Part 2: Multilevel methods and reduced order models for the optimization with variational inequalities

TECHNISCHE UNIVERSITÄT DARMSTADT

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Nonlinear Optimization

SFB 666

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Outline

Outline

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Motivation

Applications

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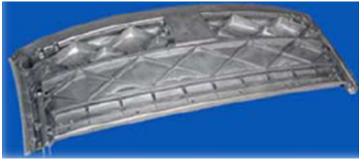
Special kind of sheet metals:

Complex curved sheet metals with bifurcated cross-sections

in aircraft industry



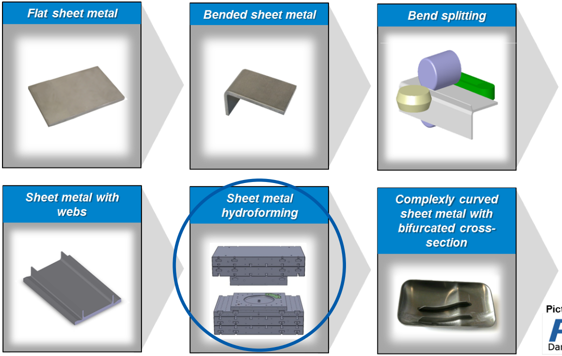
in car industry

- Light weight surface structure with additional form features
- New forming procedure (SFB 666)
  bend splitting + sheet metal hydroforming

Motivation

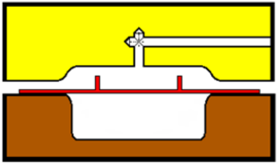
Manufacturing - New forming procedures

- **Bend splitting and sheet metal hydroforming:** manufacturing of bifurcations in integral style

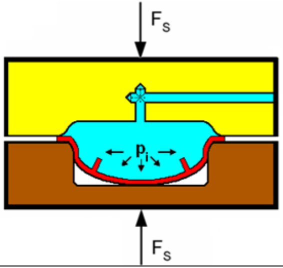


Pictures by PtU Darmstadt

- **Advantage:** avoiding the use of joining procedures completely



- Docking plate in order to avoid leakage of the tool system
- No collision between stringers and tool

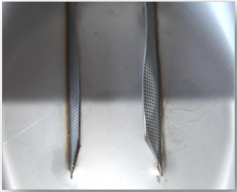


- Fluid medium performs the forming operation
- Fluid pressure and blank holder force are the main parameters

Motivation

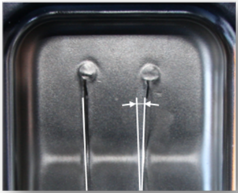
Limitations of the hydroforming process

Buckling



Collapsing

Bursting failure



Misalignment of the stringers

Basic Optimization Model

Hydroforming and cost function

Hydroforming involves:

- plasticity
- contact with friction
- large deformations, springback

Goal:

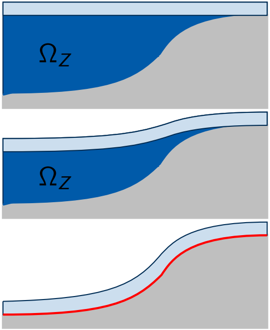
- filling of desired end shape

Controls:

- fluid pressure p_i
- blank holder force f_{BH}

Constraints:

- lower, upper bounds on the controls
- avoid buckling and folding



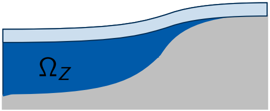
Basic Optimization Model

Optimization problem

$$\min \quad \mathrm{vol}\,(\Omega_Z(y(t)))$$

$$\text{s. t.} \quad C(y(t), q(t)) = 0$$

$$0 \le q(t) \le q_{max}$$



$y(t)$ - state variables (displacement,plastic strain,...)

$q(t)$ - control variables (pressure,blank holder force)

$C(y(t), q(t))$ - simulation model of the hydroforming process

Basic Optimization Model

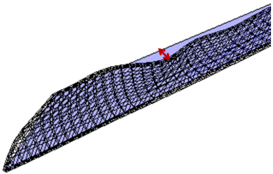
Optimization problem

Geometry-based penalty term

$$\min \quad \mathrm{vol}\,(\,\Omega_Z(y(t)\,) + \frac{c}{N}\sum_{i=1}^{N}(x_{3,i} - f(x_{1,i}, x_{2,i}))^2$$

$$\text{s. t.} \quad C(y(t), q(t)) = 0$$

$$0 \le q(t) \le q_{max}$$



$y(t)$ - state variables (displacement,plastic strain,...)

$q(t)$ - control variables (pressure,blank holder force)

$C(y(t), q(t))$ - simulation model of the hydroforming process

f - bilinear model function

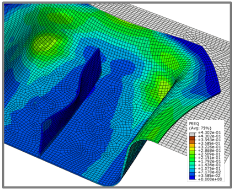
N - number of nodes in the middle plane of a stringer

Optimization example

First steps - derivative-free methods

FEM (ABAQUS 6.10)

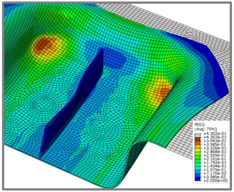
- Solver: Explicit
- Contact: penalty friction (0.05)
- Elements: 20k, solid hex
- Material: E-Modul, Poisson-Ratio, Flow Curve DC04 (steel)



initial control parameters

OPT (NLOpt Library)

- Solver: COBYLA of Powell
- Controls: p_i, f_{BH}
- Opt parameters: 8
- Iterations: 84, $\sim$ 22 days (4 CPUs)



geometric penalty term

FEM (ABAQUS 6.10)

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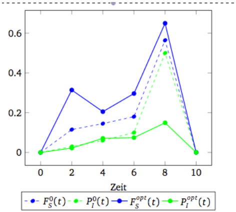

optimal control

OPT (NLOpt Library)

- Solver: COBYLA of Powell
- Controls: p_i, f_{BH}
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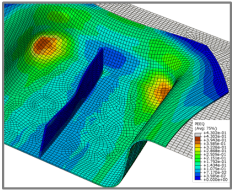

geometric penalty term

Optimization example

Next step - derivative-based methods

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$$
\begin{aligned}
\min \quad & J(y,q) \\
\text{s.t.} \quad & C(y,q) = 0, \quad q \in Q_{ad} \\
& (\textit{e.g., plasticity} \\
& \textit{frictional contact} \\
& \textit{large deformation})
\end{aligned}
$$

- Need for highly efficient derivative-based optimization methods
- Requires analysis and subgradient calculations or MPEC-formulation
- Efficient optimization based on reduced order models for hydroforming
- **Necessary**: Verified reduced order models for (finite) plasticity and frictional contact

