First and Second Order Variational Inclusions and Necessary Optimality Conditions for Deterministic Optimal Control Problems in the Presence of State Constraints

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Outline of the talk

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   - Directional "Derivatives" of Set-Valued Maps
   - Variational Differential Inclusions
2. **Control Systems under State Constraints**
3. **The Mayer Optimal Control Problem**
   - Relaxed Mayer Problem
   - Approximation of Control Systems
   - Second Order Maximum Principle

Tangents to Sets

M - a subset of a Banach space X. The tangent cones to M at $x \in M$ are defined via the **Peano-Kuratowski set limits** :

adjacent cone to M at x

$$T_M^\flat(x) := \mathrm{Liminf}_{h\to 0+} \frac{M-x}{h} = \{u \in X : \lim_{h\to 0+} \mathrm{dist}\left(u, \frac{M-x}{h}\right) = 0\}$$

Clarke tangent cone to M at x

$$C_M(x) := \mathrm{Liminf}_{y\to_M x,\ h\to 0+} \frac{M-y}{h}$$

$C_M(x)$ is convex. Normal cone to M at x

$$N_M(x) := \{p \in X^* \ : \ \langle p, u\rangle \le 0 \ \ \forall\, u \in C_M(x)\}$$

Second Order Tangents

The second-order adjacent subset to M at $(x, u) \in M \times X$:

$$T_M^{\flat(2)}(x, u) := \mathrm{Liminf}_{h \to 0+} \frac{M - x - hu}{h^2}$$

$$T_M^{\flat(2)}(x, u) = T_M^{\flat(2)}(x, u) + C_M(x)$$

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Example

Let $M = \cap_{j=1}^m M_j$, where $M_j = \{x \ : \ h_j(x) \leq 0\}$, $h_j \in C^2(\mathbb{R}^n;\mathbb{R})$
$0 \notin co\{\nabla h_j(x) \ : j \in I_{active}(x)\}$ for all $x \in \partial M$.
Then for every $x_0 \in \partial M$,

$$T_M^\flat(x_0) = \{u \in \mathbb{R}^n \ : \ \langle \nabla h_j(x_0), u\rangle \leq 0 \ \ \forall\, j \in I_{active}(x_0)\}$$

For every $u \in T_M^\flat(x_0)$, a vector $v \in T_M^{\flat(2)}(x_0,u)$ **if and only if**

$$\langle \nabla h_j(x_0), v\rangle + \frac{1}{2}h_j''(x_0)uu \leq 0 \qquad \forall\, j \in I^{(1)}(x_0,u),$$

where $I^{(1)}(x_0,u) = \{j \in I_{active}(x_0) \mid \langle \nabla h_j(x_0), u\rangle = 0\}$. That is if $I^{(1)}(x_0,u) \neq \emptyset$, then $T_M^{\flat(2)}(x_0,u)$ is a closed convex polytope in $\mathbb{R}^n$.

A Second Order Necessary Optimality Condition

Primal Approach to Necessary Conditions :

$$\min_{x \in M} \phi(x),$$

where $\phi : X \to \mathbb{R}$ is a C^2 function. Let $\bar{x} \in M$ be a local minimizer. **Fermat rule** :

$$\phi'(\bar{x})u \geq 0 \quad \forall\, u \in T_M^\flat(\bar{x}) \quad \Rightarrow \quad -\phi'(\bar{x}) \in N_M(\bar{x}).$$

Second order rule :

$$\phi'(\bar{x})v + \frac{1}{2}\phi''(\bar{x})(u,u) \geq 0$$

for all $u \in T_M^\flat(\bar{x}),\ v \in T_M^{\flat(2)}(\bar{x}, u)$ such that $\phi'(\bar{x})u = 0$.

Directional "Derivatives" of Set-Valued Maps

Let $F\colon \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be a set-valued map, locally Lipschitz around $x \in \mathbb{R}^n$ and let $y \in F(x)$.

