On the Solution of Optimization and Variational Problems with Imperfect Information

Uday V. Shanbhag (with Hao Jiang (@Illinois) and Hesam Ahmadi (@PSU))

Harold and Inge Marcus Department of Industrial and Manufacturing Engineering
Pennsylvania State University
University Park, PA

6^{th} International Conference on Complementarity Problems (ICCP)
Humboldt-Universität Zu Berlin
Berlin
August 8, 2014

A misspecified optimization problem I

A prototypical **misspecified*** convex program where $\theta^* \in \mathbb{R}^m$ is misspecified:

$$\mathcal{C}(\theta^*) \qquad \underset{x \in X}{\text{minimize}} \quad f(x, \theta^*)$$

Generally, θ^* captures problem characteristics that may require estimation.

- Parameters of cost/price functions
- Efficiencies
- Representation of uncertainty

Generally, this is part of the model building process.

- Traditionally, a dichotomy in the roles of statisticans and optimizers
  1. Statisticians Learn – (Build **model**, estimate **parameters**)
  2. Optimizers Search – (Use **model/parameters** to obtain **solution**)
- Increasingly, the serial nature cannot persist.

*This is parametric misspecification (as opposed to model misspecification)

Offline learning I

- One avenue lies in collecting observations a priori
- Learning problem $\mathcal{L}_\theta$ unaffected by the computational problem $\mathcal{C}(\theta^*)$:

$$\mathcal{L}_\theta \qquad \operatorname*{minimize}_{\theta \in \Theta} \; g(\theta)$$

Concerns:

- Exact solutions generally unavailable in finite time; solution error can be bounded in expected-value sense (at best) in stochastic regimes
- Premature termination of learning process leads to $\widehat{\theta}$; Error cascades into computational problem;

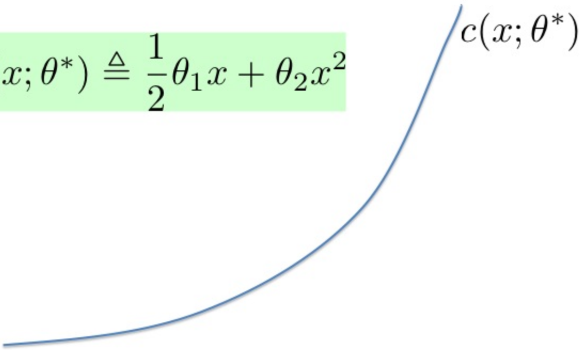
$$\widehat{x} \in \mathrm{SOL}(\mathcal{C}(\widehat{\theta})).$$

- Unclear how to develop[a] implementable scheme that produces x^*:
  - (First-order) schemes that produce x^* and θ^* asymptotically
  - Non-asymptotic error bounds

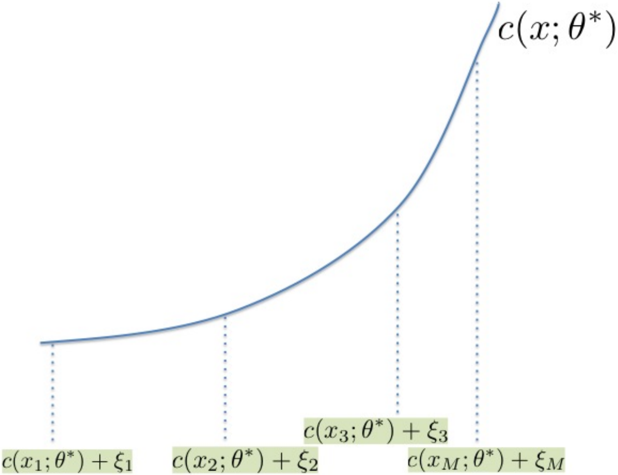
[a] Note that schemes that produce approximations are available based on Lipschitzian properties

