Generalized Nash-Equilibrium Problems in Banach Space

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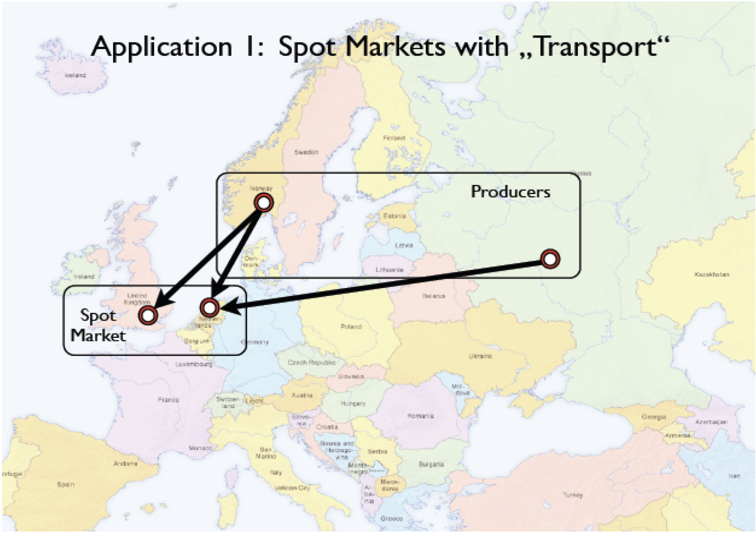
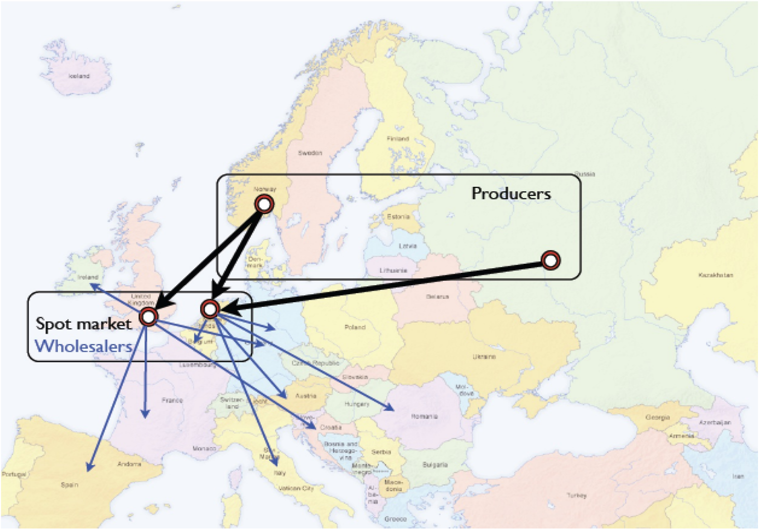
START PROJECT INTERFACES/FREE BOUNDARIES

FWF Der Wissenschaftsfonds.


Application I: Spot Markets with „Transport“
Producers
Spot
Market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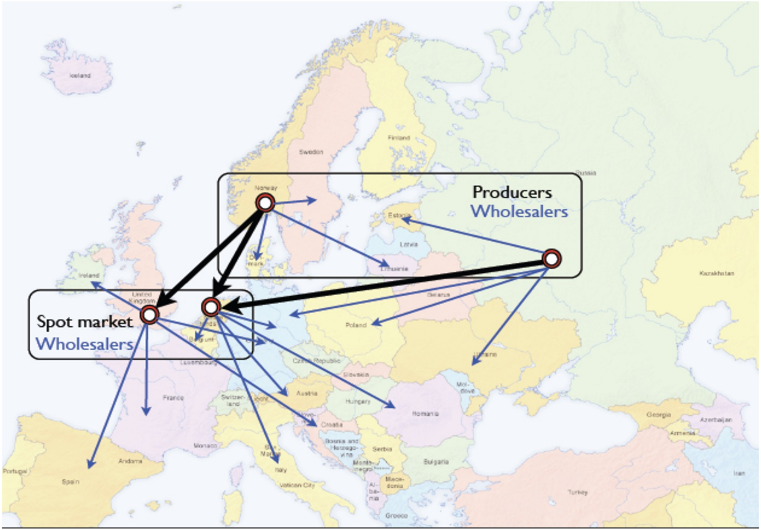
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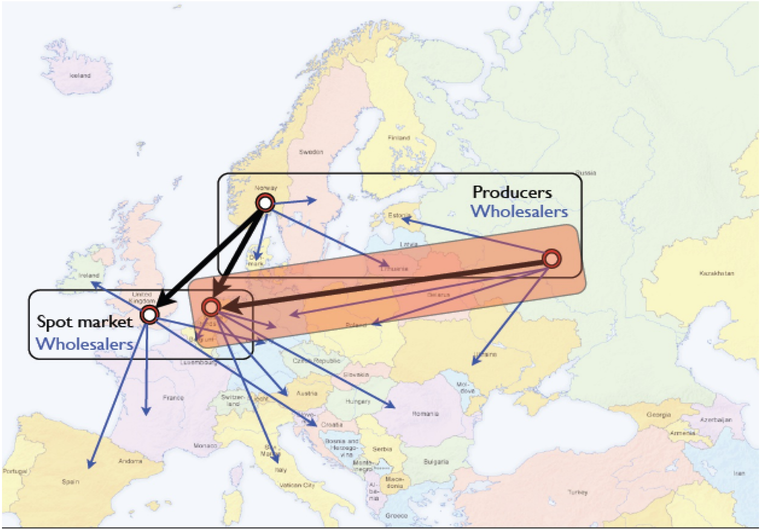
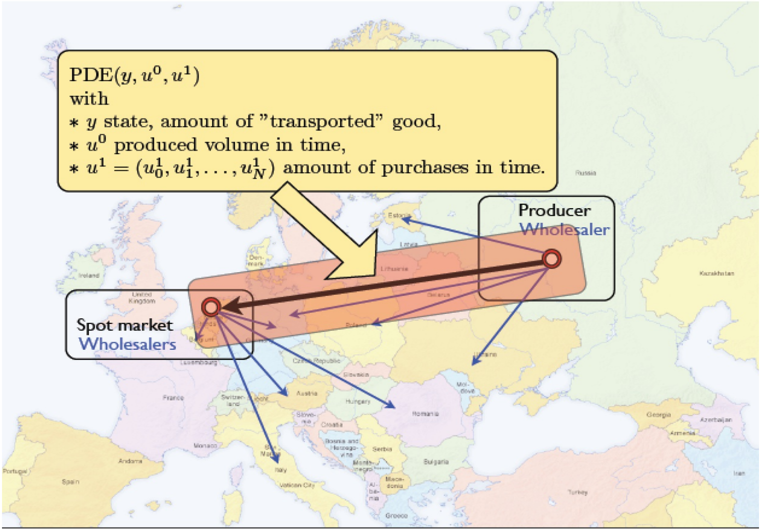

