Topologically relevant stationarity concepts

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Survey

1. The unconstrained smooth case
2. The constrained smooth case
3. Mathematical programs with complementarity constraints
4. Mathematical programs with vanishing constraints

Four reasons to look at stationary points

- Candidates for local minimizers
- Design of homotopy methods
- Understanding the problem structure (Morse theory)
- Convergence results for KKT points

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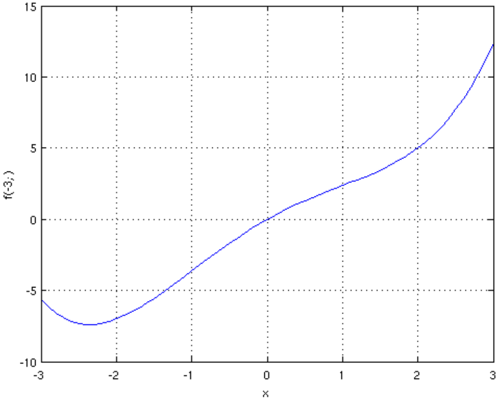
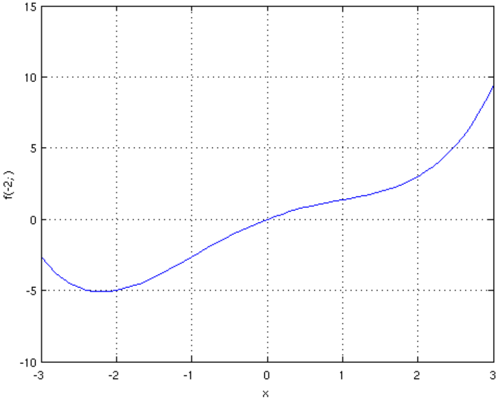
Example: parametric unconstrained smooth optimization



$$f(t,x) = \frac{x^4}{8} - \frac{3}{4}x^2 - tx \quad \text{for} \quad t = -3$$

Example: parametric unconstrained smooth optimization



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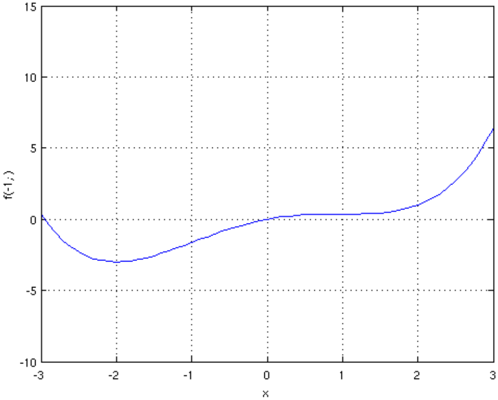

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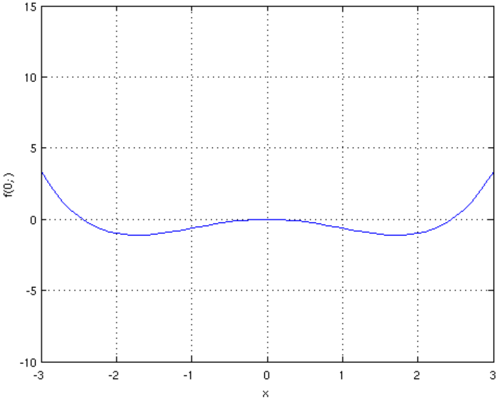
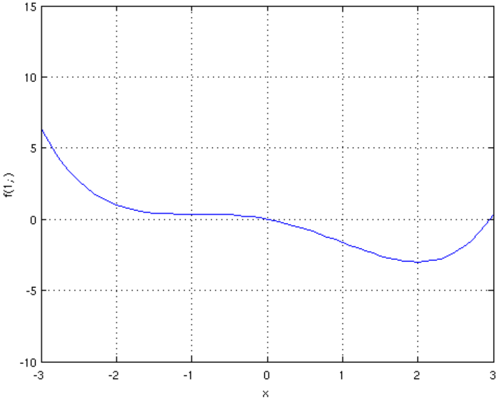