Definition

$dF(x,y)\colon \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is the set-valued map defined by

$$dF(x,y)u := \mathrm{Liminf}_{h\to 0+} \frac{F(x+hu)-y}{h} \quad \forall\, u \in \mathbb{R}^n.$$

For $v \in dF(x,y)u$, the second-order variation $d^2F(x,y,u,v)$ is the set-valued map defined by: $\forall\, z \in \mathbb{R}^n$

$$d^2F(x,y,u,v)z := \mathrm{Liminf}_{h\to 0+} \frac{F(x+hu+h^2z)-y-hv}{h^2}$$

Linearization of Differential Inclusions

Let $\widetilde{F}: \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ have nonempty compact images and $K_0 \subset \mathbb{R}^n$. Consider the differential inclusion

$$\begin{cases} x'(t) \in \widetilde{F}(x(t)) & \text{a.e. in } [0,1] \\ x(0) \in K_0 \end{cases} \qquad \text{(DI)}$$

Assume

$$\begin{cases} \exists\, \gamma > 0,\ \max_{v \in \widetilde{F}(x)} |v| \leq \gamma(1+|x|) & \forall\, x \in \mathbb{R}^n; \\ \forall\, R > 0, \exists\, c_R \geq 0 :\ \widetilde{F} \text{ is } c_R\text{-Lipschitz on } B(0,R) \end{cases} \qquad \text{(A)}$$

Define $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ by $F(x) := \mathrm{co}\, \widetilde{F}(x)$

First Order Variational Inclusion

Let $\bar{x}$ be a solution of (DI) and $y : [0,1] \to \mathbb{R}^n$ be a solution of

- $y'(t) \in dF(\bar{x}(t), \bar{x}'(t))y(t) \quad$ for a.e. $t \in [0,1]$;
- $y(0) \in T^\flat_{K_0}(\bar{x}(0))$

Theorem

Consider any $h_i \to 0+$, $y_i^0 \to y(0)$ such that $\bar{x}(0) + h_i y_i^0 \in K_0$. Then there exist solutions x_i of (DI) satisfying

$$x_i(0) = \bar{x}(0) + h_i y_i^0$$

such that $\frac{1}{h_i}(x_i - \bar{x})$ converge uniformly to y when $i \to \infty$.

If $F(x) = \{f(x)\}$ is single valued with $f \in C^1$, then the corresponding variational equation is

$$y'(t) = f_x(\bar{x}(t))y(t)$$

Second Order Variational Inclusion

For $y(\cdot)$ as above assume for some $a_2 \in L^1([0,1];\mathbb{R}_+)$ and all small $h > 0$

$$\mathrm{dist}_{F(\bar{x}(t)+hy(t))}(\bar{x}'(t) + hy'(t)) \leq a_2(t)h^2 \quad \text{a.e.}$$

We abbreviate $[t] := (\bar{x}(t), \bar{x}'(t), y(t), y'(t))$
and consider a solution $w : [0,1] \to \mathbb{R}^n$ of

- $w'(t) \in d^2F[t]w(t) \quad$ for a.e. $t \in [0,1]$;
- $w(0) \in T_{K_0}^{\flat(2)}(\bar{x}(0), y(0))$.

If $F(x) = \{f(x)\}$ is single valued with $f \in C^2$, then the corresponding variational equation is

$$w'(t) = f_x(\bar{x}(t))w(t) + \frac{1}{2} f_{xx}(\bar{x}(t))y(t)y(t)$$

Theorem

Let $h_i \to 0+$, $w_i^0 \to w(0)$ *be such that* $\bar{x}(0) + h_i y(0) + h_i^2 w_i^0 \in K_0$. *Then there exist solutions* x_i *of (DI) satisfying*

$$x_i(0) = \bar{x}(0) + h_i y(0) + h_i^2 w_i^0$$

such that

$$\frac{1}{h_i^2}(x_i - \bar{x} - h_i y)$$

converge uniformly to w *when* $i \to \infty$.

Control System under State Constraints

$$\left\{ \begin{array}{lll} x'(t) & = & f(x(t),u(t)), \quad u(t) \in U \quad \text{ a.e. in } [0,1] \\ x(0) & \in & K_0 \\ x(t) & \in & K \quad \text{ for all } \; t \in [0,1] \end{array} \right.$$

U is a complete separable metric space, $K_0, \; K \subset \mathbb{R}^n$ are nonempty and closed, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$.
Controls are Lebesgue measurable functions $u(\cdot) : [0,1] \to U$
$\mathcal{S}_K(x_0)$ denotes the set of trajectories of the control system starting at x_0.

We assume that $f(x,\cdot)$ is continuous, $f(x,U)$ are closed, $\exists \; \gamma > 0$, $\sup_{u \in U} |f(x,u)| \leq \gamma(1+|x|)$ and for every $R > 0$ there exists $c_R > 0$ such that $f(\cdot,u)$ is c_R-Lipschitz on $B(0,R)$ for any $u \in U$.