Producers
Wholesalers
Spot market
Wholesalers

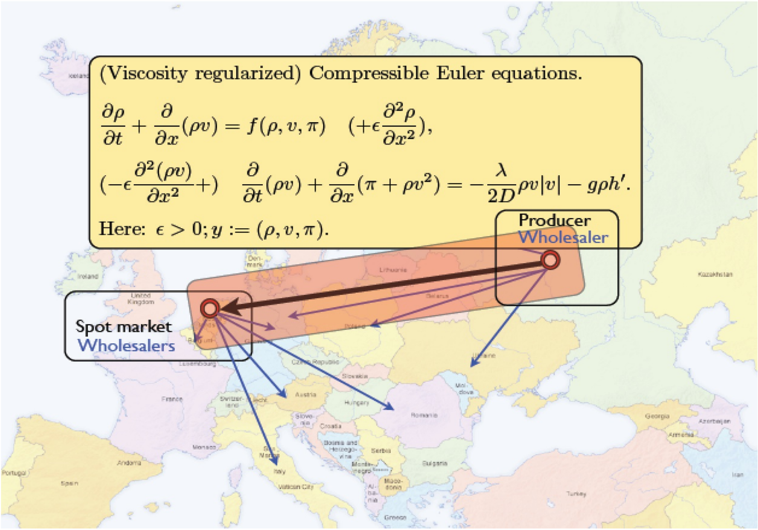

PDE(y, u^0, u^1)
with
* y state, amount of "transported" good,
* u^0 produced volume in time,
* $u^1 = (u_0^1, u_1^1, \ldots, u_N^1)$ amount of purchases in time.
Producer
Wholesaler
Spot market
Wholesalers

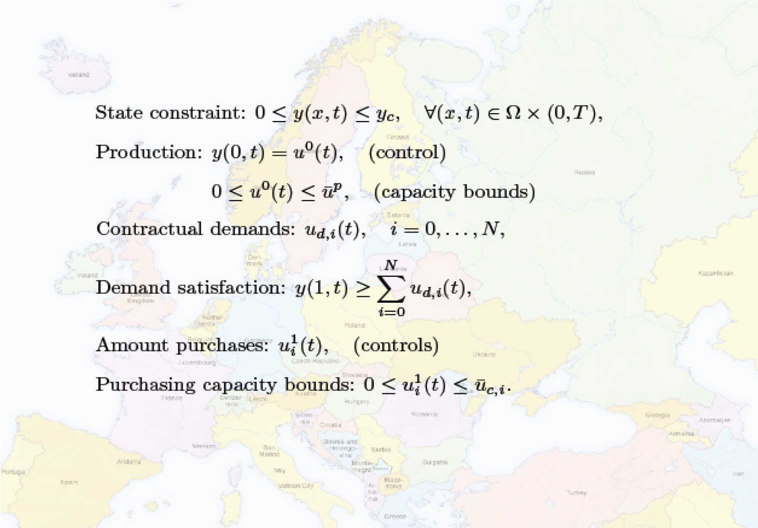
(Viscosity regularized) Compressible Euler equations.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = f(\rho, v, \pi) \quad (+\epsilon \frac{\partial^2 \rho}{\partial x^2}),$$

$$(-\epsilon \frac{\partial^2 (\rho v)}{\partial x^2} +) \quad \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\pi + \rho v^2) = -\frac{\lambda}{2D}\rho v|v| - g\rho h'.$$

Here: $\epsilon > 0; y := (\rho, v, \pi)$.





