On coupling of complementarity with friction in contact shape optimization

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Based on two joint papers with the coauthors
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Goal: To find, for and elastic body, an admissible shape of a part of its boundary such that, after applying the body forces and given surface tractions, the variables corresponding to the shape, the displacement and the multiplier associated with the Signorini condition will create a local minimizer of a given objective.

We will be dealing with 2 friction models, namely

A Coulomb friction with a fixed friction coefficient (3D);

B Coulomb friction with a solution-dependent friction ceofficient (2D).

Outline:

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(i) Backgroung from variational analysis;

(ii) Algebraic setting of the problems;

(iii) Implicit programming approach (ImP);

(iv) Computation of limiting coderivatives of the set-valued parts of the respective GEs;

(v) Sensitivity analysis;

(vi) Numerical results.

Ad (i): Background from variational analysis

Consider a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$.

$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{\tau \downarrow 0} \frac{A - \bar{x}}{\tau}$ is the *contingent (Bouligand)* cone to A at $\bar{x}$.

$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^0$ is the *regular (Fréchet) normal* cone to A at $\bar{x}$.

The *limiting (Mordukhovich) normal* cone to A at $\bar{x}$ is defined by

$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}(x) = \{x^* \in \mathbb{R}^n | \exists\, x_k \xrightarrow{A} \bar{x}, x_k^* \to x^* \text{ such that } x_k^* \in \widehat{N}_A(x_k) \forall k\}.$$

Background from variational analysis

Now consider a closed-graph multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$.
The multifunction $\widehat{D}^* \Phi(\bar{x}, \bar{y})[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ defined by

$$\widehat{D}^* \Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in \widehat{N}_{\operatorname{gph}\Phi}(\bar{x}, \bar{y})\}$$

is the *regular (Fréchet) coderivative* of Φ at $(\bar{x}, \bar{y})$.
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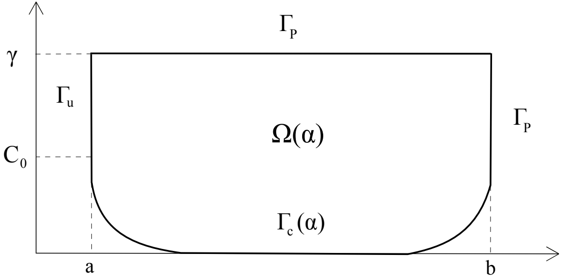
is the *limiting (Mordukhovich) coderivative* of Φ at $(\bar{x}, \bar{y})$.



Elastic body and its contact boundary:

$$\Omega(\alpha) := \{(x_1, x_2) \mid a < x_1 < b,\ \alpha(x_1) < x_2 < \gamma\}, \quad \Gamma_c(\alpha) := \operatorname{Gr} \alpha,$$

where

$$\alpha \in U_{ad} := \left\{ \alpha \in C^{0,1}([a,b]) \;\middle|\; \begin{array}{l} 0 \leq \alpha \leq C_0,\ \|\alpha'\|_{L^\infty} \leq C_1 \\ C_2 \leq \operatorname{meas} \Omega(\alpha) \leq C_3 \end{array} \right\}.$$

Ad (ii): In both considered models the discretized state problems attain the form

$$\begin{array}{ll} \text{minimize} & J(\alpha, y) \\ \text{subject to} & \\ & 0 \in F(\alpha, y) + Q(y) \\ & \alpha \in \omega \subset \mathbb{R}^l, \end{array} \tag{1}$$

where l denotes the number of nodes on the contact boundary, the state variable y amounts to (u_t, u_ν, λ), where u_t, u_ν stand for the tangential and normal displacements, respectively, λ is the multiplier associated with the Signorini condition and ω is the discretized set of admissible shapes. Concretely, the GE from (1) takes the form