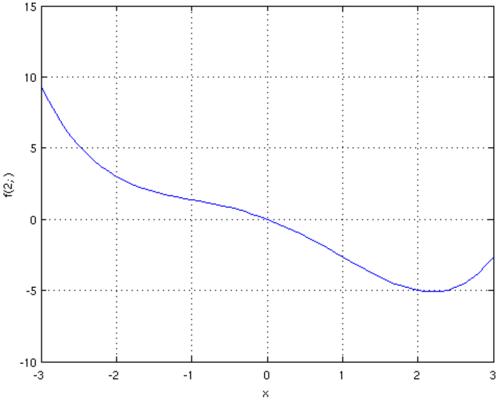
$$f(t,x) = \frac{x^4}{8} - \frac{3}{4}x^2 - tx \quad \text{for} \quad t = 0$$

Example: parametric unconstrained smooth optimization



$$f(t,x) = \frac{x^4}{8} - \frac{3}{4}x^2 - tx \quad \text{for} \quad t = 1$$

Example: parametric unconstrained smooth optimization



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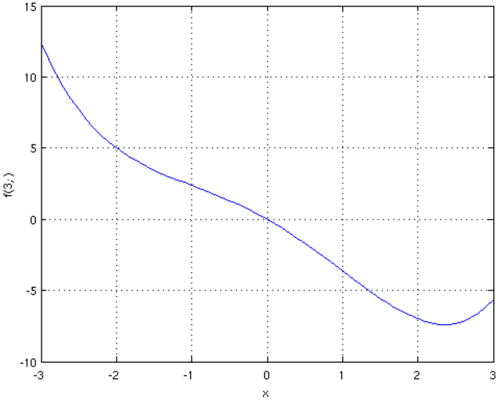
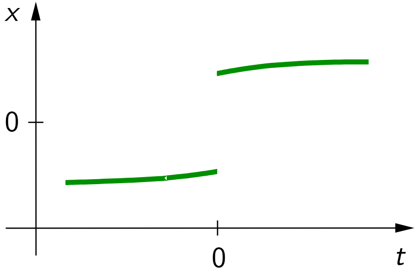


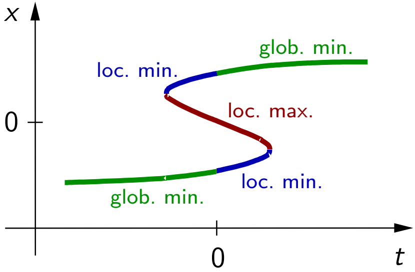
$$f(t,x) = \frac{x^4}{8} - \frac{3}{4}x^2 - tx \quad \text{for} \quad t = 3$$

Example: parametric unconstrained smooth optimization



Unfolded set of global minimizers

Example: parametric unconstrained smooth optimization



Unfolded set of critical points

Nondegenerate critical points

Necessary condition

$\bar{x} \in \mathbb{R}^n$ local minimizer of $f \;\Rightarrow\; \nabla f(\bar{x}) = 0$.

Definitions

$\bar{x} \in \mathbb{R}^n$ is called nondegenerate critical point of $f \in C^2(\mathbb{R}^n, \mathbb{R})$, if $\nabla f(\bar{x}) = 0$, and $D^2 f(\bar{x})$ is nonsingular.

The number of negative eigenvalues of $D^2 f(\bar{x})$ is called the Morse index or quadratic index of $\bar{x}$, briefly $QI(\bar{x})$.

Theorem (Jongen/Jonker/Twilt, 1983)

Generically, all critical points of $f \in C^2(\mathbb{R}^n, \mathbb{R})$ are nondegenerate.

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Nondegenerate critical points

Characterization of local minimality

For any nondegenerate critical point $\bar{x}$ of f we have

$$\bar{x} \text{ is a local minimizer of } f \;\Leftrightarrow\; QI(\bar{x}) = 0.$$

Theorem (Morse Lemma - local structure)

Let $\bar{x}$ be a nondegenerate critical point of $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then, modulo a local C^1 diffeomorphism, locally around $\bar{x}$ we have

$$f(x) \;=\; -x_1^2 - x_2^2 - \ldots - x_{QI(\bar{x})}^2 + x_{QI(\bar{x})+1}^2 + \ldots + x_n^2 \,.$$