State constraint: $0 \le y(x,t) \le y_c, \quad \forall (x,t) \in \Omega \times (0,T),$

Production: $y(0,t) = u^0(t), \quad$ (control)

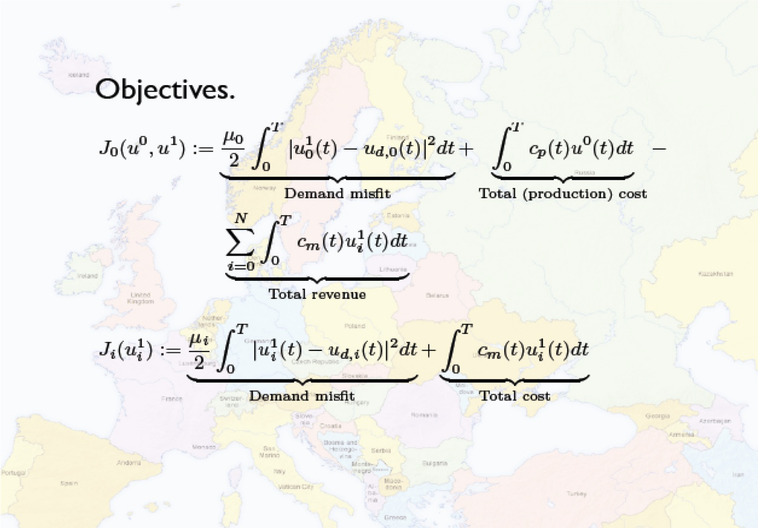
$0 \le u^0(t) \le \bar{u}^p, \quad$ (capacity bounds)

Contractual demands: $u_{d,i}(t), \quad i = 0, \dots, N,$

Demand satisfaction: $y(1,t) \ge \sum_{i=0}^{N} u_{d,i}(t),$

Amount purchases: $u_i^1(t), \quad$ (controls)

Purchasing capacity bounds: $0 \le u_i^1(t) \le \bar{u}_{c,i}.$



Objectives.

$$J_0(u^0, u^1) := \underbrace{\frac{\mu_0}{2} \int_0^T |u_0^1(t) - u_{d,0}(t)|^2 dt}_{\text{Demand misfit}} + \underbrace{\int_0^T c_p(t) u^0(t) dt}_{\text{Total (production) cost}} - \underbrace{\sum_{i=0}^{N} \int_0^T c_m(t) u_i^1(t) dt}_{\text{Total revenue}}$$

$$J_i(u_i^1) := \underbrace{\frac{\mu_i}{2} \int_0^T |u_i^1(t) - u_{d,i}(t)|^2 dt}_{\text{Demand misfit}} + \underbrace{\int_0^T c_m(t) u_i^1(t) dt}_{\text{Total cost}}$$

General Nash Equilibrium Problem - GNEP

Producer's problem.

$$\begin{aligned} &\text{minimize } J_0(u^0, u^1) \text{ over } (u^0, u_0^1) \\ &\text{subject to } PDE(y, u^0, u^1) \; + \text{state constraints} \\ &\qquad 0 \le u_0^1 \le \bar{u}_{c,0}, \quad 0 \le u^0 \le \bar{u}_p. \end{aligned}$$

i-th wholesaler's problem (i=1,...,N).

$$\begin{aligned} &\text{minimize } J_i(u_i^1) \text{ over } u_i^1 \\ &\text{subject to } PDE(y, u^0, u^1) + \text{state constraints} \\ &\qquad 0 \le u_i^1 \le \bar{u}_{c,i}. \end{aligned}$$

Optimality vs. equilibrium characterization?
Numerical solution?

Agenda

- Abstract GNEPs in Banach space.
- Existence of solutions and equilibrium conditions.
- Nikaido-Isoda based path-following.
- Numerical results.

A GNEP with Abstract Constraints

Aim of player $i = 1, \ldots, N$: Given u_{-i}, choose (u_i, y) which solves:

$$\begin{aligned} &\min J_i^1(y) + J_i^2(u_i) \text{ over } (u_i, y) \in U_i \times Y \\ &\text{subject to (s.t.)} \\ &\qquad Ay = B(u_i, u_{-i}), \\ &\qquad u_i \in U_{\mathrm{ad}}^i, \\ &\qquad y \in K. \end{aligned} \tag{P$_i$}$$

- Point (u, y) is GNE point, if no player can decrease their objective by changing unilaterally (u_i, y) to any other feasible point.

Data assumptions

- U_i $(i = 1, \ldots, N)$ reflexive sep. Banach spaces, $U := \Pi_{i=1}^N U_i$.
- Y, W reflexive B.-spaces, X B.-space.
- $Y \hookrightarrow X$ is continuous.
- If $M \subset X^*$ is bounded, then M weak-$*$ relatively compact in X^*.

Data Assumptions

- $A : Y \to W$ linear isomorphism.
- $B \in \mathcal{L}(U, W)$; $Bu = \sum_{i=1}^{m} B_i u_i$ with $B_i = B(\cdot, 0_{-i})$ with $B_i \in \mathcal{L}(U_i, W)$.
- $A^{-1}B : U \to X$ is compact.
- $K \subset X$ nonempty, closed, and convex set.
- Norm topology on X: $\exists\, x \in K$ and $\varepsilon > 0$: $\mathbb{B}_\varepsilon(x) \subset K$.
- $\exists\, u \in U_{\mathrm{ad}}$ with $A^{-1}Bu \in K$.
- $U_{\mathrm{ad}}^i \subset U_i$ nonempty, bounded, closed, and convex; and $U_{\mathrm{ad}} := \Pi_{i=1}^N U_{\mathrm{ad}}^i$.
- $J_i^1 : Y \to \mathbb{R}$ convex and completely continuous, and $J_i^2 : U_i \to \mathbb{R}$ strictly convex and continuous. In particular, if $v_k \overset{Y}{\rightharpoonup} v$, then $J_i^1(v_k) \to J_i^1(v)$.

Reduced form using solution operator $S : U \to Y$, $Su := A^{-1}(Bu)$:

$$\min \mathcal{J}_i(u_i, u_{-i}) := J_i^1(S(u_i, u_{-i})) + J_i^2(u_i) \text{ over } u_i \in U_i$$

s.t.

$$u_i \in U_{\mathrm{ad}}^i, \quad S(u_i, u_{-i}) \in K.$$

For $u \in U$ strategy u_i feasible for ith problem,

given u_{-i}, for all $i = 1, \ldots, N$ if and only if $u \in C$, where

$$C := \{u \in U_{\mathrm{ad}} \mid Su \in K\}.$$

Since C convex, problem structure of so-called jointly convex GNEP.

