrium is a MOPEC

$$\begin{aligned} \text{CE:} \qquad d_k &\in \arg\max \text{CP}(k), & k \in \mathcal{K}, \\ v_j &\in \arg\max \text{TP}(j), & j \in \mathcal{T}, \\ u_i, x_i &\in \arg\max \text{HP}(i), & i \in \mathcal{H}, \\ 0 \leq p \perp \sum_{i \in \mathcal{H}} U_i(u_i) &+ \sum_{j \in \mathcal{T}} v_j \geq \sum_{k \in \mathcal{K}} d_k. \end{aligned}$$

Agents have stochastic recourse?

- Two stage stochastic programming, x^1 is here-and-now decision, recourse decisions x^2 depend on realization of a random variable
- $\mathbb{R}$ is a risk measure (e.g. expectation, CVaR)

$$
\begin{aligned}
\text{SP: } \max \quad & c^T x^1 + \mathbb{R}[q^T x^2] \\
\text{s.t.} \quad & Ax^1 = b, \quad x^1 \geq 0, \\
& T(\omega)x^1 + W(\omega)x^2(\omega) \geq d(\omega), \\
& x^2(\omega) \geq 0, \forall \omega \in \Omega.
\end{aligned}
$$



EMP/SP extensions to facilitate these models

Two stage stochastic MOPEC

$$\text{CP}(k): \quad \max_{d_k^1 \geq 0} \quad W_k\left(d_k^1\right) - p^1 d_k^1$$

$$\text{TP}(j): \quad \max_{v_j^1 \geq 0} \quad p^1 v_j^1 - C_j(v_j^1)$$

$$\text{HP}(i): \quad \max_{u_i^1, x_i^1 \geq 0} \quad p^1 U_i(u_i^1)$$

$$\text{s.t.} \quad x_i^1 = x_i^0 - u_i^1 + h_i^1,$$

$$0 \leq p^1 \perp \sum_{i \in \mathcal{H}} U_i\left(u_i^1\right) + \sum_{j \in \mathcal{T}} v_j^1 \geq \sum_{k \in \mathcal{K}} d_k^1$$

Two stage stochastic MOPEC

$$\text{CP}(k): \quad \max_{d_k^1, d_k^2(\omega) \geq 0} \quad W_k\left(d_k^1\right) - p^1 d_k^1 + \mathbb{R}[W_k\left(d_k^2(\omega)\right) - p^2(\omega) d_k^2(\omega)]$$

$$\text{TP}(j): \quad \max_{v_j^1, v_j^2(\omega) \geq 0} \quad p^1 v_j^1 - C_j(v_j^1) + \mathbb{R}[p^2(\omega) v_j^2(\omega) - C_j\left(v_j^2(\omega)\right)]$$

$$\text{HP}(i): \quad \max_{\substack{u_i^1, x_i^1 \geq 0 \\ u_i^2(\omega), x_i^2(\omega) \geq 0}} \quad p^1 U_i(u_i^1) + \mathbb{R}[p^2(\omega) U_i(u_i^2(\omega)) + V_i(x_i^2(\omega))]$$

$$\text{s.t.} \quad x_i^1 = x_i^0 - u_i^1 + h_i^1,$$

$$x_i^2(\omega) = x_i^1 - u_i^2(\omega) + h_i^2(\omega)$$

$$0 \leq p^1 \perp \sum_{i \in \mathcal{H}} U_i\left(u_i^1\right) + \sum_{j \in \mathcal{T}} v_j^1 \geq \sum_{k \in \mathcal{K}} d_k^1$$

Two stage stochastic MOPEC

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$$0 \leq p^2(\omega) \perp \sum_{i \in \mathcal{H}} U_i\left(u_i^2(\omega)\right) + \sum_{j \in \mathcal{T}} v_j^2(\omega) \geq \sum_{k \in \mathcal{K}} d_k^2(\omega), \forall \omega$$

Results

- Suppose every agent is risk neutral and has knowledge of all deterministic data, as well as sharing the same probability distribution for inflows. SO solution is same as CE solution
- Using coherent risk measure (weighted sum of expected value and conditional value at risk), 10 scenarios for rain
  1. High initial storage: risk-averse central plan (RSO) and the risk-averse competitive equilibrium (RCE) have same solution (but different to risk neutral case)
  2. Low initial storage: RSO and RCE are very different. Since the hydro generator and the system do not agree on a worst-case outcome, the probability distributions that correspond to an equivalent risk neutral decision will not be common.
  3. Extension: Construct MOPEC models for trading risk