$$\begin{aligned} 0 &\in A_{tt}(\alpha)u_t + A_{t\nu}(\alpha)u_\nu - L_t(\alpha) + \widetilde{Q}(u_t, u_\nu, \lambda) \\ 0 &= A_{\nu t}(\alpha)u_t + A_{\nu\nu}(\alpha)u_\nu - L_\nu(\alpha) - \lambda \\ 0 &\in u_\nu + \alpha + N_{\mathbb{R}^l_+}(\lambda), \end{aligned} \tag{2}$$

where the blocks $A_{tt}, A_{t\nu}, A_{\nu t}$ and $A_{\nu\nu}$ correspond to the stiffness matrix and vectors L_t, L_ν correspond to the body forces and surface tractions. All of them depend on α in a continuously differentiable way. Further,

$$\omega = \{\alpha \in \mathbb{R}^l |$$

$$0 \le \alpha^i \le C_0, i = 1, 2, \ldots, l, |\alpha_{i+1} - \alpha_i| \le C_1 h, i = 1, 2, \ldots, l-1, C_2 \le \mathrm{meas}\, \Omega(\alpha) \le C_3\},$$

$$(\widetilde{Q}(u_t, u_\nu, \lambda))^i = \mathcal{F}\lambda^i \partial \|u_t^i\|_2, i = 1, 2, \ldots l \quad \text{(in model A)}$$

and

$$(\widetilde{Q}(u_t, u_\nu, \lambda))^i = \mathcal{F}(|u_t^i|)\lambda^i \partial |u_t^i|, i = 1, 2, \ldots l \quad \text{(in model B).}$$

It is well-known that under suitable assumptions concerning $\mathcal{F}$ (in A) or $\mathcal{F}(\cdot)$ (in B) the *solution* map

$$S(\alpha) := \{y | 0 \in F(\alpha, y) + Q(y)$$

is single-valued and Lipschitz. Moreover, for $l \to \infty$ the solutions of (1) (which exist due to the boundedness of ω) converge to a solution of the original continuous problem in the appropriate function spaces.

Ad (iii): Define $\Theta(\alpha) := J(\alpha, S(\alpha))$. Then (1) amounts to the optimization problem

$$\begin{array}{ll} \text{mimimize} & \Theta(\alpha) \\ \text{subject to} & \\ & \alpha \in \omega. \end{array} \tag{3}$$

Assume that J is continuously differentiable. Then Θ is locally Lipschitz and (3) can be numerically solved, e.g., by a bundle method of nonsmooth optimization. To this aim we must be able to compute for each $\alpha \in \omega$ the value $\Theta(\alpha)$ and a vector $\xi \in \bar{\partial}\Theta(u)$. The latter will be done by using the relationship

$$\bar{\partial}\Theta(u) = \mathrm{conv}\partial\Theta(u) \supset \partial\Theta(u) = \{\xi | \xi \in \nabla_u J(\alpha, y) + D^* S(u)(\nabla_y J(\alpha, y))\},$$

where $y = S(\alpha)$. Furthermore, for a given vector a, one has

$$\begin{aligned} D^* S(u)(a) \subset \{(\nabla_\alpha F(\alpha, y))^T b | 0 \in a + \\ (\nabla_y F(\alpha, y))^T b + D^* Q(y, -F(\alpha, y))(b)\}. \end{aligned}$$

The above inclusion becomes equality provided either

(i) $\nabla_\alpha F(\alpha, y)$ is surjective, or

(ii) $\mathrm{gph} Q$ is (normally) regular at $(y, -F(\alpha, y))$.

Ad(iv): In the computation of ξ the most difficult part consists in the computation of the limiting coderivative of Q. To facilitate this step we regroup GE (2) in such a way that

$$Q(y) = \sum_{i=1}^{l} \bar{Q}(y^i),$$

with the multifunctions

$$\bar{Q}(y^i) = \begin{bmatrix} \mathcal{F}\lambda^i \partial \|u^i_t\|_2 \\ 0 \\ N_{\mathbb{R}_+}(\lambda^i) \end{bmatrix} \quad \text{and} \quad \bar{Q}(y^i) = \begin{bmatrix} \mathcal{F}(|u^i_t|)\lambda^i \partial |u^i_t| \\ 0 \\ N_{\mathbb{R}_+}(\lambda^i) \end{bmatrix} \tag{4}$$

in the cases A and B, respectively. It follows that for $u \in Q(y)$ one has

$$d \in D^* Q(y,u)(c) \Leftrightarrow d^i \in D^* \bar{Q}(y_i, u_i)(c_i) \ \forall \ i.$$

So, everything boils down to analysis of multifunctions $\bar{Q}$ which are associated to single nodes lying on the contact part of the boundary.