Definition (Generalized Nash Equilibrium)

$\bar{u} \in C$ is Nash equilibrium provided

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \le \mathcal{J}_i(v_i, \bar{u}_{-i}), \;\; \forall v_i \in U_i : (v_i, \bar{u}_{-i}) \in C, \;\; \forall i = 1, \ldots, N.$$

Examples

Elliptic GNEP

Let $i = 1, \dots, N$.

$$\min \tfrac{1}{2}||K^i y - y_d^i||^2_{L^2(\Omega)} + \tfrac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \text{ over } (u_i, y) \in L^2(\Omega) \times W_0^{1,r}(\Omega)$$

s.t.

$$-\Delta y = \textstyle\sum_{i=1}^N \chi_{\Omega_i} u_i, \text{ in } W^{-1,r}(\Omega),$$

$$a_i \le u_i \le b_i, \text{a.e. in } \Omega,$$

$$y \le \psi, \text{ in } \Omega.$$

$\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, $r > \max(d, 2)$, $X = C(\overline{\Omega})$.

Examples

Parabolic GNEP

Let $y = Su$ for $u \in L^2(0, T; L^2(\Omega))^N$ solve the initial boundary value problem

$$\begin{aligned} y_t - \Delta y + c_0 y &= Bu, \text{ in } Q = \Omega \times (0, T), \\ y &= 0, \quad \text{on } \Sigma = \Gamma \times (0, T), \\ y(\cdot, 0) &= 0, \qquad \text{in } \Omega, \end{aligned}$$

GNEP ($i = 1, \ldots, N$):

$$\min \tfrac{1}{2} ||K^i S(u_i, u_{-i}) - y_d^i||^2_{L^2(Q)} + \tfrac{\alpha_i}{2} ||u_i||^2_{L^2(Q)} \text{ over } u_i \in L^2(Q)$$

s.t.

$$a_i \le u_i \le b_i, \text{ a.e. in } Q = \Omega \times (0, T), \quad S(u_i, u_{-i}) \le \psi, \text{ in } Q.$$

Complexity

Major complications:

- **Existence:** Classical (Ky Fan/Kakutani) theorems not directly applicable.
  ($\Rightarrow$ resort to weak topology).
- **Equilibria:** Generalized Nash vs. more tractable variational equilibria.
  ($\Rightarrow$ consider variational equilibria).

- **Numerical approach**: Handling of state constraints.
  ($\Rightarrow$ Moreau-Yosida regularization).
- **Update of path parameter**: Primal-dual path-following strategy.
  ($\Rightarrow$ Nikaido-Isoda function).

Literature

- **Finite dimensions.** Much work done for generalized Nash equilibrium problems (GNEPs); see works by Facchinei, Kanzow, Pang, Fukushima, and many more.
- **Infinite dimensions.** Significantly less in function spaces: Desideri; Ramos, Glowinski, & Periaux; Ramos & Roubicek; Kanzow & Borzì (multiobjective – monotone VI, but not GNEP!)

First-order conditions for GNEPs

Optimality Conditions for Generalized Nash Equilibria

- If a Nash equilibrium $\bar{u} \in U$ of (P) satisfies

$$\forall i = 1, \ldots, N, \exists u_i \in U_{\mathrm{ad}}^i : \quad \mathbb{B}_\varepsilon(0) \subset S(u_i, \bar{u}_{-i}) - K$$

  for some $\varepsilon > 0$, then $\exists\, \bar{y} \in Y$, $\bar{p} \in (W^*)^N$, $\bar{\lambda} \in U^*$ and $\bar{\mu} \in (X^*)^N$:

$$(\mathrm{OS}_i) \left\{ \begin{array}{rcl} \bar{y} & = & S\bar{u}, \\ -\bar{p}_i & \in & A^{-*}(\partial J_i^1(\bar{y}) + \bar{\mu}_i), \\ \bar{\lambda}_i & \in & \partial I_{U_{\mathrm{ad}}^i}(\bar{u}_i), \\ \bar{\mu}_i & \in & \partial I_K(\bar{y}), \\ 0 & \in & \partial J_i^2(\bar{u}_i) - B_i^* \bar{p}_i + \bar{\lambda}_i, \end{array} \right.$$

  is fulfilled for $i = 1, \ldots, N$. Coupled system is denoted by (OS).
- Conversely, if the tuple $(\bar{u}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{\mu}) \in U \times Y \times (W^*)^N \times U^* \times (X^*)^N$ satisfies the coupled system (OS), then $\bar{u}$ is a Nash equilibrium.

Nikaido-Isoda function

- Nikaido-Isoda function $\Psi : U \times U \to \mathbb{R}$ defined by

$$\Psi(u, v) := \sum_{i=1}^{N} \left[\mathcal{J}_i(u_i, u_{-i}) - \mathcal{J}_i(v_i, u_{-i})\right].$$

- In addition, define $V : C \to \mathbb{R}$ by

$$V(u) = \max_v \{\Psi(u, v) \mid v \in U : (v_i, u_{-i}) \in C \text{ for } i = 1, \ldots, N\}.$$

- Observation: For $v = u$ we get $V(u) \geq \Psi(u, u) = 0$ for $u \in C$.

A point $\bar{u} \in U$ is a Nash equilibrium of (P) if and only if $\bar{u} \in C$ and $V(\bar{u}) = 0$.