Contracts in MOPEC (F./Wets)

- Competing agents (consumers, or generators in energy market)
- Each agent minimizes objective independently (cost)
- Market prices are function of all agents activities
- Additional twist: model must "hedge" against uncertainty
- Facilitated by allowing contracts bought now, for goods delivered later
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancilliary services in energy market)
- Can investigate new instruments to mitigate risk, or move to system optimal solutions from equilibrium (or market) solutions

Example as MOPEC: agents solve a Stochastic Program

Buy y_i contracts in period 1, to deliver $D(\omega)y_i$ in period 2, scenario ω
Each agent i:

$$\begin{aligned} \min \quad & C(x_i^1) + \sum_{\omega} \pi_\omega C(x_i^2(\omega)) & \\ \text{s.t.} \quad & p^1 x_i^1 + v y_i \le p^1 e_i^1 & \text{(budget time 1)} \\ & p^2(\omega) x_i^2(\omega) \le p^2(\omega)(D(\omega) y_i + e_i^2(\omega)) & \text{(budget time 2)} \end{aligned}$$

$$0 \le v \perp -\sum_i y_i \ge 0 \qquad \text{(contract)}$$

$$0 \le p^1 \perp \sum_i (e_i^1 - x_i^1) \ge 0 \qquad \text{(walras 1)}$$

$$0 \le p^2(\omega) \perp \sum_i \left(D(\omega) y_i + e_i^2(\omega) - x_i^2(\omega)\right) \ge 0 \qquad \text{(walras 2)}$$

Observations

- Examples from literature solved using homotopy continuation seem incorrect - need transaction costs to guarantee solution
- Solution possible via disaggregation only seems possible in special cases
  - When problem is block diagonally dominant
  - When overall (complementarity) problem is monotone
  - (Pang): when problem is a potential game
- Progressive hedging possible to decompose in these settings by agent and scenario
- Research challenge: develop reliable algorithms for large scale decomposition approaches to MOPEC

Bilevel Program (Stackelberg)

- Assumes one leader firm, the rest follow
- Leader firm optimizes subject to expected follower behavior
- Follower firms act in a Nash manner
- All firms are price-takers in transport and demand

EMP info file

```
bilevel obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price
```

EMP = bilevel $\implies$ MPEC $\implies$ (via NLPEC) NLP(μ)

Hierarchical models

- Bilevel programs:

$$\begin{array}{ll} \min\limits_{x^*,y^*} & f(x^*,y^*) \\ \text{s.t.} & g(x^*,y^*) \leq 0, \\ & y^* \text{ solves } \min\limits_{y} v(x^*,y) \text{ s.t. } h(x^*,y) \leq 0 \end{array}$$

- model bilev /deff,defg,defv,defh/;
  empinfo: bilevel min v y defv defh
- EMP tool automatically creates the MPCC

$$\begin{array}{lll} \min\limits_{x^*,y^*,\lambda} & f(x^*,y^*) & \\ \text{s.t.} & g(x^*,y^*) \leq 0, & \\ & 0 \leq \nabla v(x^*,y^*) + \lambda^T \nabla h(x^*,y^*) & \perp y^* \geq 0 \\ & 0 \leq -h(x^*,y^*) & \perp \lambda \geq 0 \end{array}$$

The overall scheme!

- Collection of algebraic equations
- Form a bilevel program via emp
- EMP tool automatically creates the MPCC (model transformation)
- NLPEC tool automatically creates (a series of) NLP models (model transformation)
- GAMS automatically rewrites NLP models for global solution via BARON (model transformation)
- Is this global? What's the hitch?

The overall scheme!