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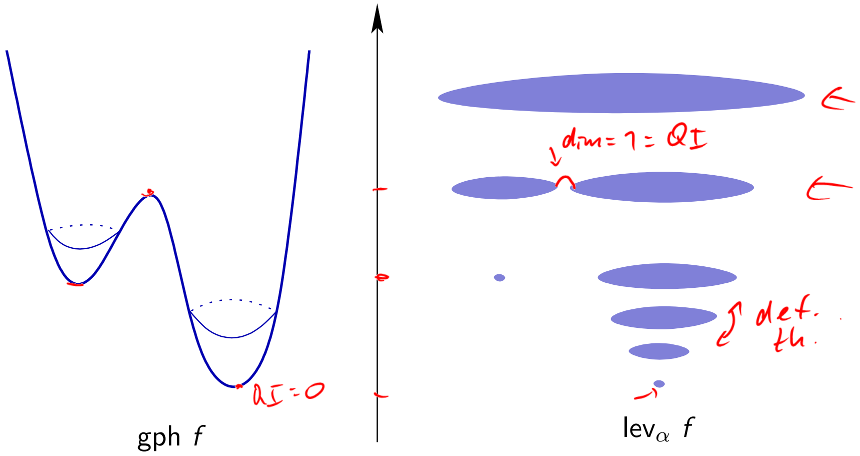
Deformation and cell attachment - global structure



Constrained smooth optimization

Consider the restriction of $f \in C^2(\mathbb{R}^n, \mathbb{R})$ to

$$M = \{g(x) \geq 0,\ h(x) = 0\}$$

with $g \in C^2(\mathbb{R}^n, \mathbb{R}^p)$, $h \in C^2(\mathbb{R}^n, \mathbb{R}^q)$, and let

$$L(x, \lambda, \mu) = f(x) - \lambda^\intercal g(x) - \mu^\intercal h(x)$$

be the Lagrangian of f on M.

Necessary condition

$\bar{x} \in \mathbb{R}^n$ local minimizer of f on M with some CQ
$\Rightarrow\ \bar{x}$ KKT point of f on M.

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Nondegenerate critical points

Definitions

$\bar{x} \in M$ is called nondegenerate critical point of f on M with multipliers $\bar{\lambda}$ and $\bar{\mu}$ if

- $\nabla_x L(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0$,
- LICQ holds at $\bar{x}$ in M,
- $\bar{\lambda}_i \neq 0$ for all active g_i,
- $D_x^2 L(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T(\bar{x},M)}$ is nonsingular.

The number of negative $\bar{\lambda}_i$ is called linear index of $\bar{x}$ ($LI(\bar{x})$), and the number of negative eigenvalues of $D_x^2 L(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T(\bar{x},M)}$ is called quadratic index of $\bar{x}$ ($QI(\bar{x})$).

Nondegenerate critical points

Characterization of local minimality

For any nondegenerate critical point $\bar{x}$ of f on M we have

$$\bar{x} \text{ is a local minimizer of } f \iff LI(\bar{x}) + QI(\bar{x}) = 0.$$

Morse theory and homotopy

- The generalizations to the constrained case of genericity, Morse lemma, deformation theorem and cell attachment theorem have been shown by Jongen/Jonker/Twilt (1983).
- For deformation and cell attachment, only the nondegenerate KKT points are relevant, that is, the nondegenerate critical points with $LI(\bar{x}) = 0$.
- Homotopy methods have been studied by, e.g., Guddat/Guerra Vázquez/Jongen (1990) ($LI(\bar{x}) \geq 0$ is relevant).

Mathematical programs with complementarity constraints

Consider the restriction of $f \in C^2(\mathbb{R}^n, \mathbb{R})$ to the set

$$M \;=\; \{G_i(x) \geq 0,\; H_i(x) \geq 0,\; G_i(x)H_i(x) = 0,\; i = 1, \ldots \ell\}$$

with $G \in C^2(\mathbb{R}^n, \mathbb{R}^\ell)$, $H \in C^2(\mathbb{R}^n, \mathbb{R}^\ell)$, and let

$$L(x, \gamma, \eta) \;=\; f(x) - \gamma^\intercal G(x) - \eta^\intercal H(x)$$

be the Lagrangian of f on M.