Variational Equilibria

- Since (P) is a jointly-convex GNEP we can use the more restrictive solution concept of variational equilibria ([Rosen '65]).
- Define $\widehat{\mathcal{R}} : C \to C$ by

$$\widehat{\mathcal{R}}(u) := \underset{v}{\operatorname{argmax}} \{\Psi(u,v) \mid v \in C\} = \underset{v}{\operatorname{argmin}} \left\{ \sum_{i=1}^{N} \mathcal{J}_i(v_i, u_{-i}) \mid v \in C \right\}$$

and $\widehat{V} : C \to \mathbb{R}$ by

$$\widehat{V}(u) := \Psi(u, \widehat{\mathcal{R}}(u)) = \max_{v} \{\Psi(u,v) \mid v \in C\} .$$

A point $\bar{u} \in U$ is called a variational equilibrium of (P) if $\bar{u} \in C$ and $\widehat{V}(\bar{u}) = 0$.

Variational Equilibria: Properties

Variational Equilibria are Nash Equilibria

Every variational equilibrium of (P) is also a Nash equilibrium of (P).

A point $\bar{u} \in C$ is a variational equilibrium if and only if $\bar{u} = \widehat{\mathcal{R}}(\bar{u})$.

Existence

The GNEP (P) admits a variational equilibrium $\bar{u} \in U$.

Proof uses Kakutani's Fixed Point Theorem applied to weak topology (yields compactness of C and upper semicontinuity of set-valued map $\widehat{\mathcal{R}}$, which then has a fixed point).

$\Rightarrow$ (P) admits Nash equilibrium.

Variational Equilibria: First-order conditions

Slater constraint qualification (weaker than previous CQ.)

$$0 \in \operatorname{int}\left(S(U_{\mathrm{ad}}) - K\right), \quad \text{interior taken in } X.$$

First-order optimality conditions

Suppose Slater CQ satisfied. Then $\bar{u} \in U$ is variational equilibrium of (P) if and only if $\exists\, \bar{y} \in Y$, $\bar{p} \in (W^*)^N$, $\bar{\lambda} \in U^*$ and $\bar{\mu} \in X^*$ such that

$$(\widehat{\mathrm{OS}}_i) \begin{cases} \bar{y} & = & S\bar{u}, \\ -\bar{p}_i & \in & A^{-*}\left(\partial J_i^1(\bar{y}) + \bar{\mu}\right), \\ \bar{\lambda}_i & \in & \partial I_{U_{\mathrm{ad}}^i}(\bar{u}_i), \\ \bar{\mu} & \in & \partial I_K(\bar{y}), \\ 0 & \in & \partial J_i^2(\bar{u}_i) - B_i^* \bar{p}_i + \bar{\lambda}_i, \end{cases}$$

is fulfilled for each $i = 1, \ldots, N$. Coupled system referred to by $(\widehat{\mathrm{OS}})$.

Reducible Case

Structural assumption.

- $J_i^1 = J_0^1 + \tilde{J}_i^1$ where J_0^1 convex and continuously Gâteaux differentiable, and $\tilde{J}_i^1$ linear-affine; w. l. o. g. we assume $\tilde{J}_i^1 \in Y^*$.

Includes typical tracking-type functionals: $J_i^1(y) = \frac{1}{2}\|y - y_i^d\|_Y^2$. Since,

$$\frac{1}{2}\|y - y_i^d\|_{L^2}^2 = \frac{1}{2}\|y\|_{L^2}^2 - (y, y_d^i)_{L^2} + \frac{1}{2}\|y_d^i\|_{L^2}^2.$$

Single objective PDE constrained optimization

Under the above assumption there exists a unique variational equilibrium $\bar{u}$ of (P), which is the unique solution of the convex optimization problem

$$\text{minimize } \widehat{J}(u) := J_0^1(Su) + \sum_{i=1}^{N} \left(J_i^2(u_i) + \langle S_i^* \tilde{J}_i^1, u_i\rangle_{U_i^*, U_i}\right) \text{ over } u \in U.$$

$$\text{s.t. } u \in C.$$

Path-Following

Penalty function (e.g. Moreau-Yosida regularization).

- $\beta : X \to \mathbb{R}_+$ is convex, continuous, and cont. Gâteaux-differentiable with $\ker \beta = K$, i.e., $\beta(y) = 0$ whenever $y \in K$, else $\beta(y) > 0$.

Consider

$$\begin{aligned} &\min J_i^1(y) + J_i^2(u_i) + \gamma\beta(y) \text{ over } (u_i, y) \in U_i \times Y \\ &\text{s.t.} \\ &\qquad Ay = B(u_i, u_{-i}), \quad u_i \in U_{\text{ad}}^i. \end{aligned} \tag{P$_{i,\gamma}$}$$

First-order conditions.

For all $i = 1, \dots, N$, u^γ is a Nash equilibrium if and only if there exist $y^\gamma \in Y$, $p^\gamma \in (W^*)^N$, $\lambda^\gamma \in U^*$ and $\mu^\gamma \in X^*$ such that

$$(\mathrm{OS}_{i,\gamma}) \begin{cases} y^\gamma = Su^\gamma, \\ -p_i^\gamma = A^{-*}((J_i^1)'(y^\gamma) + \mu^\gamma), \\ \lambda_i^\gamma \in \partial I_{U_{\text{ad}}^i}(u_i^\gamma), \\ \mu^\gamma = \gamma\beta'(y^\gamma), \\ 0 = (J_i^2)'(u_i^\gamma) - B_i^* p_i^\gamma + \lambda_i^\gamma. \end{cases}$$

Primal-Dual Path

For $\gamma > 0$, $\mathcal{S}_\gamma \subseteq U \times Y \times (W^*)^N \times U^* \times X^*$ set of solutions of (OS_γ).

$$\mathbf{C} := \left\{ \left((u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma)\right)_{\gamma>0} \mid \forall \gamma > 0 : (u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma) \in \mathcal{S}_\gamma \right\}.$$

We call every element $\mathcal{C} = \left((u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma)\right)_{\gamma>0} \in \mathbf{C}$ a primal-dual path.