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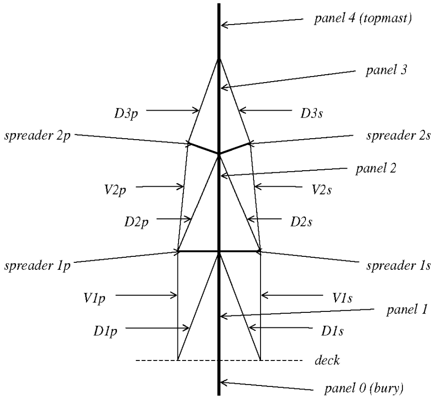
- Is this global? What's the hitch?
- Note that heirarchical structure is available to solvers for analysis or utilization

Optimal Yacht Rig Design

- Current mast design trends use a large primary spar that is supported laterally by a set of tension and compression members, generally termed the rig
- Reduction in either the weight of the rig or the height of the VCG will improve performance
- Design must work well under a variety of weather conditions

Complementarity feature

- Stays are tension only members (in practice) - Hookes Law
- When tensile load becomes zero, the stay goes slack (low material stiffness)
- $0 \geq s \perp s - k\delta \leq 0$
  - s axial load
  - k member spring constant
  - δ member extension
- Either $s_i = 0$ or $s_i = k\delta_i$



EMP(ii): MPCC: complementarity constraints

$$\begin{aligned} \min_{x,s} \quad & f(x,s) \\ \text{s.t.} \quad & g(x,s) \le 0, \\ & 0 \le s \perp h(x,s) \ge 0 \end{aligned}$$

- g, h model “engineering” expertise: finite elements, etc
- $\perp$ models complementarity, disjunctions
- Complementarity “$\perp$” constraints available in AIMMS, AMPL and GAMS

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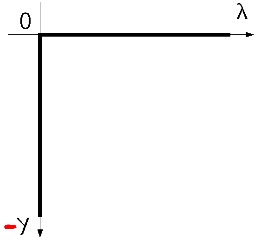
- g, h model "engineering" expertise: finite elements, etc
- $\perp$ models complementarity, disjunctions
- Complementarity "$\perp$" constraints available in AIMMS, AMPL and GAMS
- NLPEC: use the convert tool to automatically reformulate as a parameteric sequence of NLP's
- Solution by repeated use of standard NLP software
  - Problems solvable, local solutions, hard

Complementarity Problems via Graphs



$$\mathcal{T} = \mathcal{N}_{\mathbf{R}_+} = (\mathbf{R}_+ \times \{0\}) \bigcup (\{0\} \times \mathbf{R}_-)$$

$$-y \in \mathcal{T}(\lambda) \iff (\lambda, -y) \in \mathcal{T} \iff 0 \le \lambda \perp y \ge 0$$

By approximating (smoothing) graph can generate interior point algorithms for example $y\lambda = \epsilon, y, \lambda > 0$

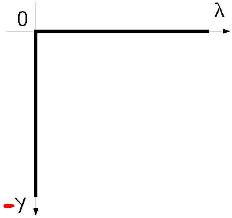
$$-F(z) \in \mathcal{N}_{\mathbf{R}^n_+}(z) \iff (z, -F(z)) \in \mathcal{T}^n \iff 0 \le z \perp F(z) \ge 0$$

Complementarity Systems (DVI)

$$\frac{dx}{dt}(t) = f(x(t), \lambda(t))$$

$$y(t) = h(x(t), \lambda(t))$$

$$0 \le y(t) \perp \lambda(t) \ge 0$$



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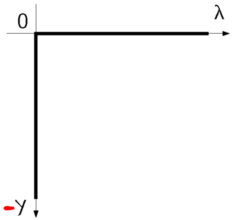
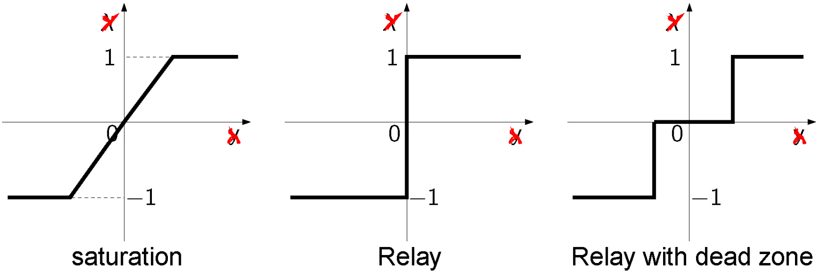