Outline

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Elastic Contact Problem

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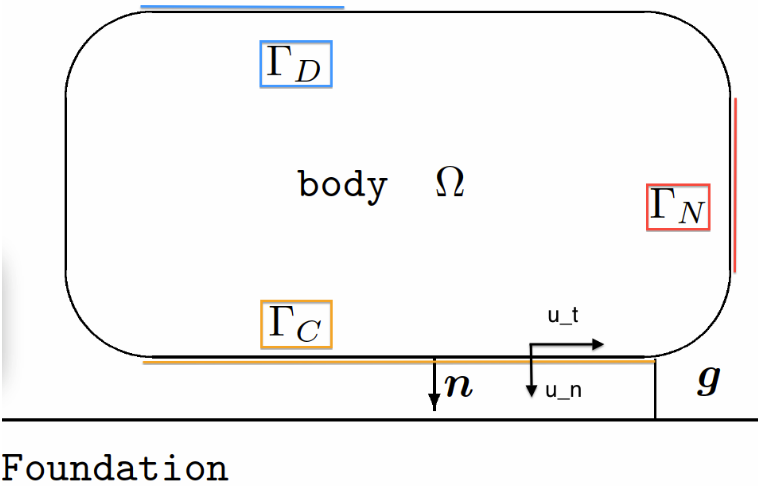

Elastic Contact Problems with Plasticity

Governing equations

Governing equations

$$
\begin{aligned}
-\operatorname{div}\sigma(t) &= f_v(t), & & & u(t) &= 0, & &\text{on } \Gamma_D,\\
\sigma(t) &= C^{el}\,\varepsilon^{el} = C^{el}(\varepsilon(u(t)) \qquad), & &\text{in } \Omega, & \sigma(t)n &= f_s(t), & &\text{on } \Gamma_N,\\
\varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T), & & & & & &
\end{aligned}
$$

u - displacement

σ - stress

ε - linearized strain

λ - contact stress

Elastic Contact Problems with Plasticity

Governing equations

Governing equations

$$\begin{aligned} -\operatorname{div}\sigma(t) &= f_v(t), \\ \sigma(t) &= C^{el}\varepsilon^{el} = C^{el}(\varepsilon(u(t)) \qquad), \qquad \text{in } \Omega, \\ \varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T), \end{aligned}$$

$$\begin{aligned} u(t) &= 0, && \text{on } \Gamma_D, \\ \sigma(t)\mathrm{n} &= f_s(t), && \text{on } \Gamma_N, \\ -\sigma(t)\mathrm{n} &= \lambda(t), && \text{on } \Gamma_C. \end{aligned}$$

contact with coulomb friction

$$\begin{aligned} \lambda_n &:= \lambda \cdot n \\ \lambda_t &:= \lambda - \lambda_n n \\ f^c(\lambda_n, \lambda_t) &:= \|\lambda_t\| - \mu\lambda_n \\ \dot{u}_t &= \begin{cases} \gamma^c \frac{\lambda_t}{\|\lambda_t\|}, & \|\lambda_t\| > 0, \\ 0, & \mu\lambda_n > \|\lambda_t\| = 0, \end{cases} \\ u_t\big|_{t=0} &= 0. \end{aligned}$$

$$\begin{aligned} \gamma^c &\geq 0, \\ f^c(\lambda_n, \lambda_t) &\leq 0, \quad \text{(sliding cond.)} \\ \gamma^c f^c(\lambda_n, \lambda_t) &= 0, \end{aligned} \qquad \begin{aligned} \lambda_n &\geq 0, \\ u_n - g &\leq 0, \quad \text{(non-penetr.)} \\ \lambda_n(u_n - g) &= 0. \end{aligned}$$

Elastic Contact Problems with Plasticity

Governing equations

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Governing equations

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-\operatorname{div}\sigma(t) &= f_v(t), & & & u(t) &= 0, && \text{on } \Gamma_D,\\
\sigma(t) &= C^{el}\,\varepsilon^{el} = C^{el}(\varepsilon(u(t)) - \varepsilon^p(t)), & &\text{in } \Omega, & \sigma(t)n &= f_s(t), && \text{on } \Gamma_N,\\
\varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T),\ \operatorname{tr}\varepsilon^p(t) = 0, & & & -\sigma(t)n &= \lambda(t), && \text{on } \Gamma_C.
\end{aligned}
$$

u - displacement

σ - stress

ε - linearized strain

ε^p - plastic strain

Elastic Contact Problems with Plasticity

Governing equations

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\varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T),\ \operatorname{tr}\varepsilon^p(t) = 0, & & & -\sigma(t)\mathrm{n} &= \lambda(t), && \text{on } \Gamma_C.
\end{aligned}
$$

Plasticity with linear and isotropic or kinematic hardening

$$
\eta = \operatorname{dev}\sigma - \tfrac{1}{a_0^2}K\varepsilon^p, \quad \operatorname{dev}\sigma = \sigma - \tfrac{1}{3}\operatorname{tr}(\sigma)I
$$

$$
f^p(\alpha,\eta) = \|\eta\| - \underbrace{\frac{1}{a_0}(k_f + H\alpha)}_{\bar{Y}^p(\alpha)}, \quad \text{(yield function)}
$$

$$
\begin{pmatrix} -H\dot{\alpha} \\ \dot{\varepsilon}^p \end{pmatrix} = \gamma^p \nabla_{(\alpha,\eta,)} f^p = \gamma^p \begin{pmatrix} -\frac{1}{a_0}H \\ \frac{\eta}{\|\eta\|} \end{pmatrix},
$$

$$
\begin{aligned}
\gamma^p &\geq 0,\\
f^p(\alpha,\eta) &\leq 0, \quad \text{(yield condition)}\\
\gamma^p f^p(\alpha,\eta) &= 0.
\end{aligned}
$$

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Find $u(t) \in V_D$, $\forall\, t \in (0, T]$

Variational inequality

$$a(u, v) \quad - F(t; v) \quad = 0, \qquad v \in V_D,$$

$$a(u, v) := \int_\Omega \mathrm{C}^{el} \varepsilon(u) : \varepsilon(v)\; dx,$$

$$F(t; v) := \int_\Omega f_v(t) \cdot v\; dx + \int_{\Gamma_N} f_s(t) \cdot v\; ds.$$

$$V_D := H^1_D(\Omega)^d$$

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

u - displacement

Elastic Contact Problems with Plasticity

Variational inequality

Find $u(t) \in V_D$, $\lambda(t) \in \mathcal{E}^c(\lambda_n(t))$, $\qquad \forall\, t \in (0, T]$