Uniform Boundedness

Let (P) fulfill Slater CQ. Then $\exists\, 0 < \rho < \infty$ such that for all $\gamma > 0$:

$$\|u^\gamma\|_U + \|y^\gamma\|_Y + \|p^\gamma\|_{(W^*)^N} + \|\lambda^\gamma\|_{U^*} + \|\mu^\gamma\|_{X^*} \leq \rho.$$

Path convergence

Let (P) fulfill Slater CQ. Then for every primal-dual path $\mathcal{C} \in \mathbf{C}$ $\exists\, \gamma_n \to \infty$:

$$u^{\gamma_n} \overset{U}{\rightharpoonup} u^*,\ y^{\gamma_n} \overset{Y}{\rightharpoonup} y^*,\ p^{\gamma_n} \overset{(W^*)^N}{\rightharpoonup} p^*,\ \lambda^{\gamma_n} \overset{U^*}{\rightharpoonup} \lambda^*,\ \mu^{\gamma_n} \overset{X^*}{\rightharpoonup^*} \mu^*.$$

Moreover, the point $(u^*, y^*, p^*, \lambda^*, \mu^*)$ fulfills the first order optimality conditions $(\widehat{\mathrm{OS}})$, in particular u^* is a variational equilibrium of (P).

Path-Following

- For any $\gamma > 0$ let $\Psi_\gamma : U \times U \to \mathbb{R}$ be the Nikaido-Isoda function for (P_γ), i.e.

$$\Psi_\gamma(u, v) := \sum_{i=1}^{N} \left[\mathcal{J}_i^\gamma(u_i, u_{-i}) - \mathcal{J}_i^\gamma(v_i, u_{-i})\right],$$

  where $\mathcal{J}_i^\gamma(u) := J_i^1(Su) + J_i^2(u) + \gamma\beta(Su)$ represents the objective of $(\mathrm{P}_{i,\gamma})$, and
- consider $V : U \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$V(u, \gamma) := \max_{v \in U_{\mathrm{ad}}} \Psi_\gamma(u, v) = \sum_{i=1}^{N} \mathcal{J}_i^\gamma(u_i, u_{-i}) - \min_{v \in U_{\mathrm{ad}}} \sum_{i=1}^{N} \mathcal{J}_i^\gamma(v_i, u_{-i}).$$

- One observes that $V(u, \gamma) \geq 0$ for all $u \in U_{\mathrm{ad}}$ and analogously to before:

$$V(u, \gamma) = 0 \text{ if and only if } u \text{ is an equilibrium.}$$

γ-Update Strategy.

Given NE u^γ for γ, find $\gamma_+ > \gamma$ based on deviations of $V(u^\gamma, \gamma')$ from zero.

- Let $\mathcal{V}(\gamma + \eta) := V(u^\gamma, \gamma + \eta)$, $\eta > 0$, and assume the directional derivative $\mathcal{V}'(\gamma, \eta)$ exists.
- We observe $\mathcal{V}(\gamma + t\eta) = \mathcal{V}(\gamma) + \mathcal{V}'(\gamma; t\eta) + o(t) = \mathcal{V}'(\gamma; t\eta) + o(t)$.
- Therefore, we can base either directly on $\mathcal{V}'(\gamma; \eta)$ or an efficient approximation thereof.

Estimate of dir. deriv.

For any $\gamma > 0$, let u^γ be the corresponding equilibrium and define $\mathcal{V}(\gamma + \eta) := V(u^\gamma, \gamma + \eta)$, $\eta > 0$. It holds that for all $\eta > 0$:

$$\eta N \beta(S(u^\gamma)) \geq \limsup_{t \downarrow 0} t^{-1}(\mathcal{V}(\gamma + t\eta) - \mathcal{V}(\gamma)) \geq \liminf_{t \downarrow 0} t^{-1}(\mathcal{V}(\gamma + t\eta) - \mathcal{V}(\gamma)) \geq 0.$$

γ-Update Strategy.

- **Redundancy**: If $\beta(S(u^\gamma)) = 0$, then there is no need to increase γ, as the current state y^γ is feasible.
- **State constraint not redundant**: Bound secants by a fixed threshold $\pi_{path} > 0$ and choosing $\eta > 0$ such that

$$\eta N\beta(S(u^\gamma)) \leq \pi_{path}.$$

For example:

$$\eta = \frac{\pi_{path}}{N\beta(S(u^\gamma))}$$

and then use the update $\gamma := \gamma + \eta$.

Solvers.

- **Reducible case**. Semi-smooth Newton method (mesh independent).
- **General case**. Projected gradient-type method.

Elliptic GNEP – Reducible

- $N = 4$, $\Omega = (0,1) \times (0,1)$, uniform grid with mesh size h.
- $A := -\Delta$ discretized by standard 5-point stencil.
- **Nested Grid-Strategy**: 9-point prolongation coarse $\rightarrow$ fine mesh.
- **Stopping criterion**: H^{-1}-norm of reduced system, with tol $= 10^{-6}$ (for sufficiently fine mesh sizes).
- γ**-Update**: Given h, we update γ until $C\gamma^{-1} > h^2$, $C > 0$. Then refine mesh and continue.