Theorem 1.

Consider the multifunction $\Psi[\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^o \rightrightarrows \mathbb{R}^p \times \mathbb{R}^s]$ defined by

$$F(x, y, z) = \begin{bmatrix} G(x, y) \\ H(y, z) \end{bmatrix},$$

where $G[\mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p]$ and $H[\mathbb{R}^m \times \mathbb{R}^o \rightrightarrows \mathbb{R}^s]$ are closed-graph multifunctions. Assume that $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}) \in \mathrm{gph} F$ and the qualification condition

$$\left.\begin{array}{l} \begin{bmatrix} 0 \\ w_2 \end{bmatrix} \in D^* G(\bar{x}, \bar{y}, \bar{u})(0), \\ \begin{bmatrix} -w_2 \\ 0 \end{bmatrix} \in D^* H(\bar{y}, \bar{z}, \bar{v})(0) \end{array}\right\} \Rightarrow w_2 = 0 \tag{5}$$

holds true. Then for any $d_1^*, d_2^* \in \mathbb{R}^p \times \mathbb{R}^s$ one has

$$\begin{aligned} D^* F(\bar{x}, \bar{y}, \bar{z})(d_1^*, d_2^*) \subset \{(w_1, w_2 + w_3, w_4) | (w_1, w_2) &\in D^* G(\bar{x}, \bar{y}, \bar{u})(d_1^*), \\ (w_3, w_4) &\in D^* H(\bar{y}, \bar{z}, \bar{v})(d_2^*)\}. \end{aligned} \tag{6}$$

Remark

Qualification condition (5) can be weakened on the basis of the calmness of respective perturbation maps.

Theorem 2.

Inclusion (6) becomes equality provided

(i) G is single-valued and continuously differentiable near $(\bar{x}, \bar{y})$. In this case condition (5) is automatically fulfilled;

(i) In addition to the assumptions of Theorem 1, for each sequence $y^{(i)} \to \bar{y}$ and each $\eta \in D^* G(\bar{x}, \bar{y})(d_1^*) \exists$ sequences $x^{(i)} \to \bar{x}, u^{(i)} \to \bar{u}, d_1^{*(i)} \to d_1^*$ such that

$$(x^{(i)}, y^{(i)}, u^{(i)}) \in \operatorname{gph} G$$
$$\eta \in \operatorname*{Lim\,sup}_{i \to \infty} \widehat{D}^* G(x^{(i)}, y^{(i)}, u^{(i)})(d_1^{*(i)}).$$

In verification of the assumptions in (ii) one may use the following statement.

Lemma.

Assume that $G(x, y) = f(x)g(y)$, where $f[\mathbb{R}^n \to \mathbb{R}]$ and $g[\mathbb{R}^m \to \mathbb{R}^p]$ are Lipschitz near $\bar{x}$ and $\bar{y}$, respectively. Then for any (x, y) close to $(\bar{x}, \bar{y})$ and any d^* one has

$$\widehat{D}^* G(x, y)(d^*) = \begin{bmatrix} \widehat{D}^* f(x)(\langle g(y), d^* \rangle) \\ \widehat{D}^* g(y)(f(x)d^*) \end{bmatrix}.$$

The above assertion enables us to prove that such mapping G fulfills the assumptions in (ii) whenever g is continuously differetiable near $\bar{y}$.