Applications of MPCCs

- Game theory
- Obstacle problems
- Truss topology design
- Network equilibria
- Bilevel optimization
- Semi-infinite optimization
- ...

C-stationarity

C-stationarity

$\bar{x} \in M$ is called C-stationary point of f on M with multipliers $\bar{\gamma}$ and $\bar{\eta}$ if

- $\nabla_x L(\bar{x}, \bar{\gamma}, \bar{\eta}) = 0$,
- $\bar{\gamma}_i = 0$ for all i with $G_i(\bar{x}) > 0$, $H_i(\bar{x}) = 0$,
- $\bar{\eta}_i = 0$ for all i with $G_i(\bar{x}) = 0$, $H_i(\bar{x}) > 0$,
- $\bar{\gamma}_i \bar{\eta}_i \geq 0$ for all i with $G_i(\bar{x}) = H_i(\bar{x}) = 0$.

Necessary condition

$\bar{x} \in \mathbb{R}^n$ local minimizer of f on M with some MPEC-CQ
$\Rightarrow$ $\bar{x}$ C-stationary for f on M.

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$\Rightarrow$ $\bar{x}$ C-stationary for f on M.

Nondegenerate C-stationary points

Definitions (Ralph/St., 2006, Jongen/Rückmann/Shikhman, 2009)

A C-stationary point $\bar{x}$ of f on M with multipliers $\bar{\gamma}$ and $\bar{\eta}$ is called nondegenerate if

- MPEC-LICQ holds at $\bar{x}$,
- $D_x^2 L(\bar{x}, \bar{\gamma}, \bar{\eta})|_{T(\bar{x},M)}$ is nonsingular,
- $\bar{\gamma}_i \bar{\eta}_i > 0$ for all i with $G_i(\bar{x}) = H_i(\bar{x}) = 0$. $\quad (\star)$

The number of pairs $(\bar{\gamma}_i, \bar{\eta}_i)$ with negative entries in $(\star)$ is called biactive index of $\bar{x}$ ($BI(\bar{x})$), the number of negative eigenvalues of $D_x^2 L(\bar{x}, \bar{\gamma}, \bar{\eta})|_{T(\bar{x},M)}$ is called quadratic index of $\bar{x}$ ($QI(\bar{x})$), and their sum $BI(\bar{x}) + QI(\bar{x})$ is called C-index of $\bar{x}$.

Nondegenerate C-stationary points

Characterization of local minimality

For any nondegenerate C-stationary point $\bar{x}$ of f on M we have

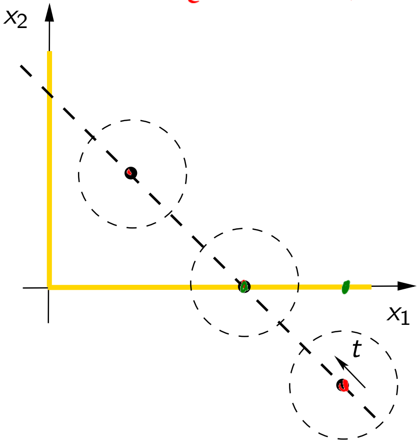
$$\bar{x} \text{ is a local minimizer of } f \;\Leftrightarrow\; BI(\bar{x}) + QI(\bar{x}) = 0.$$

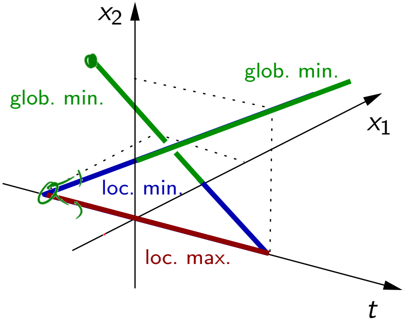
Morse theory and homotopy for MPCCs

- The (full) generalizations to MPCCs of genericity, Morse lemma, deformation theorem and cell attachment theorem have been shown by Jongen/Rückmann/Shikhman (2009).
- Homotopy methods for (special) generic MPCCs have been studied by Ralph/St. (2006).

