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Calmness of solution mappings in parametric optimization problems

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Based on:

[KK14] D. Klatte, B. Kummer, On calmness of the argmin mapping in parametric optimization problems, *Optimization online*, February 2014.

[KKK12] D. Klatte, A. Kruger, B. Kummer, From convergence principles to stability and optimality conditions, *J. Convex Analysis*, 19 (2012) 1043-1073.

[KK09] D. Klatte, B. Kummer, Optimization methods and stability of inclusions in Banach spaces, *Math. Program. Ser. B* 117 (2009) 305-330.

[KK02] D. Klatte, B. Kummer, *Nonsmooth Equations in Optimization*, Kluwer 2002.

Contents:

1. Basic model and main purpose

Consider the parametric optimization problem

$$f(x,t) \to \min_x \quad \text{s.t.} \quad x \in M(t)\,,\; t \text{ varies near } t^*, \tag{1}$$

where M is the **feasible set mapping** of (1). We assume throughout:

T is a normed linear space, $M : T \rightrightarrows \mathbb{R}^n$ has closed graph $\operatorname{gph} M$,
$\boldsymbol{(t^*, x^*) \in \operatorname{gph} M}$ is a given reference point,
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For (1), define the **infimum value function φ** by

$$\varphi(t) := \inf_x \{ f(x,t) \mid x \in M(t) \}\,, \; t \in T$$

and the **argmin mapping Ψ** by

$$\Psi(t) \; := \; \operatorname*{argmin}_x \{ f(x,t) \mid x \in M(t) \}\,, \; t \in T\,. \tag{2}$$

We are interested in conditions for **calmness** of the argmin mapping

$$\boxed{t \mapsto \Psi(t) = \{x \in M(t) \mid f(x,t) \le \varphi(t)\}\,,}$$

for t near t^*, and to relate this to calmness of the **auxiliary mappings**

$$\begin{array}{rcl} (t,\mu) \;\mapsto\; L(t,\mu) & = & \{x \in M(t) \mid f(x,t^*) \le \mu\}\,, \\ \mu \;\mapsto\; L(t^*,\mu) & = & \{x \in M(t^*) \mid f(x,t^*) \le \mu\}\,. \end{array} \tag{3}$$

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If $M(t)$ is described by inequalities, then $L(t,\mu)$ is so, too, and moreover, $L(t^*,\mu)$ is given by inequalities perturbed only at the right-hand side.

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Main purpose of the paper:

To show under suitable conditions and for a large class of problems that

$$\boxed{L \text{ calm} \;\Rightarrow\; \Psi \text{ calm}} \qquad (5)$$

and to discuss inspired by Canovas et al. (JOTA '14) whether (or not)

$$\boxed{\Psi \text{ calm} \;\Rightarrow\; L \text{ calm.}} \qquad (6)$$

Canovas et al. proved (6) for <u>canonically perturbed linear SIPs.</u>

2. Definition of calmness and motivations

Definitions

Let T be a normed linear space,
B closed unit ball (in T or X), $B(x, \varepsilon) := \{x\} + \varepsilon B$.

Given a multifunction $\Phi : T \rightrightarrows \mathbb{R}^n$ and $x^* \in \Phi(t^*)$,

Φ is called **calm** at (t^*, x^*) if there are $\varepsilon, \delta, L > 0$ such that

$$\Phi(t) \cap B(x^*, \varepsilon) \subset \Phi(t^*) + L\|t - t^*\|B \quad \forall t \in B(t^*, \delta), \tag{7}$$

in particular, $\Phi(t) \cap B(x^*, \varepsilon) = \emptyset$ for $t \neq t^*$ possible.

Example: If $T = \mathbb{R}^m$ and $\operatorname{gph} \Phi$ is the union of finitely many convex polyhedral sets, then Φ is calm at each $(t^*, x^*) \in \operatorname{gph} \Phi$. (Robinson '81)

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In contrast, we say that

Φ **has the Aubin property** at (t^*, x^*) if for some $\varepsilon, \delta, L > 0$,

$$\emptyset \neq \Phi(t) \cap B(x^*, \varepsilon) \subset \Phi(t') + L\|t' - t\|B \quad \forall t, t' \in B(t^*, \delta). \tag{8}$$

Example: If $T = \mathbb{R}^m$ and $\operatorname{gph} \Phi$ is the union of finitely many convex polyhedral sets, then Φ is calm at each $(t^*, x^*) \in \operatorname{gph} \Phi$. (Robinson '81)