Denote by Φ the friction terms in the definitions of $\bar{Q}$ in (4), i.e.,

$$\Phi(y^i) = \mathcal{F}\lambda^i \partial \|u_t^i\|_2 \quad \text{(in the case A)}$$
$$\Phi(y^i) = \mathcal{F}(|u_t^i|)\lambda^i \partial |u_t^i| \quad \text{(in the case B)}.$$

Let $\bar{z} \in \mathrm{gph}\Phi$ and $\exists$ neighborhood O of $\bar{z}$ such that

$$\mathrm{gph}\Phi \cap O = \Gamma \cup \Xi \cup \Lambda,$$

where Γ and Ξ are open in the relative topology of $\mathrm{gph}\Phi$ and $\bar{z} \in \Lambda \subset \mathrm{bd}\,\Gamma \cap \mathrm{bd}\,\Xi$. Then, by the definition,

$$\begin{aligned} N_{\mathrm{gph}\Phi}(\bar{z}) = & \operatorname*{Lim\,sup}_{z \xrightarrow{\Gamma} \bar{z}} \widehat{N}_\Gamma(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Xi} \bar{z}} \widehat{N}_\Xi(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Lambda} \bar{z}} \widehat{N}_{\mathrm{gph}\Phi}(z) = \\ & \operatorname*{Lim\,sup}_{z \xrightarrow{\Gamma} \bar{z}} \widehat{N}_\Gamma(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Xi} \bar{z}} \widehat{N}_\Xi(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Lambda} \bar{z}} (T_\Gamma(z) \cup T_\Xi(z))^\circ = \\ & \operatorname*{Lim\,sup}_{z \xrightarrow{\Gamma} \bar{z}} \widehat{N}_\Gamma(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Xi} \bar{z}} \widehat{N}_\Xi(z) \cup \operatorname*{Lim\,sup}_{z \xrightarrow{\Lambda} \bar{z}} (\widehat{N}_\Gamma(z) \cap \widehat{N}_\Xi(z)). \end{aligned}$$

Computation of $D^*\bar{Q}$ is based on the following partition of $\mathrm{gph}\bar{Q}$. Fix i and consider $\bar{b} \in \bar{Q}(\bar{a})$, where $\bar{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) = (\bar{u}^i_t, \bar{u}^i_\nu, \bar{\lambda}^i) \in \mathbb{R}^4$. To simplify the notation, put $\bar{a}_{12} = (\bar{a}_1, \bar{a}_2)$ and $\bar{b}_{12} = (\bar{b}_1, \bar{b}_2)$.

	no contact: $a_4 = 0,\ b_4 < 0$	weak contact: $a_4 = 0,\ b_4 = 0$	strong contact: $a_4 > 0,\ b_4 = 0$
sliding: $a_{12} \neq 0$, $b_{12} = \mathcal{F} a_4 a_{12} \|a_{12}\|^{-1}$	L	M_2	M_1
weak sticking: $a_{12} = 0$, $\|b_{12}\| = \mathcal{F} a_4$		M_4	M_3^-
strong sticking: $a_{12} = 0$, $\|b_{12}\| < \mathcal{F} a_4$	$\times\times\times$	$\times\times\times$	M_3^+

Table: Possible positions of $(\bar{a}, \bar{b})$ in $\mathrm{gph}\bar{Q}$

In the case of L, M_1 and M_3^+, $D^*\bar{Q}(\bar{a},\bar{b})(b^*)$ can be computed easily by standard calculus rules.

Proposition 1.

Let $(\bar{a},\bar{b}) \in M_2 \subset \mathbb{R}^4 \times \mathbb{R}^4$. Then for any $b^* \in \mathbb{R}^4$ one has

$$D^*\bar{Q}(\bar{a},\bar{b})(b^*) =$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\bar{a}_1}{\|a_{12}\|} b_1^* + \dfrac{\bar{a}_2}{\|\bar{a}_{12}\|} b_2^* + w \end{bmatrix} \middle| \, w \in \begin{cases} \mathbb{R} & \text{if } b_4^* = 0 \\ \mathbb{R}_- & \text{if } b_4^* < 0 \\ 0 & \text{otherwise} \end{cases} \right\}.$$

Proposition 2.