An example I

$$c(x;\theta^*) \triangleq \frac{1}{2}\theta_1 x + \theta_2 x^2$$



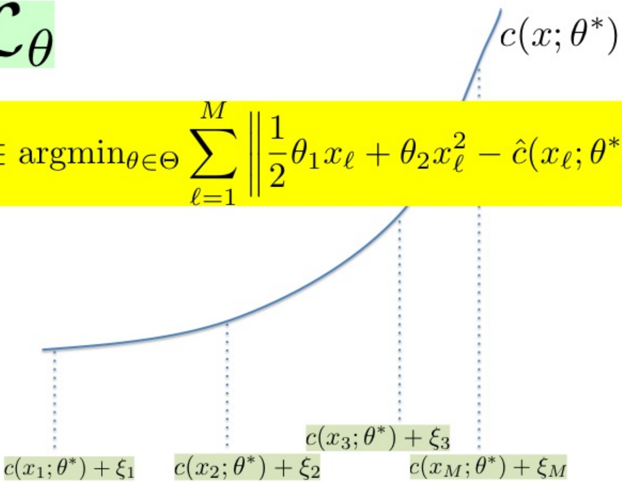
An example II



An example III

$$\mathcal{L}_\theta$$

$$\theta^* \in \text{argmin}_{\theta\in\Theta} \sum_{\ell=1}^{M} \left\| \frac{1}{2}\theta_1 x_\ell + \theta_2 x_\ell^2 - \hat{c}(x_\ell;\theta^*) \right\|^2$$



Data-driven stochastic programming I

- Consider the following static stochastic program

$$\min_{x \in X} \quad \mathbb{E}[f(x, \xi_{\theta^*}(\omega))], \qquad (\mathcal{C}_{\theta^*})$$

where $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, $\xi_{\theta^*} : \Omega \to \mathbb{R}^d$ and $(\Omega, \mathcal{F}, \mathbb{P}_{\theta^*})$ represents the probability space.

- Traditionally, the parameters of this distribution are estimated a priori (by MLE approaches for instance). Often a challenging problem (such as covariance selection)

Misspecified production planning problems I

- The production planner solves the following problem:

$$\begin{aligned} \min_{x_{fi} \geq 0} \quad & \sum_{f=1}^{N} \sum_{i=1}^{W} c_{fi}(x_{fi}) \\ \text{subject to} \quad & x_{fi} \leq \text{cap}_{fi}, \qquad \text{for all } f, i, \\ & \sum_{f=1}^{N} x_{fi} = d_i. \end{aligned} \tag{1}$$

- Machine type f's production cost at node i $c_{fi}^{(l)}(x_{fi}^{(l)})$ at time l, $l = 1, \ldots, T$:

$$c_{fi}^{(l)}(x_{fi}^{(l)}) = d_{fi}(x_{fi}^{(l)})^2 + h_{fi} x_{fi}^{(l)} + \xi_{fi}^{(l)}$$

- The planner will solve the following problem to estimate d_{fi} and h_{fi}:

$$\min_{\{d_{fi}, h_{f,i}\} \in \Theta} \quad \sum_{l=1}^{T} \sum_{f=1}^{N} \sum_{i=1}^{W} (d_{fi}(x_{fi}^{(l)})^2 + h_{fi} x_{fi}^{(l)} - c_{fi}^{(l)}(x_{fi}^{(l)}))^2.$$

A framework for learning and computation I

$$\mathcal{C}(\theta^*) \qquad \underset{x \in X}{\text{minimize}} \;\; f(x, \theta^*)$$

$$\mathcal{L}_\theta \qquad \underset{\theta \in \Theta}{\text{minimize}} \;\; g(\theta)$$

Our focus is on general purpose algorithms that jointly generate sequences $\{x_k\}$ and $\{\theta_k\}$ with the following goals:

$$\lim_{k \to \infty} x_k = x^* \text{ and } \lim_{k \to \infty} \theta_k = \theta^* \qquad \text{(Global convergence)}$$

$$\|f(x_K, \theta_K) - f(x^*, \theta^*)\| \leq \mathcal{O}\left(h(K)\right), \qquad \text{(Rate statements)}$$

where $h(K)$ specifies the rate.

A serial approach

1. Compute a solution $\tilde{\theta}$ to ($\mathcal{L}_\theta$)
2. Use solution to solve ($\mathcal{C}(\tilde{\theta})$)

Challenges:

- Given the stage-wise nature, step 1. needs to provide accurate/exact $\tilde{\theta}$ in finite time; possible for small problems;
- In stochastic regimes, solution bounds available in expected-value sense:

$$\mathbb{E}[\|\theta_K - \theta^*\|^2] \leq \mathcal{O}(1/K).$$

- In fact, unless the learning problem is solvable via a finite termination algorithm, asymptotic statements are unavailable

A complementarity approach

- A direct variational approach: under convexity assumptions, equilibrium conditions are given by VI(Z, H) where

$$H(z) \triangleq \begin{pmatrix} F(x,\theta) \\ \nabla_\theta g(\theta) \end{pmatrix} \text{ and } Z \triangleq X \times \Theta.$$

Challenges:

- Problem rarely monotone and low-complexity first-order projection/stochastic approximation schemes cannot accommodate such problems.