Mayer Problem

For $\varphi : \mathbb{R}^n \to \mathbb{R}$ the **Mayer problem under state constraints** is

$$\text{minimize}\left\{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0),\;\; x_0 \in K_0\right\}$$

$(\bar{x}, \bar{u})$ is called a strong local minimizer if for some $\varepsilon > 0$ and for all $x(\cdot) \in \mathcal{S}_K(x_0)$, $x_0 \in K_0$ satisfying $||\bar{x} - x||_\infty < \varepsilon$ we have $\varphi(x(1)) \geq \varphi(\bar{x}(1))$.

Let $(\bar{x}, \bar{u})$ be a strong local minimizer, $\varphi \in \mathcal{C}^2$ on a neighborhood of $\bar{x}(1)$,

$$f_x(\bar{x}(t), \cdot) \text{ is continuous on a neighborhood of } \bar{u}(t) \text{ for a.e. } t$$

and for some $\varepsilon > 0$, $c \geq 0$ and for a.e. $t \in [0,1]$, $f_x(\cdot, \bar{u}(t))$ is Lipschitz on $B(\bar{x}(t), \varepsilon)$ with Lipschitz constant c.

Relaxed trajectories of control system are solutions of

$$x'(t) \in \mathbf{co}\, f(x(t), U) \text{ a.e. in } [0,1].$$

Set $F(x) = \mathbf{co}\, f(x, U)$. The Inward Pointing Condition :

$$F(x) \cap \mathrm{Int}\, C_K(x) \neq \emptyset \quad \forall x \in \partial K \qquad (IPC)$$

Theorem

If (IPC) holds true and $(\bar{x}, \bar{u})$ is a strong local minimizer, then $\bar{x}$ is a strong local minimizer for the relaxed Mayer problem

$$\textit{minimize } \varphi(x(1))$$

$$x'(t) \in F(x(t)) \text{ a.e. in } [0,1], \;\; x(0) \in K_0, \;\; x(t) \in K \;\; \forall\, t \in [0,1].$$

HF+ F. Rampazzo, JDE 2000; HF + M. Mazzola, NoDEA 2013, measurably t-dependent f and a different (IPC)

First Order Approximation of Control Systems

Denote by $\mathcal{V}^{(1)}(\bar{x},\bar{u})$ the set of solutions y of the following "linearized" along $(\bar{x},\bar{u})$ system :

$$\begin{cases} y'(t) = f_x(\bar{x}(t),\bar{u}(t))y(t) + v(t), \quad v(t) \in F(\bar{x}(t)) - \bar{x}'(t) \quad \text{a.e.} \\ y(0) \in T^{\flat}_{K_0}(\bar{x}(0)) \\ y(t) \in C_K(\bar{x}(t)) \text{ for all } t \in [0,1]. \end{cases}$$

$\forall z \in \mathbb{R}^n$ we have

$$f_x(\bar{x}(t),\bar{u}(t))z + F(\bar{x}(t)) - \dot{\bar{x}}(t) \subset dF(\bar{x}(t),\dot{\bar{x}}(t))z$$

Second Order Approximation of Control Systems

Let $\mathcal{K} := \{x \in \mathcal{C}([0,1];\mathbb{R}^n) \mid x(t) \in K \quad \forall t \in [0,1]\}$.

For $y \in \mathcal{V}^{(1)}(\bar{x},\bar{u})$ we introduce the sets $\mathcal{E}(y;t), \mathcal{F}(y;t) \subset \mathbb{R}^n$ defined for almost all $t \in [0,1]$ by

$$\mathcal{E}(y;t) := T_{dF(\bar{x}(t),\bar{x}'(t))y(t)}(y'(t))$$

$$\mathcal{F}(y;t) := \{v \,|\, \exists\, u_h \in U,\ \lim_{h\to 0+} u_h = \bar{u}(t) \ \text{ such that } \ \forall\, h>0$$

$$\bar{x}'(t) + hy'(t) + h^2 v = f(\bar{x}(t)+hy(t),u_h) + o(h^2)\}$$

Second Order Approximation of Control Systems

Consider the set $\mathcal{V}^{(2)}(\bar{x},\bar{u},y)$ of all $w \in W^{1,1}([0,1];\mathbb{R}^n)$ satisfying

$$\begin{cases} w'(t) \in f_x(\bar{x}(t),\bar{u}(t))w(t) + \mathcal{F}(y;t) + \mathcal{E}(y;t) \quad \text{for a.e. } t \in [0,1] \\ w(0) \in T_{K_0}^{\flat(2)}(\bar{x}(0),y(0)) \\ w \in T_{\mathcal{K}}^{\flat(2)}(\bar{x},y). \end{cases}$$