Let $(\bar{a}, \bar{b}) \in M_3^-$ and $\bar{w} = \frac{\bar{b}_{12}}{\mathcal{F}\bar{a}_4}$. Then one has, with $b_{12}^* = (b_1^*, b_2^*)$, that for any b^*

$$D^*\bar{Q}(\bar{a}, \bar{b})(b^*) =$$

$$\begin{cases} \{a^* \in \mathbb{R}^4 | a_{12}^* = 0, a_3^* = 0, a_4^* = \mathcal{F}\alpha\} \text{ if } b_{12}^* = \alpha\bar{w}, \alpha \geq 0, \\ \{a^* \in \mathbb{R}^4 | \langle a_{12}^*, \bar{w}\rangle \leq 0, a_3^* = 0, a_4^* = \mathcal{F}\alpha\} \text{ if } b_{12}^* = \alpha\bar{w}, \alpha < 0, \\ \mathbb{R}_2 \times \{0\}_2 \text{ if } b_{12}^* = 0 \\ \emptyset \text{ otherwise.} \end{cases}$$

Sensitivity analysis in the case B

Computation of $D^*\bar{Q}$ is based on the following partition of $\mathrm{gph}\bar{Q}$: [1] Fix i and consider $\bar{b} \in \Phi(\bar{a})$, where $\bar{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) = (\bar{u}^i_t, \bar{u}^i_\nu, \bar{\lambda}^i) \in \mathbb{R}^3$.

	no contact: $a_3 = 0,\ b_3 < 0$	weak contact: $a_3 = 0,\ b_3 = 0$	strong contact: $a_3 > 0,\ b_3 = 0$
sliding: $a_1 \neq 0$, $b_1 = \mathrm{sgn}(a_1)\mathcal{F}(a_1)a_3$	L	M_2	M_1
weak sticking: $a_1 = 0$, $\lvert b_1\rvert = \mathcal{F}(0)a_3$		M_4	M_3^-
strong sticking: $a_1 = 0$, $\lvert b_1\rvert < \mathcal{F}(0)a_3$	$\times\times\times$	$\times\times\times$	M_3^+

Table: Possible positions of $(\bar{a}, \bar{b})$ in $\mathrm{gph}\bar{Q}$

[1] In the sequel we shall work with the even extension of $\mathcal{F}$ into $\mathbb{R}$, i.e. $\mathcal{F}(x) = \mathcal{F}(-x)\ \forall x < 0$.

Sensitivity analysis in the case B

In the case of L, M_1 and M_3^+, $D^*\bar{Q}(\bar{a},\bar{b})(b^*)$ can be computed easily by standard calculus rules.

Proposition 3.

Let $(a,b) \in M_2 \subset \mathbb{R}^3 \times \mathbb{R}^3$. Then for any $b^* \in \mathbb{R}^3$ one has

$$D^*\bar{Q}(a,b)(b^*) = \begin{cases} \{0\} \times \{0\} \times \mathbb{R} & \text{if } b_3^* = 0, \\ \{0\} \times \{0\} \times (-\infty, \operatorname{sgn}(a_1)\mathcal{F}(a_1)b_1^*] & \text{if } b_3^* < 0, \\ \{0\} \times \{0\} \times \{\operatorname{sgn}(a_1)\mathcal{F}(a_1)b_1^*\} & \text{if } b_3^* > 0. \end{cases}$$

Proposition 4.

Let $(a,b) \in M_3^-$, and assume that $\mathcal{F}$ is *weakly semismooth* at 0. Then for any b^*

$$D^*\bar{Q}(a,b)(b^*) =$$

$$\begin{cases} \mathbb{R} \times \{0\} \times \{0\} & \text{if } b_1^* = 0, \\ (\mathcal{F}'_+(0)\bar{a}_3 b_1^* + \operatorname{sgn}(\bar{b}_1)\mathbb{R}_+) \times \{0\} \times \{\operatorname{sgn}(\bar{b}_1)\mathcal{F}(0)b_1^*\} & \text{if } b_1^* \operatorname{sgn}(\bar{b}_1) < 0, \\ \{\mathcal{F}'_+(0)\bar{a}_3 b_1^*\} \times \{0\} \times \{\operatorname{sgn}(\bar{b}_1)\mathcal{F}(0)b_1^*\} & \text{otherwise.} \end{cases}$$