Variational inequality

$$
\begin{aligned}
a(u, v) \qquad\qquad + b^c(\lambda, v) - F(t; v) &= 0, \qquad v \in V_D,\\
b^c(\mu - \lambda, (u_n - g)\, n) &\le 0, \qquad \mu \in \mathcal{E}^c(\lambda_n),\\
b^c(\mu - \lambda, \dot{u}_t) &\le 0, \qquad \mu \in \mathcal{E}^c(\lambda_n).
\end{aligned}
$$

where $\eta = \mathrm{dev}(\mathrm{C}^{el}(\varepsilon(u) - \varepsilon^p)) - a_0^{-2} K\, \varepsilon^p$,

$$b^c(\mu, v) := \int_{\Gamma_C} \mu \cdot v \; ds, \qquad \mathcal{E}^c(\lambda_n) := \{(\mu_n, \mu_t) \in W : \mu_n \ge 0, f^c(\lambda_n, \mu_t) \le 0\},$$

$$W := ([H^{\frac{1}{2}}(\Gamma_C)]^d)'$$

u - displacement $\qquad \lambda$ - contact stress

Find $u(t) \in V_D$, $\lambda(t) \in \mathcal{E}^c(\lambda_n(t))$, $(\alpha, \eta)(t) \in \mathcal{E}^p \; \forall\, t \in (0, T]$

Variational inequality

$$
\begin{aligned}
a(u, v) + b^p(\varepsilon^p, \varepsilon(v)) + b^c(\lambda, v) - F(t; v) &= 0, && v \in V_D,\\
-\, b^p({C^{el}}^{-1}(\tau - \eta), \dot{\varepsilon}^p) - \int_\Omega \mathrm{H}\, \dot{\alpha}(\beta - \alpha)\, dx &\le 0, && (\beta, \tau) \in \mathcal{E}^p,\\
b^c(\mu - \lambda, (u_n - g)\, n) &\le 0, && \mu \in \mathcal{E}^c(\lambda_n),\\
b^c(\mu - \lambda, \dot{u}_t) &\le 0, && \mu \in \mathcal{E}^c(\lambda_n).
\end{aligned}
$$

where $\eta = \mathrm{dev}(C^{el}(\varepsilon(u) - \varepsilon^p)) - a_0^{-2} K\, \varepsilon^p$,

$$
b^p(\varepsilon, \tau) := -\int_\Omega C^{el} \varepsilon : \tau \; dx,
$$

$$
\begin{aligned}
\mathcal{E}^p &:= \{(\beta, \tau) \in M^p : f^p(\beta, \tau) \le 0\},\\
M^p &:= L^2(\Omega) \times \{\tau \in L^2(\Omega; \mathbb{R}^{d,d}_{sym}) : \; \mathrm{tr}\, \tau = 0\}.
\end{aligned}
$$

u - displacement λ - contact stress α - inner variable ε^p - plastic strain

Elastic Contact Problems with Plasticity

Time discretization

Let $t_k = k\Delta t,\ k = 0, \dots, K$ with $\Delta t = \frac{T}{K}$,

$$\Delta u^k := u^k - u^{k-1}, \quad \Delta\varepsilon^{p,k} := \varepsilon^{p,k} - \varepsilon^{p,k-1}, \Delta\alpha^k := \alpha^k - \alpha^{k-1}.$$

Implicit Euler method:

Find $u^k \in V_D$, $(\alpha^k, \eta^k) \in \mathcal{E}^p$, $\lambda^k \in \mathcal{E}^c(\lambda_n^k)$ $k = 1, \dots, K$

Variational inequality

$$\begin{aligned} a(u^k, v) + b^p(\varepsilon^{p,k}, \varepsilon(v)) + b^c(\lambda^k, v) - F(t_k; v) \quad &= 0, && v \in V_D, \\ -\, b^p(\,\mathrm{C}^{el^{-1}}(\tau - \eta^k), \Delta\varepsilon^{p,k}\,) - \int_\Omega \mathrm{H}\,\Delta\alpha^k(\beta - \alpha^k)\, dx \quad &\le 0, && (\beta, \tau) \in \mathcal{E}^p, \\ b^c(\mu - \lambda^k, u_n^k\, n) - \int_{\Gamma_C} d_n(\mu_n - \lambda_n^k)\, ds \quad &\le 0, && \mu \in \mathcal{E}^c(\lambda_n^k), \\ b^c(\mu - \lambda^k, \Delta u_t^k) \quad &\le 0, && \mu \in \mathcal{E}^c(\lambda_n^k). \end{aligned}$$

where $\eta^k = \mathrm{dev}(\mathrm{C}^{el}(\varepsilon(u^k) - \varepsilon^{p,k})) - a_0^{-2} K\, \varepsilon^{p,k}$.

u^k - displacement $\quad \lambda^k$ - contact stress $\quad \alpha^k$ - inner variable $\quad \varepsilon^{p,k}$ - plastic strain

Analytical results

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Quasistatic contact problem without friction:

- $F \in L^p(0, T; V_D') \mapsto u \in L^p(0, T; V_D)$ is Lipschitz continuous

Analytical results

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Quasistatic contact problem without friction:

- $F \in L^p(0, T; V_D') \mapsto u \in L^p(0, T; V_D)$ is Lipschitz continuous

Quasistatic contact problem with Tresca friction:

- $F \in H^1(0, T; V_D') \mapsto u \in H^1(0, T; V_D)$ is Lipschitz continuous
- Existence of optimal controls $q = F$, optimality conditions, smooth regularization (e.g. Amassad, Chenais, Fabre 02)

Analytical results

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Quasistatic plasticity without contact: (Gröger 79, Krejci 96, Han, Reddy 99, Herzog et al. 11, G. Wachsmuth 12)

- There exists a unique Lipschitz continuous solution operator $F \in W^{1,1}(0, T; V_D') \mapsto (u, \varepsilon^p) \in L^\infty(0, T; V_D \times L^2(\Omega; \mathbb{R}^{d,d}_{sym}))$.
- Optimality theory for optimal control problems (e.g. Herzog, Meyer, G. Wachsmuth; Hintermüller, Surowiec; Outrata)

Semismooth Newton methods for regularized problems (Hintermüller, M. Ulbrich, Kunisch, Stadler, S.U.,...)