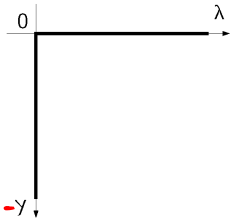


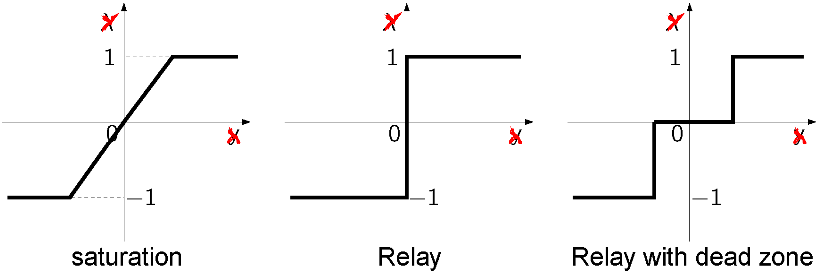
Complementarity Systems (DVI)

$$\frac{dx}{dt}(t) = f(x(t), \lambda(t))$$

$$y(t) = h(x(t), \lambda(t))$$

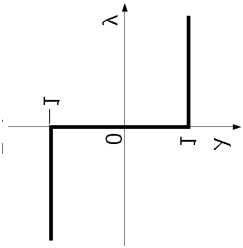
$$(\lambda(t), -y(t)) \in \mathcal{T}$$

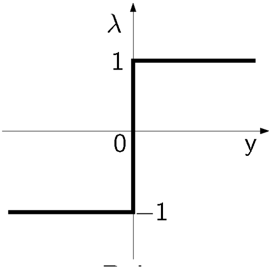


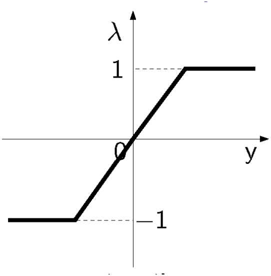


Operators and Graphs ($\mathcal{C} = [-1, 1], \mathcal{T} = \mathcal{N}_{\mathcal{C}}$)

$$z_i = -1, -F_i(z) \leq 0 \text{ or } z_i \in (-1, 1), -F_i(z) = 0 \text{ or } z_i = 1, -F_i(z) \geq 0$$







$\mathcal{T}(\lambda)$ $\mathcal{T}^{-1}(y)$ $(\mathcal{I} + \mathcal{T})^{-1}(y) = P_{\mathcal{T}}(y)$

$P_{\mathcal{T}}(y)$ is the projection of y onto $[-1, 1]$

Generalized Equations

- Suppose $\mathcal{T}$ is a maximal monotone operator

$$0 \in F(z) + \mathcal{T}(z) \qquad (GE)$$

- Define $P_{\mathcal{T}} = (\mathcal{I} + \mathcal{T})^{-1}$
- If $\mathcal{T}$ is polyhedral (graph of $\mathcal{T}$ is a finite union of convex polyhedral sets) then $P_{\mathcal{T}}$ is piecewise affine (continous, single-valued, non-expansive)

$$\begin{aligned} 0 \in F(z) + \mathcal{T}(z) \quad &\Longleftrightarrow \quad z \in F(z) + \mathcal{I}(z) + \mathcal{T}(z) \\ &\Longleftrightarrow \quad z - F(z) \in (\mathcal{I} + \mathcal{T})(z) \Longleftrightarrow P_{\mathcal{T}}(z - F(z)) = z \end{aligned}$$

Use in fixed point iterations (cf projected gradient methods)

Normal Map

- Suppose $\mathcal{T}$ is a maximal monotone operator

$$0 \in F(z) + \mathcal{T}(z) \quad (GE)$$

- Define $P_{\mathcal{T}} = (I + \mathcal{T})^{-1}$

$$\begin{aligned} 0 \in F(z) + \mathcal{T}(z) &\iff z \in F(z) + \mathcal{I}(z) + \mathcal{T}(z) \\ &\iff z - F(z) = x \text{ and } x \in (\mathcal{I} + \mathcal{T})(z) \\ &\iff z - F(z) = x \text{ and } P_{\mathcal{T}}(x) = z \\ &\iff P_{\mathcal{T}}(x) - F(P_{\mathcal{T}}(x)) = x \\ &\iff 0 = F(P_{\mathcal{T}}(x)) + x - P_{\mathcal{T}}(x) \end{aligned}$$

This is the so-called Normal Map Equation

Key idea of algorithm $\mathcal{T} = \mathcal{N}_\mathcal{C}$

Homotopy: Easy solution for μ large, drive $\mu \to 0$.