An MPCC homotopy

$x_1 x_2 = 0, x_1 \geq 0, x_2 \geq 0$



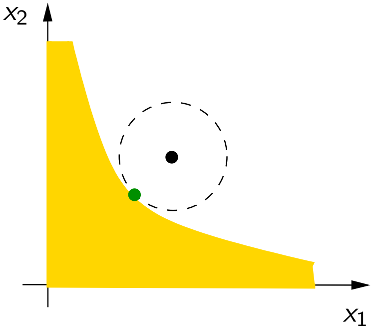


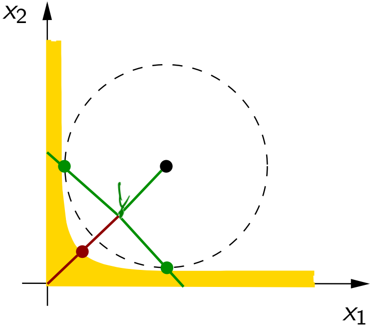
Limits of KKT points

Smoothed MPCCs (Scholtes 2001, Steffensen/Ulbrich 2010, Hoheisel/Kanzow/Schwartz 2011)

For a sequence of smoothing parameters $t_k \searrow 0$ and a sequence of KKT points x^k of some smoothing problem $NLP(t_k)$ with $x^k \to \bar{x}$, under some CQ the point $\bar{x}$ is C-stationary for MPCC.

Example: Scholtes smoothing for an MPCC





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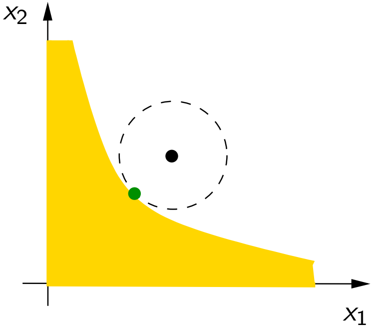
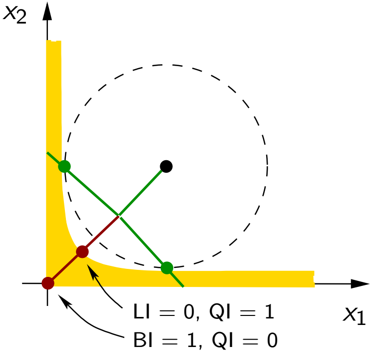




First order descent directions at C-stationary points

Nondegenerate C-stationary points with positive biactive index allow first order descent directions.

This is due to the nonsmoothness of MPCCs and cannot be avoided in a topologically relevant stationarity concept.

T-stationarity

Definition

For a given class of optimizations problems we call a set of conditions a stationarity concept, if these conditions hold (under some CQ) at each local minimizer, and we call the stationarity concept topologically relevant, if it admits

- a nondegeneracy concept
- the definition of a (Morse) index,
- a Morse lemma,
- a deformation theorem,
- and a cell attachment theorem.

The stationarity concept is then also called T-stationarity.

T-stationarity

Examples:

- Unconstrained smooth optimization:

  T-stationarity = stationarity

- Constrained smooth optimization:

  T-stationarity = KKT-stationarity

- MPCCs:

  T-stationarity = C-stationarity

- Disjunctive optimization:

  T-stationarity = stationarity (Jongen/Rückmann/St. 1997)

Limits of KKT points for MPCCs revisited

Smoothed MPCCs (Scholtes 2001, Steffensen/Ulbrich 2010, Hoheisel/Kanzow/Schwartz 2011)

For a sequence of smoothing parameters $t_k \searrow 0$ and a sequence of KKT-stationary points x^k of some smoothing problem $NLP(t_k)$ with $x^k \to \bar{x}$, under some CQ the point $\bar{x}$ is C-stationary for MPCC.

Limits of KKT points for MPCCs revisited

Smoothed MPCCs (Scholtes 2001, Steffensen/Ulbrich 2010, Hoheisel/Kanzow/Schwartz 2011)

For a sequence of smoothing parameters $t_k \searrow 0$ and a sequence of T-stationary points x^k of some smoothing problem $NLP(t_k)$ with $x^k \to \bar{x}$, under some CQ the point $\bar{x}$ is T-stationary for MPCC.