Outline

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Simplification:
Elastic Contact Problems without Friction

Governing equations

$$
\begin{aligned}
-\operatorname{div}\sigma(t) &= f_v(t), \\
\sigma(t) &= C^{el}\varepsilon(u(t)), \qquad \text{in } \Omega, \\
\varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T)
\end{aligned}
\qquad
\begin{aligned}
u(t) &= 0, && \text{on } \Gamma_D, \\
\sigma(t)n &= f_s(t), && \text{on } \Gamma_N, \\
-\sigma(t)n &= \lambda(t), && \text{on } \Gamma_C.
\end{aligned}
$$

u - displacement

σ - stress

ε - linearized strain

Simplification:
Elastic Contact Problems without Friction

Governing equations

$$\begin{aligned} -\operatorname{div}\sigma(t) &= f_v(t), \\ \sigma(t) &= C^{el}\varepsilon(u(t)), \\ \varepsilon(u(t)) &= \tfrac{1}{2}(\nabla u(t) + \nabla u(t)^T) \end{aligned} \quad \text{in } \Omega,$$

$$\begin{aligned} u(t) &= 0, && \text{on } \Gamma_D, \\ \sigma(t)\mathrm{n} &= f_s(t), && \text{on } \Gamma_N, \\ -\sigma(t)\mathrm{n} &= \lambda(t), && \text{on } \Gamma_C. \end{aligned}$$

contact without friction

$$\begin{aligned} \lambda_n &:= \lambda \cdot n, \\ \lambda_t &:= \lambda - \lambda_n n, \\ \lambda_t &= 0, \\ \lambda_n &\geq 0, \\ u_n - g &\leq 0, \ \text{(non-penetr.)} \quad \text{on } \Gamma_C \\ \lambda_n(u_n - g) &= 0. \end{aligned}$$

Elastic Contact Problems without Friction

Variational inequality

Find $u(t) \in V_D$, $\lambda_n(t) \in \mathcal{K}^c \ \forall\, t \in (0, T]$

Variational inequality

$$\begin{aligned} a(u, v) + b^c(\lambda_n, v_n) - F(t; v) &= 0, \qquad v \in V_D, \\ b^c(\mu_n - \lambda_n, u_n - g) &\leq 0, \qquad \mu_n \in \mathcal{K}^c. \end{aligned}$$

$$a(u, v) := \int_\Omega C^{el}\varepsilon(u) : \varepsilon(v)\ dx,$$

$$F(t; v) := \int_\Omega f_v(t) \cdot v\ dx + \int_{\Gamma_N} f_s(t) \cdot v\ ds.$$

$$b^c(\mu_n, v_n) := \int_{\Gamma_C} \mu_n \cdot v_n\ ds,$$

$$V_D := H^1_D(\Omega)^d$$

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\mathcal{K}^c := \{\mu_n \in W : \mu_n \geq 0\},$$

$$W := H^{\frac{1}{2}}(\Gamma_C)'$$

u - displacement $\quad \lambda_n$ - normal contact stress

Elastic Contact Problems without Friction

Reformulation as nonsmooth equation

Find $u(t) \in V_D$, $\lambda_n(t) \in \mathcal{K}^c \; \forall\, t \in (0, T]$

Variational inequality

$$\begin{aligned} a(u, v) + b^c(\lambda_n, v_n) - F(t; v) &= 0, \qquad v \in V_D, \\ b^c(\mu_n - \lambda_n, u_n - g) &\le 0, \qquad \mu_n \in \mathcal{K}^c. \end{aligned}$$

Regularity assumption: $\lambda_n \in L^2(0, T; L^2(\Gamma_C))$ (Necas 75, Kikuchi, Oden 88)

Then VI can be rewritten pointwise as

$$\lambda_n \ge 0, \quad g - u_n \ge 0, \quad \lambda_n(g - u_n) = 0 \iff \min(\lambda_n, \gamma^{-1}(g - u_n)) = 0, \gamma > 0 \text{ fixed}$$

Reformulation as nonsmooth equation

$$\begin{aligned} Au + B^*\lambda_n - F(t) &= 0, \\ \min(\lambda_n, \gamma^{-1}(g - Bu)) &= 0. \end{aligned}$$

$Au = a(u, \cdot)$, $Bu = b^c(\cdot, u_n)$.

Elastic Contact Problems without Friction

Regularized problem as semismooth equation

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Find $u(t) \in V_D$, $\lambda_n(t) \in L^2(\Gamma_C)\ \forall\, t \in (0, T]$

Reformulation as nonsmooth equation

$$Au + B^*\lambda_n - F(t) = 0,$$
$$\min(\lambda_n, \gamma^{-1}(g - Bu)) = 0.$$

$Au = a(u, \cdot),\ Bu = b^c(\cdot, u_n)$.

Problem: The operator ist not semismooth from $V_D \times L^2(\Gamma_C) \to V_D' \times L^2(\Gamma_C)$.

Regularized semismooth problem

$$Au + B^*\lambda_n - F(t) = 0,$$
$$\min(\lambda_n, \lambda_n + \gamma^{-1}(g - Bu)) = 0.$$

- Error is $o(\sqrt{\gamma})$ (variant of normal compliance regularization)
- Semismooth Newton methods locally superlinearly convergent

(e.g., Stadler; M. Ulbrich; Hintermüller; Kunisch; M. Ulbrich, S.U., Bratzke)

Elastic Contact Problems without Friction

Space discretization

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Spatial discretization: Finite element method

- P2-elements for the displacement u
- dual basis functions $\{\psi_j\}_{j \in I^C}$ for λ via the biorthogonality relation

$$\int_{\Gamma_C} \phi_i \, \psi_j \, \mathrm{ds} = \delta_{ij} \int_{\Gamma_C} \phi_i \, \mathrm{ds} =: \delta_{ij} d_j^2, \quad i, j \in I^C$$

with $\{\phi_j\}_{j \in I^C}$ restrictions of basis functions of u_n onto Γ_C

cf. Wohlmuth, Krause, Stadler

Elastic Contact Problems without Friction

Discretized problem

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Let $V_D^h \subset V_D$, $\mathcal{K}_h^c \subset \mathcal{K}^c$. Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c$ $\forall\, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, && v^h \in V_D^h, \\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\le 0, && \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

Reformulation as nonsmooth equation

$$\begin{aligned} A_h \vec{u}^h + B_h^T \vec{\lambda}_n^h - F_h(t) &= 0, \\ \min(\vec{\lambda}_n^h, \gamma^{-1}(g_h - B_h \vec{u}^h)) &= 0. \end{aligned}$$