$$\mu r = F(\pi_\mathcal{C}(x(\mu))) + x(\mu) - \pi_\mathcal{C}(x(\mu))$$

Define $z(\mu) = \pi_\mathcal{C}(x(\mu))$, then

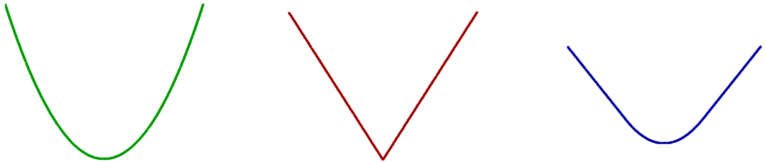
$$\mu r = F(z(\mu)) + x(\mu) - z(\mu)$$

$$\begin{aligned}
x - z \quad & \in N_\mathcal{C}(z) \\
N_\mathcal{C}(z) \quad & = \{-A^T u - w + v\} \\
\text{such that} \quad & Az(\geq, =, \leq)a \perp u(\geq, \text{free}, \leq)0 \\
& 0 \leq w \perp z - l \geq 0 \\
& 0 \leq v \perp u - z \geq 0
\end{aligned}$$

EMP(iv): Extended nonlinear programs

$$\min_{x \in X} f_0(x) + \theta(f_1(x), \ldots, f_m(x))$$

Examples of different θ

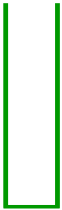


least squares, absolute value, Huber function

Solution reformulations are very different

Huber function used in robust statistics.

More general θ functions



In general any piecewise linear penalty function can be used: (different upside/downside costs).

General form:

$$\theta(u) = \sup_{y \in Y} \{ y^T u - k(y) \}$$

θ nonsmooth due to the max term; θ separable in example.

θ is always convex.

Solution Procedures

- Solution uses reformulation - one way: first order conditions

$$\text{VI}\left(\begin{bmatrix} \nabla_x \mathcal{L}(x,y) \\ -\nabla_y \mathcal{L}(x,y) \end{bmatrix}, X \times Y\right)$$

based on extended form of the Lagrangian:

$$\mathcal{L}(x,y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x) - k(y)$$

- EMP: allows "annotation" of constraints to facilitate library of different θ functions to be applied
- EMP tool automatically creates an MCP (or a smooth NLP)

Choices for θ

$$\inf_{x \in X} f_0 + \theta[f(x)], \;\; \theta(u) = \sup_{y \in Y} \{y^T u - k(y)\}$$

θ is convex with values in $(-\infty, +\infty]$; may be nonsmooth

- L_2: $k(u) = \frac{1}{4\lambda} u^2$, $Y = (-\infty, +\infty)$
- L_1: $k(u) = 0$, $Y = [-\rho, \rho]$
- L_∞: $k(u) = 0$, $Y = \Delta$, unit simplex in $\mathbf{R}^m_+$
- Linear-quad (Huber 1981): $k(u) = \frac{1}{4\lambda} u^2$, $Y = [-\rho, \; rho]$
- Second order cone constraint: $k(y) = 0$, $Y = C^\circ$
- The new feature here is implementation and solution within the GAMS modeling language framework, which produces a tool usable without advanced knowledge in convex analysis and without cumbersome "hand tailoring" to accommodate different penalizations [Ferris, Dirkse, Jagla, and Meeraus 2008]
- This makes the theoretical benefits accessible to users from a wide variety of different fields

Alternative Reformulations

Convert does symbolic/numeric reformulations. Alternative NLP formulations also possible.

$$k(y) = \frac{1}{2} y' Q y, \quad X = \{x : Rx \le r\}, \quad Y = \{y : S'y \le s\}$$

Defining

$$Q = DJ^{-1}D', \quad F(x) = (f_1(x), \ldots, f_m(x))$$

$$\begin{array}{ll} \text{min} & f_0(x) + s'z + \frac{1}{2} wJw \\ \text{s.t.} & Rx \le r, z \ge 0, F(x) - Sz - Dw = 0 \end{array}$$

Can set up better (solver) specific formulation.

Does it work at realistic scales: GTAP?

- The latest GTAP database represents global production and trade for 113 country/regions, 57 commodities and 5 primary factors.
- Data characterizes intermediate demand and bilateral trade in 2007, including tax rates on imports/exports and other indirect taxes.
- The core GTAP model is a static, multi-regional model which tracks the production and distribution of goods in the global economy.
- In GTAP the world is divided into regions (typically representing individual countries), and each region's final demand structure is composed of public and private expenditure across goods.