Special cases

1. Calmness and error bounds: For $g : X \to T$, let Φ be defined by

$$\Phi(t) := \{x \in X \mid g(x) + t \in T^0\}, \;\; T^0 \subset T \text{ closed}, \; g \text{ continuous},$$

then Φ is calm at $(0, x^*) \in \operatorname{gph} \Phi$ if and only if for some $L, \varepsilon > 0$,

$$\operatorname{dist}(x, \Phi(0)) \leq L \operatorname{dist}(g(x), T^0) \quad \forall x \in B(x^*, \varepsilon)\,. \qquad \text{(local error bound)}$$

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2. Canonically perturbed linear SIPs: Consider the special case of (1) with I - a compact metric space, $a \in (C(I, \mathbb{R}))^n$ given,

$$f(x, c) = c^\top x \to \min_x \quad \text{s.t.} \quad a_i^\top x \le b_i,\ i \in I, \tag{9}$$

$t = (c, b)$ varies in $T = \mathbb{R}^n \times C(I, \mathbb{R})$ (i.e. $b: I \to \mathbb{R}$ continuous, max-norm).

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$t = (c, b)$ varies in $T = \mathbb{R}^n \times C(I, \mathbb{R})$ (i.e. $b : I \to \mathbb{R}$ continuous, max-norm).

Theorem 1 (Canovas et al. '14): Given $(t^*, x^*) \in \operatorname{gph}\Psi$, $t^* = (c^*, b^*)$, and under Slater CQ at b^*, Ψ is calm at (t^*, x^*) if and only if

$$\mu \mapsto L(b, \mu) = \{x \mid a_i^{\mathsf{T}} x \le b_i, \; i \in I, c^{*\mathsf{T}} x \le \mu\} \text{ is calm at } ((t^*, \varphi(t^*)), x^*).$$

Every nonempty closed convex set S can be represented by a linear semi-infinite system of the type as given in (9), see Goberna-Lopez '98.

Question: Does Proposition 1 also hold for a problem e.g. of the type

$$f(x,c) = c^\top x \to \min_x \quad \text{s.t.} \quad g_i(x) \leq b_i,\ i = 1,\dots,m,$$

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No! The "only if"-direction fails.

Example 1:*) Consider

$$\min\ y - c_1 x - c_2 y \quad \text{s.t. } x^2 - y \le b, \quad (c_1,c_2,b) \text{ close to } \underline{o} = (0,0,0).$$

Its argmin mapping Ψ is Lipschitz near $\underline{o}$, and hence calm at $(\underline{o},(0,0))$:

$$\Psi(c_1,c_2,b) = \left\{ \left(\frac{c_1}{2(1-c_2)}, \frac{c_1^2}{4(1-c_2)^2} - b \right) \right\}.$$

However, $\boxed{L(0,\mu) = \{(x,y) \mid y \le \mu,\ x^2 \le y\}}$ is not calm at the origin.

*) For this and a 2nd example, with quadratic f and linear g_i, see [KK14].

3. Calmness of the argmin map via calmness of auxiliary maps

Consider again the parametric optimization problem (1),

$$f(x,t) \to \min_x \quad \text{s.t.} \quad x \in M(t)\,,\ t \text{ varies near } t^*,$$

and assume

$$\begin{array}{l} M \text{ is closed, } (t^*, x^*) \in \operatorname{gph} \Psi \text{ is a given point, and} \\ f \text{ is Lipschitzian near } (x^*, t^*) \text{ with modulus } \varrho_f > 0. \end{array} \tag{10}$$

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Define for given $V \subset \mathbb{R}^n$,

$$\begin{aligned} \Psi_V(t) &:= \text{argmin}_x\{f(x,t) \mid x \in M(t) \cap V\}, \quad t \in T, \\ \varphi_V(t) &:= \text{inf}_x\{f(x,t) \mid x \in M(t) \cap V\}. \qquad t \in T, \end{aligned}$$

Definition: M is called **Lipschitz l.s.c.** at $(t^*, x^*) \in \operatorname{gph} M$ if there are constants $\delta, \varrho > 0$ such that

$$\operatorname{dist}(x^*, M(t)) \leq \varrho \|t - t^*\| \quad \forall t \in B(t^*, \delta).$$

Obviously, the Aubin property implies both calmness and Lipschitz l.s.c.