Regularized semismooth problem

$$\begin{aligned} A_h \vec{u}^h + B_h^T \vec{\lambda}_n^h - F_h(t) &= 0, \\ \min(\vec{\lambda}_n^h, \vec{\lambda}_n^h + \gamma^{-1}(g_h - B_h \vec{u}^h)) &= 0. \end{aligned}$$

Elastic Contact Problems without Friction

Error estimators

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Let $V_D^h \subset V_D$, $\mathcal{K}_h^c \subset \mathcal{K}^c$. Find $u(t) \in V_D$, $\lambda_n^h(t) \in \mathcal{K}^c$ $\forall\, t \in (0, T]$

Variational inequality

$$\begin{aligned} a(u, v) + b^c(\lambda_n, v_n) - F(t; v) &= 0, \qquad v \in V_D, \\ b^c(\mu_n - \lambda_n, u_n - g) &\leq 0, \qquad \mu_n \in \mathcal{K}^c. \end{aligned}$$

Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c$ $\forall\, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, \qquad v^h \in V_D^h, \\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\leq 0, \qquad \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

A priori estimate: (Schröder, Wiedemann 10, Haasdonk, Salomon, Wohlmuth 11)

$$\|(u - u^h)(t)\|_{V_D}^2 + \|(\lambda_n - \lambda_n^h)(t)\|_W^2$$

$$\leq C\Big(\inf_{v^h \in V_D^h} \|u(t) - v^h\|_{V_D}^2 + \inf_{\mu_n^h \in \mathcal{K}_h^c} \|\lambda_n(t) - \mu_n^h\|_W^2 + b^c(\lambda_n(t) - \mu_n^h, u_n(t) - g) \Big).$$

Let $V_D^h \subset V_D$, $\mathcal{K}_h^c \subset \mathcal{K}^c$. Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c$ $\forall\, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, & v^h \in V_D^h,\\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\le 0, & \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

Let $u(t) \in V_D$, $\lambda_n(t) \in \mathcal{K}^c$ be the exact solution and $\overline{u}(t) \in V_D$ be the solution of

$$a(\overline{u}(t), v) + b^c(\lambda_n^h(t), v) - F(t; v) = 0 \quad \forall\, v \in V_D.$$

- $u^h(t)$ is the classical FE-approximation of $\overline{u}(t)$
- $\|\overline{u}(t) - u^h(t)\|_{V_D}$ can be estimated by PDE error estimators.

A posteriori estimate: (e.g. Schröder, Wiedemann 10)

$$\|(u - u^h)(t)\|_{V_D} + \|(\lambda_n - \lambda_n^h)(t)\|_W$$

$$\le C\left(\|\overline{u}(t) - u^h(t)\|_{V_D} + \|(u_n^h(t) - g)_+\|_{H^{1/2}(\Gamma_C)} + |b^c(\lambda_n^h(t), (u_n(t) - g)_+)|^{1/2}\right).$$

Model Reduction

Construction of a reduced basis - Φ_u, Φ_{λ_n}

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FE discretization: Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c$ $\forall\, t \in (0, T]$

Variational inequality FE discretization

$$
\begin{aligned}
a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, & v^h \in V_D^h,\\
b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\leq 0, & \mu_n^h \in \mathcal{K}_h^c.
\end{aligned}
$$

- λ_n is restricted to be nonnegative
- Extract positive reduced bases $\Phi_{\lambda_n} \subset \mathrm{span}\{\lambda_n(t_k)\}_{k=1}^K$
- Capture as much information as possible
  $\Rightarrow$ Angle-greedy algorithm [Haasdonk, Salomon, Wohlmuth '12]
- Extract orthonormal bases $\widetilde{\Phi}_u \subset \mathrm{span}\{u(t_k)\}_{k=1}^K$ from the snapshots by POD with energy level δ
- **But:** Taking $\widetilde{\Phi}_u$ as basis for u might result in an ill posed problem
- To guarantee the inf-sup stability: enrichment by supremizers
  - $\Phi_u := \widetilde{\Phi}_u \cup$ Riesz respresentation of $B^*\Phi_{\lambda_n}$ (Patera et al., Haasdonk et al. 11)

Elastic Contact Problems without Friction

Reduced problem by Galerkin projection (ROM 1)

Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c \; \forall \, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, & v^h \in V_D^h, \\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\leq 0, & \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

Find $u^{red}(t) = \Phi_u \vec{u}^{red}(t)$, $\lambda_n^{red}(t) = \Phi_{\lambda_n} \vec{\lambda}_n^{red}(t)$, $\vec{\lambda}_n^{red}(t) \geq 0 \; \forall \, t \in (0, T]$

Variational inequality reduced model (ROM 1)

$$\begin{aligned} a(\Phi_u \vec{u}^{red}, \Phi_u \vec{v}^{red}) + b^c(\Phi_{\lambda_n} \vec{\lambda}_n^{red}, \Phi_u \vec{v}^{red}) - F(t; \Phi_u \vec{v}^{red}) &= 0, & \forall \, \vec{v}^{red}, \\ b^c(\Phi_{\lambda_n}(\vec{\mu}_n^{red} - \vec{\lambda}_n^{red}), (\Phi_u \vec{u}^{red})_n - g) &\leq 0, & \forall \, \vec{\mu}_n^{red} \geq 0. \end{aligned}$$

Elastic Contact Problems without Friction

Reduced problem by Galerkin projection (ROM 1)

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Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c \ \forall\, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, \qquad v^h \in V_D^h, \\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\le 0, \qquad \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

Find $u^{red}(t) = \Phi_u \vec{u}^{red}(t)$, $\lambda_n^{red}(t) = \Phi_{\lambda_n} \vec{\lambda}_n^{red}(t)$, $\vec{\lambda}_n^{red}(t) \ge 0 \ \forall\, t \in (0, T]$

Variational inequality reduced model (ROM 1)

$$\begin{aligned} a(\Phi_u \vec{u}^{red}, \Phi_u \vec{v}^{red}) + b^c(\Phi_{\lambda_n} \vec{\lambda}_n^{red}, \Phi_u \vec{v}^{red}) - F(t; \Phi_u \vec{v}^{red}) &= 0, \qquad \forall\, \vec{v}^{red}, \\ b^c(\Phi_{\lambda_n}(\vec{\mu}_n^{red} - \vec{\lambda}_n^{red}), (\Phi_u \vec{u}^{red})_n - g) &\le 0, \qquad \forall\, \vec{\mu}_n^{red} \ge 0. \end{aligned}$$