We abbreviate $[t] := (\bar{x}(t),\bar{x}'(t),y(t),y'(t))$

Proposition

Let $y \in \mathcal{V}^{(1)}(\bar{x},\bar{u})$. Then for almost all $t \in [0,1]$ and all $z \in \mathbb{R}^n$,

$$f_x(\bar{x}(t),\bar{u}(t))z + \mathcal{F}(y;t) + \mathcal{E}(y;t) \subset d^2F[t]z.$$

Maximum Principle

Theorem (well known maximum principle)

There exist $p \in W^{1,1}([0,1];\mathbb{R}^n)$, $\lambda \in \{0,1\}$, *a non-negative Borel measure* μ *on* $[0,1]$ *and a Borel measurable selection*

$$\nu(t) \in N_K(\bar{x}(t)) \cap B \quad \mu\text{-a.e. in } [0,1] \quad \Rightarrow \quad \textit{Complementarity}$$

such that for $\psi : [0,1] \to \mathbb{R}^n$ *defined by* $\psi(t) := \int_{[0,t]} \nu(s) d\mu(s)$ *if* $t \in]0,1]$ *and* $\psi(0) = 0$ *we have* $(p, \psi, \lambda) \neq 0$,

$$-p'(t) = f_x(\bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) \quad \text{a.e.}$$

$$p(0) \in N_{K_0}(\bar{x}(0)), \quad -p(1) = \lambda \nabla \varphi(\bar{x}(1)) + \psi(1)$$

$$\langle p(t) + \psi(t), f(\bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U} \langle p(t) + \psi(t), f(\bar{x}(t), u) \rangle \text{a.e.}$$

Second Order Maximum Principle

Theorem

Let $y \in \mathcal{V}^{(1)}(\bar{x}, \bar{u})$, $\langle \nabla \varphi(\bar{x}(1)), y(1) \rangle = 0$ *and* $\mathcal{V}^{(2)}(\bar{x}, \bar{u}, y) \neq \emptyset$. *If (IPC) holds true, then* $\exists\ (\lambda, p, \psi)$ *as in the* ***maximum principle*** *such that in addition* $\langle p(t) + \psi(t), f_x(\bar{x}(t), \bar{u}(t)) y(t) \rangle\ =$

$$\max\{\langle p(t) + \psi(t), \sum_{i=1}^k \lambda_i f_x(\bar{x}(t), u_i) y(t) \rangle \mid$$
$$\lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1,\ u_i \in U, f(\bar{x}(t), \bar{u}(t)) = \sum_{i=1}^k \lambda_i f(\bar{x}(t), u_i)\}$$

for a.e. $t \in [0,1]$.
In particular, for almost all $t \in [0,1]$ *and for every* $u \in U$ *satisfying* $f(\bar{x}(t), \bar{u}(t)) = f(\bar{x}(t), u)$ *we have*

$$\langle p(t) + \psi(t), f_x(\bar{x}(t), \bar{u}(t)) y(t) \rangle\ \geq\ \langle p(t) + \psi(t), f_x(\bar{x}(t), u) y(t) \rangle$$

Normality of the Maximum Principle

$\lambda = 1$, if $C_{K_0}(\bar{x}(0)) \cap \operatorname{Int} C_K(\bar{x}(0))) \neq \emptyset$ and a pointwise inward pointing condition holds true : for a.e. $t \in [0,1]$

$$\mathcal{E}(y;t) \cap \operatorname{Int} C_K(\bar{x}(t)) \neq \emptyset$$

Conclusions and Future Work

1. To get necessary optimality conditions it is enough to know subsets of tangents. "Linearizations" provide such subsets.

2. Inward pointing conditions imposed on state constraints allow to relax control systems and to show that solutions of "linearized systems" are in tangents.

3. The infinite dimensional case is under investigation and relaxation theorems with state constraints are already proved.

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