Elliptic GNEP: Example 1

Example

- Objectives: $J_i(u_i, y) = \frac{1}{2}\|y - y_d^i\|_{L^2}^2 + \frac{\alpha_i}{2}\|u_i\|_{L^2}^2$,
- $a_i \leq u_i \leq b_i$ with $a_i = -10$, $b_i = 10$ for $i = 1, \ldots, 4$,
- $K = \{y \geq \psi\}$.
- $\alpha_1 = 2.1853, \alpha_2 = 2.0942, \alpha_3 = 2.8730, \alpha_4 = 2.0866$.
- For $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega$ we defined the obstacle ψ by

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \cos(5\sqrt{(\mathbf{x}_1^2 - 0.5)^2 + (\mathbf{x}_2^2 - 0.5)^2}).$$

- $O_1 =]0, \frac{1}{2}[\times]0, \frac{1}{2}[$, $O_2 =]\frac{1}{2}, 1[\times]0, \frac{1}{2}[$, $O_3 =]0, \frac{1}{2}[\times]\frac{1}{2}, 1[$, $O_4 =]\frac{1}{2}, 1[\times]\frac{1}{2}, 1[$, , $B_i := \chi_{O_i}$.
- $T := [0.25, 0.75] \times [0.25, 0.75] \subset \mathbb{R}^2$, we define $f = -\chi_T \Delta\psi - 11$.
- $y_d^i = \chi_T \psi$

Elliptic GNEP

Example

$$q_l := ||y_{l+1} - y_l||_{H_0^1} / ||y_l - y_{l-1}||_{H_0^1},$$

$$r_l := ||y_{l+1} - y_l||_{H_0^1}$$

$$s_l := ||\Delta y_l + \sum_{i=1}^{N} \chi_{B_i} \left(\tilde{\alpha}_i p_l - (\tilde{\alpha}_i \chi_{B_i} p_l - \tilde{b}_i)_+ + (\tilde{a}_i - \tilde{\alpha}_i \chi_{B_i} p_l)_+ \right) + f||_{H^{-1}} + ||\Delta p_l + \gamma(\psi - y_l)_+ + y_l||_{H^{-1}}$$

- $\gamma_{\max}$ represents the penalty parameter at which $C\gamma^{-1} > h^2$, $C = 1e3$
- 'iter' total number of iterations on mesh size h

Elliptic GNEP

	$h = 1/128$	$h = 1/256$	$h = 1/512$
q_l	0.25911	0.305	0.29818
	0.065435	0.013243	0.016935
	4.4273e-10	7.9809e-09	2.868e-08
r_l	0.035755	0.041051	0.035415
	0.0023397	0.00054362	0.00059974
	1.0358e-12	4.3386e-12	1.7201e-11
s_l	6.9274	5.1778	13.9437
	1.0617e-10	9.1398e-10	7.502e-09
	9.9346e-11	8.586e-10	7.5513e-09

Table: Behavior of the Newton Step in Example 1.

Elliptic GNEP

γ_{max}	iter	h
8192	16	1/4
32768	5	1/8
131072	2	1/16
524288	6	1/32
2097152	5	1/64
8388608	8	1/128
33554432	8	1/256
134217728	8	1/512

Table: Behavior of the Outer Loop in Example 2

Elliptic GNEP

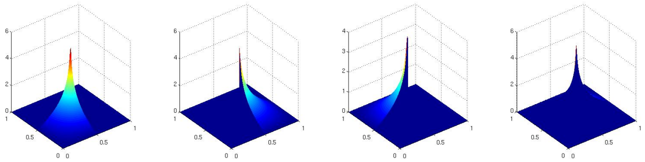


Figure: $(u_1^*, u_2^*, u_3^*, u_4^*)$ for Example 1.

Elliptic GNEP

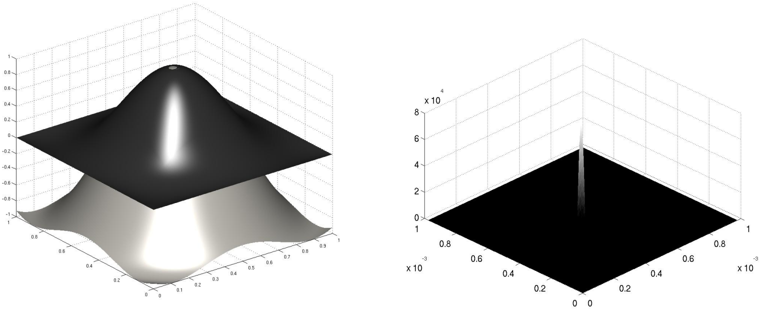


Figure: l: y^* (dark) and ψ (light); r: λ^* for Example 1.

Elliptic GNEP: Example 2

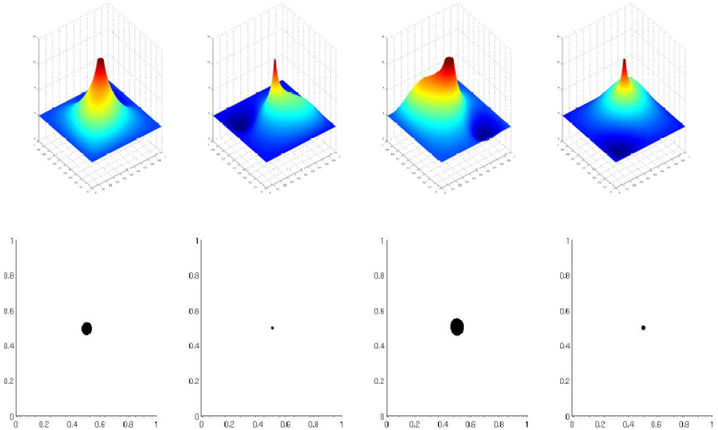


Figure: I: Optimal controls and active sets for Example 2.