Reformulation as semismooth equation (ROM 1)

$$\begin{aligned} \vec{\Phi}_u^T A_h \vec{\Phi}_u \vec{u}^{red} + \vec{\Phi}_u^T B_h^T \vec{\Phi}_{\lambda_n} \vec{\lambda}_n^{red} - \vec{\Phi}_u^T F_h(t) &= 0, \\ \min(\vec{\lambda}_n^{red}, \gamma^{-1} \vec{\Phi}_{\lambda_n}^T (g_h - B_h \vec{\Phi}_u \vec{u}^{red})) &= 0. \end{aligned}$$

Elastic Contact Problems without Friction

Reduced problem by Galerkin projection (ROM 1)

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Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c$ $\forall\, t \in (0, T]$

Variational inequality FE discretization

$$\begin{aligned} a(u^h, v^h) + b^c(\lambda_n^h, v_n^h) - F(t; v^h) &= 0, \qquad v^h \in V_D^h, \\ b^c(\mu_n^h - \lambda_n^h, u_n^h - g) &\leq 0, \qquad \mu_n^h \in \mathcal{K}_h^c. \end{aligned}$$

Find $u^{red}(t) = \Phi_u \vec{u}^{red}(t)$, $\lambda_n^{red}(t) = \Phi_{\lambda_n} \vec{\lambda}_n^{red}(t)$, $\vec{\lambda}_n^{red}(t) \geq 0$ $\forall\, t \in (0, T]$

Variational inequality reduced model (ROM 1)

$$\begin{aligned} a(\Phi_u \vec{u}^{red}, \Phi_u \vec{v}^{red}) + b^c(\Phi_{\lambda_n} \vec{\lambda}_n^{red}, \Phi_u \vec{v}^{red}) - F(t; \Phi_u \vec{v}^{red}) &= 0, \qquad \forall\, \vec{v}^{red}, \\ b^c(\Phi_{\lambda_n}(\vec{\mu}_n^{red} - \vec{\lambda}_n^{red}), (\Phi_u \vec{u}^{red})_n - g) &\leq 0, \qquad \forall\, \vec{\mu}_n^{red} \geq 0. \end{aligned}$$

A priori estimate: (Haasdonk, Salomon, Wohlmuth 11)

$$\begin{aligned} \|(u^h - u^{red})(t)\|_{V_D}^2 + \|(\lambda_n^h - \lambda_n^{red})(t)\|_W^2 \leq C\Big(&\inf_{\vec{v}^{red}} \|u^h(t) - \Phi_u \vec{v}^{red}\|_{V_D}^2 \\ &+ \inf_{\vec{\mu}_n^{red} \geq 0} \|\lambda_n^h(t) - \Phi_{\lambda_n} \vec{\mu}_n^{red}\|_W^2 + b^c(\lambda_n^h(t) - \Phi_{\lambda_n} \vec{\mu}_n^{red}, u_n^h(t) - g)\Big). \end{aligned}$$

Elastic Contact Problems without Friction

Reduced problem by projected semismooth equ. (ROM 2)

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Find $u^h(t) \in V_D^h$, $\lambda_n^h(t) \in \mathcal{K}_h^c \; \forall \, t \in (0, T]$

FE discretization semismooth reformulation

$$A_h \vec{u}^h + B_h^T \vec{\lambda}_n^h - F_h(t) = 0,$$
$$\min(\vec{\lambda}_n^h, \gamma^{-1}(g_h - B_h \vec{u}^h)) = 0.$$

Reduced problem, projected semismooth formulation (ROM 2)

$$\vec{\Phi}_u^T A_h \vec{\Phi}_u \vec{u}^{red} + \vec{\Phi}_u^T B_h^T \vec{\Phi}_{\lambda_n} \vec{\lambda}_n^{red} - \vec{\Phi}_u^T F_h(t) = 0,$$
$$\vec{\Phi}_{\lambda_n}^T \min(\vec{\Phi}_{\lambda_n} \vec{\lambda}_n^{red}, \gamma^{-1}(g_h - B_h \vec{\Phi}_u \vec{u}^h)) = 0.$$

A priori and a posteriori error estimate:

Error estimates based on the residual of the reduced semismooth system (Bratzke, S.U., work in progress)

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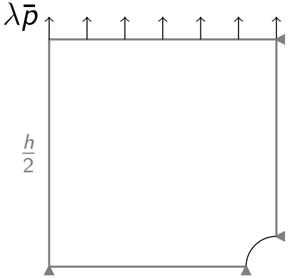
Benchmark problem: Quadratic plate with hole [Ramm et al.]

- elastic-perfectly plastic material
- plane strain setting
- material parameters steel
- monotonic loading



Model Reduction (ROM 2)

Example: Plasticity

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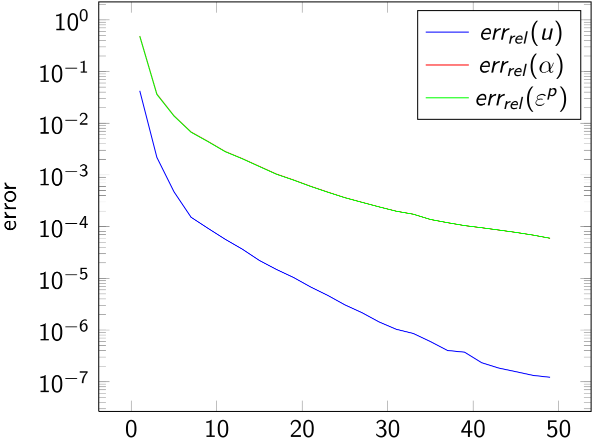


m number of basis elements $\widetilde{\Phi}_u$

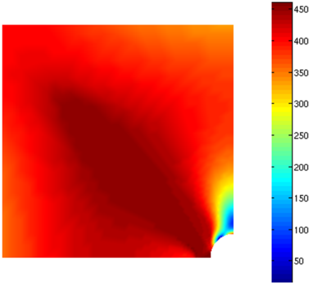
Φ_α (64), Φ_{ε^p} (64) fix

$$err_{rel}(x) = \frac{\sqrt{\Delta t \sum_{k=0}^{N} \left\| y_{FE}^k - y_{ROM,m}^k \right\|_{L^2}^2}}{\sqrt{\Delta t \sum_{k=0}^{N} \left\| y_{FE}^k \right\|_{L^2}^2}}$$

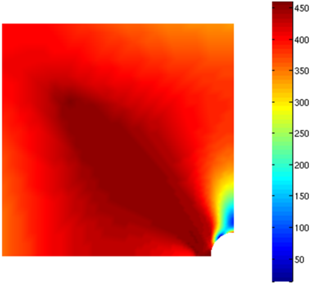
(ROM bases created from FE solutions $\{y_{FE}^k\}_{k=1}^N$)