We consider the problem data $a = 0, b = 2, \gamma = 1, l = 81$, and assume that the admissible control set is given by the constants $C_0 = 0.75, C_1 = 3, C_2 = 1.8, C_3 = 2$. The objective is $\mathcal{J}(\alpha, y) = \|\lambda\|_6^6$. Further we suppose that the friction coefficient $\mathcal{F}$ is defined by

$$\mathcal{F}(t) = 0.25 \cdot \frac{1}{t^2 + 1} \quad \forall t \in \mathbb{R}_+,$$

The state problem is discretized by isoparametric quadrilateral elements of Lagrange type and solved with MatSol, developed at the TU Ostrava. Problem (3) is minimized by the Bundle Trust method.

Example 1

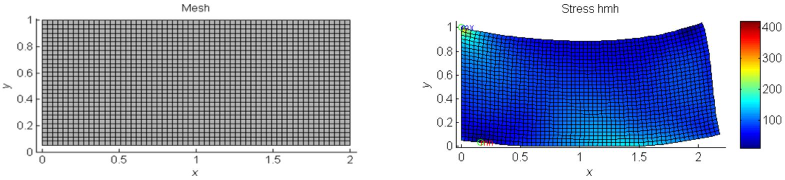


Figure: Initial design: unloaded and deformed body; $\mathcal{J}(\boldsymbol{\alpha}_0) = 2.1159 \cdot 10^{11}$

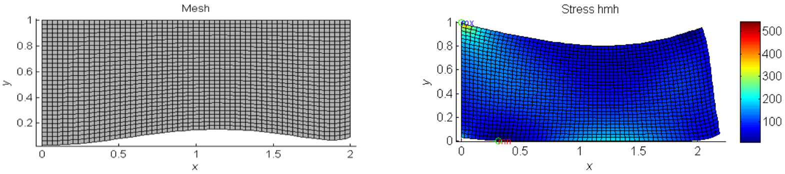


Figure: Optimal design: unloaded and deformed body; $\mathcal{J}(\boldsymbol{\alpha}_{\text{opt}}) = 2.6513 \cdot 10^{8}$

Example 1

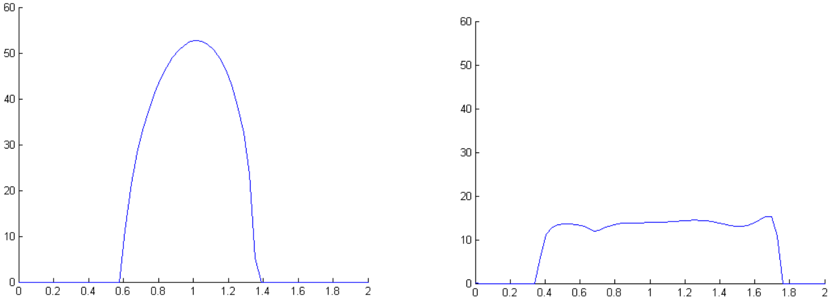


Figure: Normal stress for initial (left) and optimal (right) design.

Conclusion:

Both considered problems belong to the class of MPECs solvable by ImP. To compute the needed subgradient information, one has to deal with a complicated set-valued mapping coupled with complementarity constraints. To its analysis we applied the generalized calculus of B. Mordukhovich which does contain suitable rules for this kind of computations. Nevertheless, in these rules one mostly has inclusions, which contradicts our intention to compute the mentioned subgradient information as exact as possible. To overcome this hurdle, we have exploited the available special structure

(1) to obtain "strenthened" variants of one from these rules which are valid as equalities,

(2) to compute some limiting coderivatives "almost from the scratch".

The results of the computed test examples correspond to the expertize of the engineers and demonstrate the efficiency of the proposed method.

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