Mathematical programs with vanishing constraints

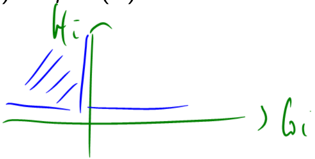
Consider the restriction of $f \in C^2(\mathbb{R}^n, \mathbb{R})$ to the set

$$M \;=\; \{H_i(x) \geq 0,\;\; G_i(x)H_i(x) \leq 0,\;\; i = 1, \ldots \ell\}$$

with $G \in C^2(\mathbb{R}^n, \mathbb{R}^\ell)$, $H \in C^2(\mathbb{R}^n, \mathbb{R}^\ell)$, and let

$$L(x, \gamma, \eta) \;=\; f(x) - \gamma^\mathsf{T} G(x) - \eta^\mathsf{T} H(x)$$

be the Lagrangian of f on M.



Application of MPVCs

- MPVC was introduced as a model for structural and topology optimization.
- It is motivated by the fact that the constraint G_i does not play any role whenever H_i is active.

T-stationarity for MPVCs

from scratch

T-stationarity (Dorsch/Shikhman/St., 2010)

$\bar{x} \in M$ is called T-stationary point of f on M with multipliers $\bar{\gamma}$ and $\bar{\eta}$ if

- $\nabla_x L(\bar{x}, \bar{\gamma}, \bar{\eta}) = 0$,
- $\bar{\gamma}_i = 0$ for all i with $G_i(\bar{x}) < 0$, $H_i(\bar{x}) \geq 0$,
- $\bar{\gamma}_i = 0$ for all i with $G_i(\bar{x}) > 0$, $H_i(\bar{x}) = 0$,
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T-stationarity for MPVCs

Necessary condition

$\bar{x} \in \mathbb{R}^n$ local minimizer of f on M with some MPVC-CQ
$\Rightarrow \;\; \bar{x}$ T-stationary for f on M.

Nondegenerate T-stationary points

Definitions (Dorsch/Shikhman/St., 2010)

A T-stationary point $\bar{x}$ of f on M with multipliers $\bar{\gamma}$ and $\bar{\eta}$ is called nondegenerate if

- MPVC-LICQ holds at $\bar{x}$,
- $D_x^2 L(\bar{x}, \bar{\gamma}, \bar{\eta})|_{T(\bar{x},M)}$ is nonsingular,
- $\bar{\gamma}_i < 0$ for all i with $G_i(\bar{x}) = 0$, $H_i(\bar{x}) \geq 0$,
- $\bar{\eta}_i > 0$ for all i with $G_i(\bar{x}) < 0$, $H_i(\bar{x}) = 0$,
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Nondegenerate T-stationary points

Characterization of local minimality

For any nondegenerate T-stationary point $\bar{x}$ of f on M we have

$$\bar{x} \text{ is a local minimizer of } f \iff BI(\bar{x}) + QI(\bar{x}) = 0.$$

Morse theory for MPVCs

- The generalizations to MPVCs of genericity, Morse lemma, deformation theorem and cell attachment theorem have been shown by Dorsch/Shikhman/St. (2010).

Limits of KKT points

Smoothed MPVCs (Hoheisel/Kanzow/Schwartz 2011)

For a sequence of smoothing parameters $t_k \searrow 0$ and a sequence of KKT-stationary points x^k of some smoothing problem $NLP(t_k)$ with $x^k \to \bar{x}$, under some CQ the point $\bar{x}$ is T-stationary for MPVC.

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Observation and a conjecture

In the known smoothing methods for MPCC, as well as for MPVC, any nondegenerate T-stationary point $\bar{x}$ of the nonsmooth problem

- locally 'unfolds' into a smooth curve $\{x(t) | \, t \in U(0)\}$ of KKT points of the smoothing problems (with smoothing parameter t, $x(0) = \bar{x}$, via the implicit function theorem),
- and the T-index of $\bar{x}$ coincides with the Morse index of $x(t)$, $t \in U(0)$ (via continuity arguments).

Conjecture 1

These effects occur for 'a large class of smoothing methods' for 'a large class of nonsmooth problems'.

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Another conjecture

Conjecture 2

T-stationarity is uniquely defined for ‘a large class of nonsmooth problems’.

Conclusion

For any class of optimization problems, the T-stationarity concept is the natural one for

- topological considerations
- design of homotopy methods
- limits of KKT points
- (design of Newton methods).

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