Model Reduction (ROM 2)

Example: Plasticity



FE - von Mises stress



ROM - von Mises stress

	u	α	ε^p
$\#\ \Phi$.	$8+sup$	10	8
$err_{rel}(\cdot)$	5.0996e-04	0.0504	0.0164

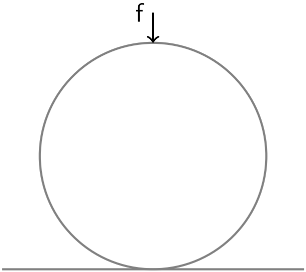
Model Reduction (ROM 2)

Example: Frictional contact

Problem:

Hertzian problem with friction

- plane strain setting
- material parameters steel
- monotonic loading



Model Reduction (ROM 2)

Example: Frictional contact

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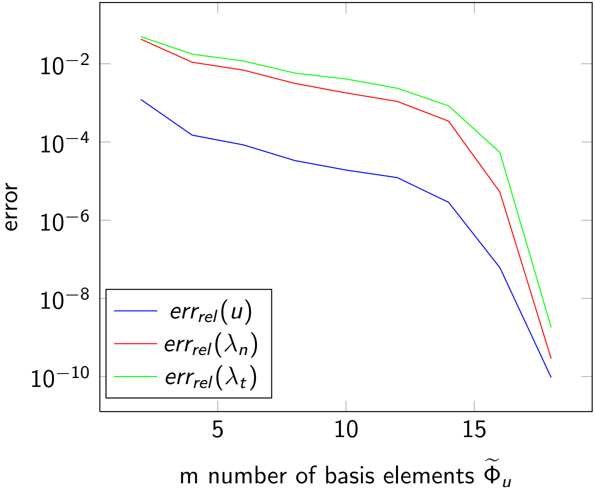


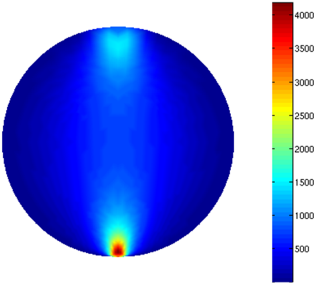
Φ_{λ_n} (17), Φ_{λ_t} (17) fix

$$err_{rel}(x) = \frac{\sqrt{\Delta t \sum_{k=0}^{N} \left\| y_{FE}^k - y_{POD,m}^k \right\|_{L^2}^2}}{\sqrt{\Delta t \sum_{k=0}^{N} \left\| y_{FE}^k \right\|_{L^2}^2}}$$

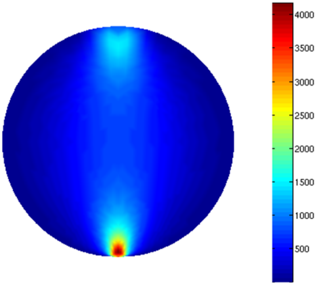
(ROM bases created from FE solutions $\{y_{FE}^k\}_{k=1}^N$)

Model Reduction (ROM 2)

Example: Frictional contact



FE - von Mises stress



ROM - von Mises stress

	u	λ_n	λ_t
$\#\ \Phi.$	$30+sup$	19	19
$err_{rel}(\cdot)$	1.6511e-09	1.3304e-09	3.6600e-09

Model Reduction (ROM 2)

Example: simplified hydroforming example

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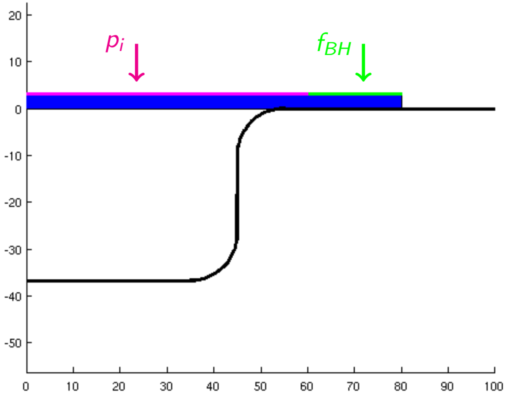
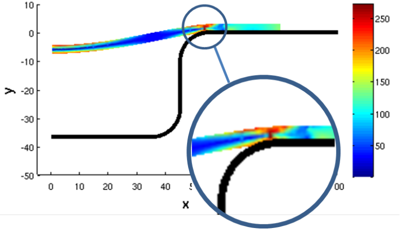
- plane strain setting
- material steel
- controls p_i and f_{BH}
- monotonic loading

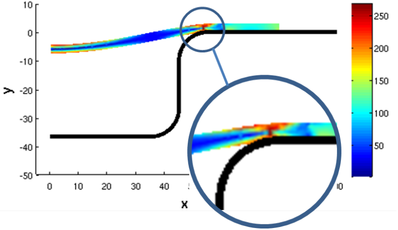


Model Reduction (ROM 2)

Experiment: simplified hydroforming example



FE - von Mises stress

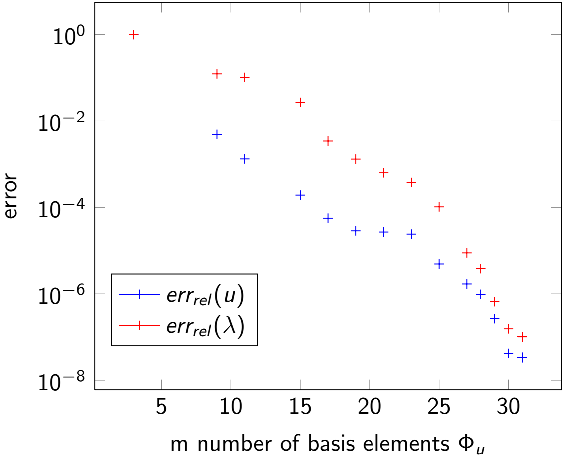


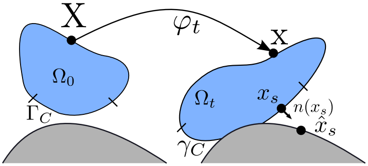
ROM - von Mises stress

	u	α	ε^p	λ_n	λ_t
$\#\ \Phi.$	19+*sup*	17	22	10	10
$err_{rel}(\cdot)$	2.1200e-04	0.0125	0.0124	0.0197	0.0451

Model Reduction (ROM 2)

Example: nonlinear contact





$$err_{rel}(x) = \frac{\sqrt{\Delta t \sum_{k=0}^{N} \left\| x_k^{FE} - x_k^{POD,m} \right\|^2_{H^1 \times H^{-\frac{1}{2}}}}}{\sqrt{\Delta t \sum_{k=0}^{N} \left\| x_k^{FE} \right\|^2_{H^1 \times H^{-\frac{1}{2}}}}}$$