Example 2

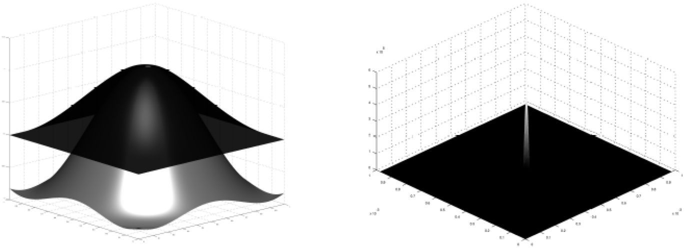


Figure: I: Optimal state for Example 2.

Example 2

	$h = 1/128$	$h = 1/256$	$h = 1/512$
q_l	0.86665	2.7241	1.2221
	0.30953	0.40752	0.25405
	0.77642	0.31822	0.3131
	1.3829e-11	0.26581	0.40521
		1.1597e-10	0.0026788
			3.1146e-07
r_l	0.51824	1.6909	1.058
	0.16041	0.68906	0.26878
	0.12455	0.21927	0.084154
	1.7224e-12	0.058284	0.0341
		6.7594e-12	9.1348e-05
			2.8451e-11
s_l	21.8022	332.9043	222.4632
	4.9505e-10	8.6492	3.2809e-08
	5.0033e-10	4.0237e-09	0.21287
	4.7248e-10	3.9826e-09	0.25647
		4.0435e-09	3.2643e-08
			3.2628e-08

$\gamma_{\max}$	iter	h
8192	17	1/4
32768	3	1/8
131072	4	1/16
524288	6	1/32
2097152	7	1/64
8388608	11	1/128
33554432	12	1/256
134217728	13	1/512

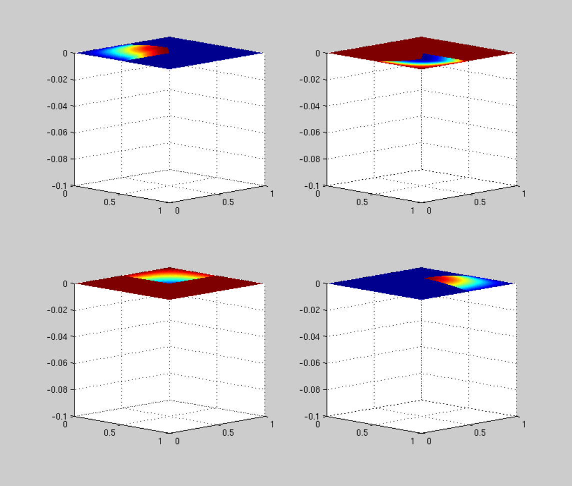
Figure: I: Convergence behavior for Example 2.

Parabolic GNEP

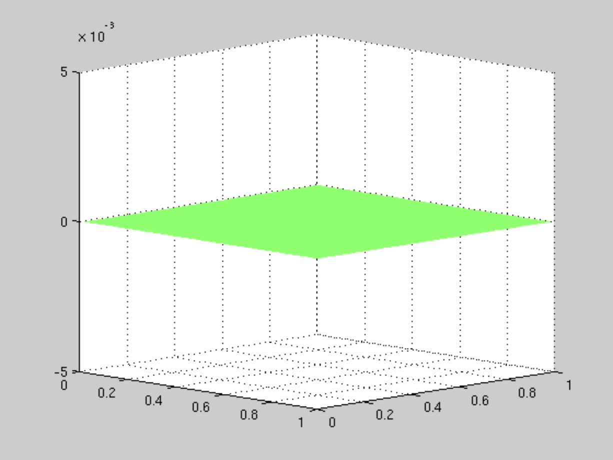
Here, we consider the settings:

- $U_i = L^2(0, T; L^2(\Omega))$,
  $V = \{ y \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \,|\, y_t \in L^2(0, T; L^2(\Omega)) \}$,
  $W = L^2(0, T; L^2(\Omega)) \times H_0^1(\Omega)$, $X = C(\overline{Q})$,
- $Ay = (y_t - \Delta y + c_0 y, y(0))^T$, $B \in \mathcal{L}(L^2(0, T; L^2(\Omega))$ analogous to elliptic case,
- $\psi \in V$, $\psi|_{\partial\Omega} > 0$,
- $a_i, b_i \in L^2(Q)$, $a_i \leq b_i$, a.e. in Q,
- $y_d^i \in L^2(\Omega)$, $\alpha_i > 0$,
- $U_{ad}^i := \{ v \in L^2(Q) \,|\, a_i \leq v \leq b_i, \text{a.e. in } Q \}$,
- $K = \psi + C(Q)_-$ (the cone of nonpositive continuous functions on Q),
- $\mathcal{J}_i(u) = \frac{1}{2}||S(u_i, u_{-i}) - y_d^i||^2_{L^2(L^2)} + \frac{\alpha_i}{2}||u_i||^2_{L^2(L^2)}$.

Equilibrium Controls u_i, $i = 1, \ldots, 4$



Equilibrium State y



Summary

- Motivating applications: Spot markets, sub/trans-sonic airfoil design.
- Abstract GNEPs
  - Uniform Slater CQ, Slater CQ;
  - Ky Fan with weak topology;
  - Moreau-Yosida regularization of state constraint;
  - sequence of NEPs.
- Nikaido-Isoda-based primal-dual path following.