Definition: Given a function $F : T \to \overline{\mathbb{R}}$ and $t^* \in \operatorname{dom} F$,

F is called **calm** at t^* if there are $\delta, L > 0$ such that

$$|F(t) - F(t^*)| \leq L\|t - t^*\| \quad \forall t \in \operatorname{dom} F \cap B(t^*, \delta),$$

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Lemma 1. [KK14]*)

If M is calm and Lipschitz l.s.c. at $(t^*, x^*) \in \operatorname{gph} \Psi$, then there exists a closed neighborhood V of x^* such that the function $\boxed{\varphi_V \text{ is calm at } t^*.}$

*) **Proof** based on ideas in Alt '83 and Klatte '84.

Theorem 2. [KK14]

Consider the problem (1) under the assumptions (10). Suppose that for the reference point $(t^*, x^*) \in \text{gph}\,\Psi$,

(i) the feasible set map M is calm and Lipschitz l.s.c. at (t^*, x^*),

(ii) $L(t, \mu) = \{x \in M(t) \mid f(x, t^*) \le \mu\}$ is calm at $((t^*, \varphi(t^*)), x^*)$.

Then the argmin mapping Ψ is calm at (t^*, x^*).

Note. In general, one cannot avoid to assume **M l.s.c.**, even if $M(t)$ is given by convex inequalities with rhs perturbations (see examples in Bank-Guddat-Klatte-Kummer-Tammer, *Nonlinear Parametric Optimization* '82).

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The proof of Theorem 2 essentially uses Lemma 1 and

$$\Psi(t) \cap V \neq \emptyset \;\Rightarrow\; \Psi_V(t) = \Psi(t) \cap V \qquad \boxed{\text{(hence, } \varphi_V(t)) = \varphi(t))}$$

for given $t \in T$ and $V \subset \mathbb{R}^n$, as well as

$$\boxed{\Psi(t) = L(t, \mu(x, t))} \quad \text{with } \mu(x, t) := \varphi(t) + f(x, t^*) - f(x, t).$$

Corollary 1. [KK14]

Suppose that for the reference point $(t^*, x^*) \in \operatorname{gph} \Psi$,

(i) the feasible set map M is calm and Lipschitz l.s.c. at (t^*, x^*),

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(iii) the level set map $\boxed{F(\mu) = \{x \mid f(x, t^*) \le \mu\}}$ is calm at $(\varphi(t^*), x^*)$.

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Proof: By Theorem 2, one has to prove that L is calm. To show this, apply Thm. 2.5 in [KK02] (calm intersection theorem) to

$$L(t, \mu) := \{x \in M(t) \mid f(x, t^*) \le \mu\} = \boxed{M(t) \cap F(\mu).}$$

By the intersection thm, one has to check (at the corresponding points)

M, F and $L(t^*, \cdot)$ are calm, and F^{-1} has Aubin property.

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Calmness is guaranteed by (i)–(iii), while $F^{-1}(x) = \{\mu \,|\, \mu \ge f(x, t^*)\}$ has the Aubin property since f is locally Lipschitz. □

4. Application to an inequality constrained setting

Consider the canonically perturbed program $P(t)$, $t = (c, b) \in \mathbb{R}^n \times C(I, \mathbb{R})$ varies near $t^* = (c^*, b^*)$,

$$\min_x f(x, c) = h(x) + c^\top x \quad \text{s.t.} \quad g_i(x) \le b_i \quad \forall\, i \in I, \tag{11}$$

where the mappings M, Ψ, L are as above, and (11) satisfies $^{*)}$

- I compact metric space (including finite I),
- $(t^*, x^*) \in \operatorname{gph} \Psi$ is a given reference point,
- $(i, x) \in I \times \mathbb{R}^n \mapsto g_i(x) \in \mathbb{R}$ is continuous,
- $h, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex $\quad (i \in I)$.

$C(I, \mathbb{R})$ = space of continuous fcts $b : I \to \mathbb{R}$ (normed by $\|b\| = \max_{i \in I} |b_i|$).

$^{*)}$ For h, g_i linear, this is the setting of Theorem 1 (Canovas et al '14)

Application of Corollary 1 to the parametric problem (11),

$$\min_x f(x,c) = h(x) + c^\top x \quad \text{s.t.} \quad g_i(x) \le b_i, \ \forall\ i \in I.$$

Suppose (as in Theorem 1) the Slater CQ at $M(b^*)$, i.e.

$$\exists \widetilde{x}\ \forall i \in I : g_i(\widetilde{x}) < b_i^*,$$

and let $\mu^* = f(x^*, c^*) = \varphi(c^*, b^*)$. Let $F(\mu) = \{x \,|\, h(x) + (c^*)'x \le \mu\}$.