The Model

The GTAP model may be posed as a system of *nonlinear equations*:

$$F(w, z; t) = 0$$

in which: where

- w_r is a vector of regional *welfare* levels
- $z \in \mathrm{R}^N$ represents a vector of endogenous economic variables, e.g. prices and quantities, $z = \begin{pmatrix} P \\ Q \end{pmatrix}$.
- t represents matrices of trade tax instruments – import tariffs (t^M_{irs}) and export taxes (t^X_{irs}) for each commodity i and region r

Optimal Sanctions

Coalition member states strategically choose trade taxes which *minimize* Russian welfare:

$$\min_{t_r : r \in \mathcal{C}} w_{rus}$$

s.t.

$$F(w, z; t) = 0$$

$$t_r = \bar{t}_r \quad \forall r \notin \mathcal{C}$$

$$t^M_{i,rus,r} \leq \bar{t}^M_{i,r,rus} \quad \forall r \in \mathcal{C}$$

$$t^X_{i,r,rus} \leq \bar{t}^X_{i,rus,r} \quad \forall r \in \mathcal{C}$$

Optimal Retaliation

Russia choose trade taxes which *maximize* Russian welfare in response to the coalition actions:

$$\max_{t_{rus}} w_{rus}$$

s.t.

$$F(w, z; t) = 0$$

$$t_r = \begin{cases} \hat{t}_r & r \in \mathcal{C} \\ \bar{t}_r & r \notin \mathcal{C} \end{cases}$$

where $\hat{t}_r$ represents trade taxes for coalition countries ($r \in \mathcal{C}$) from the optimal sanction calculation.

Coalition Member States for Illustrative Calculation

USA	United States
ANZ	Australia and New Zealand
CAN	Canada
FRA	France
DEU	Germany
ITA	Italy
JPN	Japan
GBR	United Kingdom
REU	Rest of the European Union

Welfare Changes (% Hicksian EV)

	sanction	retaliation	tradewar
RUS	-4.4	-3.5	-9.8
$\mathcal{C}$ AVERAGE	0.03	0.05	0.03
CAN	0.021	0.033	0.032
USA	0.007	-0.017	0.032
FRA	0.042	0.020	0.032
DEU	0.119	-0.047	0.032
ITA	0.069	0.050	0.032
GBR	0.045	-0.002	0.032
REU	0.058	0.365	0.032
ANZ	0.011	0.003	0.032
JPN	0.012	-0.020	0.032
CHN	0.115	0.057	0.290
SAU	0.240	1.865	-0.892

Scenarios and Key Insights

SANCTION If coalition states were to increases tariffs and export taxes on Russia to the same level which is currently applied by Russia on bilateral trade flows with the coalition, Russian welfare could be substantially impacted with no economic cost for any coalition members.

RETALIATION Russia could respond to such sanctions by changing it's own trade taxes, but optimal "retaliation" largely results in a *reduction* rather than an increase in trade taxes on trade flows to and from coalition states. These tariff changes can only partially offset the adverse impact of the sanctions.

TRADEWAR If sanctions and retaliation were to result in an unconstrained trade war, Russia faces a drastic economic cost while the coalition countries could even be slight better off.

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS

But who cares?

- Why aren't you using my *********** algorithm?
  (Michael Ferris, Boulder, CO, 1994)

But who cares?

- Why aren't you using my *********** algorithm?
  (Michael Ferris, Boulder, CO, 1994)
- Show me on a problem like mine
- Must run on defaults
- Must deal graciously with poorly specified cases
- Must be usable from my environment (Matlab, R, GAMS, ...)
- Must be able to model my problem easily

EMP provides annotations to an existing optimization model that convey new model structures to a solver

NEOS is soliciting case studies that show how to do the above, and will provide some tools to help

Conclusions

- Optimization helps understand what drives a system
- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers
- EMP model type is clear and extensible, additional structure available to solver
- Extended Mathematical Programming available within the GAMS modeling system
- Able to pass additional (structure) information to solvers
- Embedded optimization models automatically reformulated for appropriate solution engine
- Exploit structure in solvers
- Extend application usage further
- Modeling, optimization, statistics and computation embedded within the application domain is critical