(ROM bases created from FE solutions $\{x_k^{FE}\}_{k=1}^{N}$)

Outline

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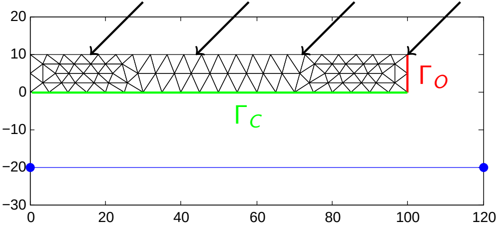
Optimal control problem

Problem setting

Optimal control of nonlinear contact problem (P)

$$\begin{aligned} \min \quad & J^h(u,q) = \tfrac{1}{2}\,\|u(\cdot,T) - u_{d,T}\|^2_{L^2(\Gamma_0)} + \tfrac{\alpha}{2}\int_0^T \|q\|^2_{L^2(\Gamma_N)}\,dt \\ \text{s.t.} \quad & C^h(u,\lambda_n,q) = 0. \end{aligned}$$



$q(t) = c(t)\cdot\binom{-1}{-1}$

- Nonlinear contact without friction, large deformation, no plasticity
- Reduced objective function $\hat{J}^h(q) = J^h(u(q),q)$
- Reduced subgradient with discrete adjoint approach $g^k \in \partial\hat{J}^h(q_k)$

ROM bundle trust region algorithm with Levenberg regularization

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Choose $q_0 \in \mathbb{R}^n, \eta \in (0,1)$ and set $k=0$. For $k=0,1,2,\dots$:

1. Compute time FE-snapshots for u^h and λ_n^h corresponding to q_k.
2. Compute reduced bases Φ_u, Φ_{λ_n} and build ROM model.
3. Minimize the ROM model function with Levenberg regularization ρ_L^k
$$s_k = \arg\min \hat{J}_k^{ROM}(q_k+s_k) + \frac{\rho_L^k}{2}\|s_k\|_2^2, \qquad \text{(Bundle TR)}$$
4. Compute $\hat{J}_k^h(q_k+s_k)$ and set
$$\alpha = \frac{\hat{J}_k^h(q_k) - \hat{J}_k^h(q_k+s_k)}{\hat{J}_k^{ROM}(q_k) - \hat{J}_k^{ROM}(q_k+s_k)}$$
5. If $\eta < \alpha$: set $q_{k+1} = q_k + s_k$ and decrease $\rho_L^{k+1} = \rho_L^k/4$ (successful) GOTO 1.
6. If $\alpha \le \eta$: set $q_{k+1} = q_k$ and increase $\rho_L^{k+1} = 8\rho_L^k$ (non-successful) GOTO 3.

(cf. TR-POD: Arian, Fahl, Sachs; Bundle TR: Schramm, Zowe)

Optimal control problem

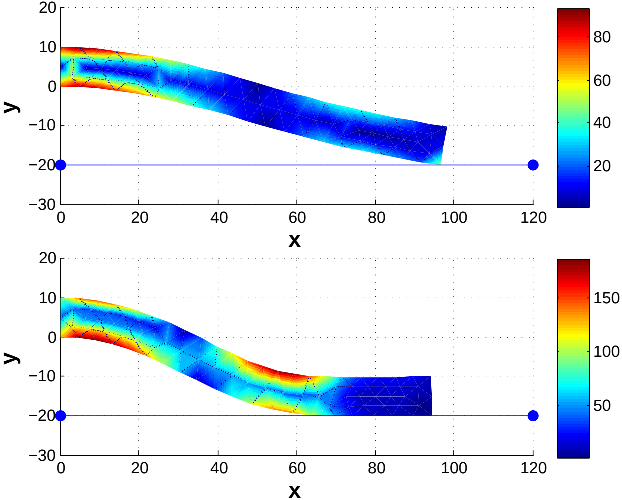
Convergence history

It.	$\#\hat{\Phi}_u$	$\#\Phi_{\lambda_n}$	J	ROM-BTR-Lev (s/n)	ρ_L	# Bundle it. (s/n)
0	5	2	304.1981	-	-	-
1	5	2	298.8423	s	60	(1/0)
2	6	3	206.4340	s	15	(4/0)
3	6	3	173.5778	s	3.750	(3/0)
4	6	3	138.6091	s	0.937	(7/2)
5	6	3	130.9544	s	0.234	(5/0)
6	6	3	129.9420	s	0.059	(2/0)

- Accuracy Reduced Basis: $\delta = 0.999999999$

Optimal control problem

Numerical results



Initial control

$J_0 = 304.1981$

Optimal control

$J_{opt} = 129.9420$

Multilevel ROM bundle trust region algorithm with Levenberg regularization

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Choose $q_0 \in \mathbb{R}^n, \eta \in (0,1)$ and set $k = 0$. For $k = 0, 1, 2, \ldots$:

1. Compute time FE-snapshots for u^h and λ_n^h corresponding to q_k.
   Check that state and adjoint residuals are small enough compared to criticality measure. Else refine.
2. Compute reduced bases Φ_u, Φ_{λ_n} and build ROM model.
3. Minimize the ROM model function with Levenberg regularization ρ_L^k

$$s_k = \arg\min \hat{J}_k^{ROM}(q_k + s_k) + \frac{\rho_L^k}{2} \|s_k\|_2^2, \qquad \text{(Bundle TR)}$$

4. Compute $\hat{J}_k^h(q_k + s_k)$ and set $\alpha = \dfrac{\hat{J}_k^h(q_k) - \hat{J}_k^h(q_k + s_k)}{\hat{J}_k^{ROM}(q_k) - \hat{J}_k^{ROM}(q_k + s_k)}$
5. If $\eta < \alpha$: set $q_{k+1} = q_k + s_k$ and decrease $\rho_L^{k+1} = \rho_L^k/4$ (successful) GOTO 1.
6. If $\alpha \leq \eta$: set $q_{k+1} = q_k$ and increase $\rho_L^{k+1} = 8\rho_L^k$ (non-successful)
   Adjust ROM-error based on $\hat{J}_k^{ROM}(q_k) - \hat{J}_k^{ROM}(q_k + s_k)$. GOTO 3.

Outline

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Summary

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- Engineering application for optimal control of evolutionary VIs
- Modeling and semismooth reformulation of plasticity and (frictional) contact problems
- Discretization and error estimators
- Reduced order model
- Optimization results with reduced order models