Then

- M has the Aubin property at (b^*, x^*) (consequence of the Robinson-Ursescu theorem), cf. e.g. Canovas-Dontchev et al.'05.

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- If $x^* \notin \operatorname{argmin}_x f(x, c^*)$, then $F(\mu^*)$ fulfills SlaterCQ ($\Rightarrow$ calm). Otherwise, see error bound literature (e.g. Li '97, Pang '97).

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- F^{-1} has Aubin property since f is convex.

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- F^{-1} has Aubin property since f is convex.
- To check that

  $$\mu \mapsto L(c^*, b^*, \mu) = M(b^*) \cap F(\mu)$$

  is calm at (μ^*, x^*) reduces to calmness of a (semi-infinite) inequality system with right-hand side perturbations, see e.g. Henrion-Outrata'05, [KK09], Canovas et al.'14 and the following.

Calmness for solution maps of inequality systems

Let $h : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and consider the level sets

$$S_h(q) = \{x \in \mathbb{R}^n \,|\, h(x) \le q\},\ q \in \mathbb{R}.$$

Calmness of S_h is obviously equivalent to

calmness of the inverse multifunction to $h^+(x) = \max\{0, h(x)\}$

Theorem 3. [KK09] (see also [KKK12] for generalizations to Hölder calmness and l.s.c. functions on complete metric spaces).

Given a zero x^* of h, S_h is calm at $(0, x^*)$ if and only if for $H(x) = h^+(x)$,

> there are $\lambda, \delta > 0$ such that for all $x \in B(x^*, \delta)$ there is some x' satisfying $H(x') - H(x) \le -\lambda \|x' - x\|$ and $\|x' - x\| \ge \lambda H(x)$.

Application to the semi-infinite setting (11)

Replace in the setting (11) "g_i *convex*" by "g_i *locally Lipschitz*".

Then Theorem 3 applies to the solution set map S of the system

$$g_i(x) \le b_i, \; i \in I, \quad \text{and for } b^* = 0,$$

since calmness of S is equivalent to calmness of

$$\Sigma(q) = \left\{ x \;\middle|\; H(x) := \left(\max_{i \in I} g_i(x) \right)^+ = q \right\}, \; q \text{ real.}$$

For

$$H(x) = \left(\max_{i \in I} g_i(x) \right)^+ > 0$$

define the *relative slack of g_i* by

$$s_i(x) = \frac{H(x) - g_i(x)}{H(x)} \quad (\ge 0).$$

Suppose here for simplicity even $g_i \in C^1$,

see also Henrion-Outrata '05 for different conditions, and for more general cases see [KK09] and [KKK12].

Theorem 4 (slope condition) [KK09].

S is calm at $(b^*, x^*) = (0, x^*)$ if and only if

for some $\lambda \in]0,1[$ and some nbhd Ω of x^*, one has

$$\text{For all } x \in \Omega \quad \text{with} \quad H(x) = (\max_{i \in I} g_i(x))^+ > 0$$

$$\text{there is some } u \in \text{bd}\, B : \qquad Dg_i(x)u \ \leq \ \frac{s_i(x)}{\lambda} - \lambda \quad \forall i \in I\,.$$

Note: the right-hand side of the latter inequality may be positive also for active i (in contrast to the extended MFCQ).

5. Final remarks

1. At first glance, calmness seems to be a very weak Lipschitz stability concept for the argmin mapping, since solvability can disappear under small perturbations. However, it is useful as a kind of minimal requirement for the lower level in bi-level problems (CQ).

2. We have shown that calmness of

$$L^*(\mu) := L(t^*, \mu) := \{x \in M(t^*) \mid f(x, t^*) \le \mu\}$$

is essential for checking calmness of the argmin map Ψ. Note that calmness of L^* at $(f(x^*, t^*), x^*)$ **for each** $x^* \in \Psi(t^*)$ (provided $\Psi(t^*)$ is compact) implies: $\Psi(t^*)$ is a weak sharp minimizing set of the problem $f(x, t^*) \to \min_x$ s.t. $x \in M(t^*)$, cf. Henrion, Jourani, Outrata '02.

3. The calm intersection theorem used in the proof of Theorem 2 is a powerful tool also in other situations, see recent papers by Henrion, Outrata, Surowiec and the authors.

Some further references mentioned in the talk

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