Research questions

- First-order schemes available for solution of deterministic/stochastic convex optimization and monotone variational problems
- Can we develop analogous schemes that guarantee global/a.s. convergence†
- Can rate statements be provided for such schemes:
  - Are the original rates preserved?
  - What is the price of learning in terms of the modification/degradation in rates?

† **not immediate since problems can be viewed as non-monotone VIs/SVIs.**

Outline

Part I: Deterministic problems:

- Gradient methods for smooth/nonsmooth and strongly convex/convex optimization
- Extragradient and regularization methods for monotone variational inequality problems

Part II: Stochastic problems:

- Stochastic approximation schemes for strongly convex/convex stochastic optimization with stochastic learning problems
- Regularized stochastic approximation for monotone stochastic variational inequality problems with stochastic learning problems

Literature Review

Static decision-making problems with perfect information

- Optimization: convex programming [BNO03], integer programming [NW99], stochastic programming [BL97]
- Variational inequality problems [FP03a]

Learning

- Linear and nonlinear regression, support vector machines (SVMs), etc. [HTF01]

Joint schemes for related problems:

- Adaptive control [AW94], Iterative learning (tracking) control [Moo93]
- Bandit problems [Git89], regret problems [Zin03]
- Relatively less on joint schemes focusing on stylized problems in revenue management [CHdMK06, HKZ, CHdMK12]

Misspecified deterministic optimization

Consider the *static misspecified convex optimization problem* $(\mathcal{C}(\theta^*))$:

$$\min_{x \in X} f(x, \theta^*), \qquad (\mathcal{C}(\theta^*))$$

where $x \in \mathbb{R}^n$, $f : X \times \Theta \to \mathbb{R}$ is a convex function in x for every $\theta \in \Theta \subseteq \mathbb{R}^m$. Suppose θ^* denotes the solution to a convex learning problem denoted by $(\mathcal{L})$:

$$\min_{\theta \in \Theta} g(\theta), \qquad (\mathcal{L})$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a convex function in θ and is defined on a closed and convex set Θ.

A joint gradient algorithm

Algorithm 1 (Joint gradient scheme)

Given $x_0 \in X$ *and* $\theta_0 \in \Theta$ *and sequences* $\gamma_{f,k}, \gamma_{g,k}$,

$$x_{k+1} := \Pi_X \left(x_k - \gamma_{f,k} \nabla_x f(x_k, \theta_k)\right), \qquad \forall k \geq 0, \qquad (\mathrm{Opt}(\theta_k))$$

$$\theta_{k+1} := \Pi_\Theta \left(\theta_k - \gamma_{g,k} \nabla_\theta g(\theta_k)\right), \qquad \forall k \geq 0. \qquad (\mathrm{Learn})$$

Assumptions

Assumption 1

The function $f(x, \theta)$ is continuously differentiable in x for all $\theta \in \Theta$ and function g is continuously differentiable in θ.

Assumption 2

The gradient map $\nabla_x f(x; \theta)$ is Lipschitz continuous in x with constant $G_{f,x}$ uniformly over $\theta \in \Theta$ or

$$\|\nabla_x f(x_1, \theta) - \nabla_x f(x_2, \theta)\| \leq G_{f,x}\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in X, \quad \forall \theta \in \Theta.$$

Additionally, the gradient map $\nabla_\theta g$ is Lipschitz continuous in θ with constant G_g.

Assumption 3

Let $\{\gamma_{f,k}\}$ and $\{\gamma_{g,k}\}$ be diminishing nonnegative sequences chosen such that $\sum_{k=1}^{\infty} \gamma_{f,k} = \infty$, $\sum_{k=1}^{\infty} \gamma_{f,k}^2 < \infty$, $\sum_{k=1}^{\infty} \gamma_{g,k} = \infty$, and $\sum_{k=1}^{\infty} \gamma_{g,k}^2 < \infty$.

Constant steplength schemes for strongly convex problems I

Assumption 4

The function f is strongly convex in x with constant η_f for all $\theta \in \Theta$ and the function g is strongly convex with constant η_g.

Assumption 5

The gradient $\nabla_x f(x^*, \theta)$ is Lipschitz continuous in θ with constant L_θ.

Proposition 1 (**Rate analysis in strongly convex regimes)**

Let Assumptions 1, 2, 4 and 5 hold. In addition, assume that γ_f and γ_g are chosen such that $\gamma_f \leq \min(2\eta_f / G_{f,x}^2, 1/L_\theta)$ and $\gamma_g \leq 2/G_g$. Let $\{x_k, \theta_k\}$ be the sequence generated by Algorithm 1. Then for every $k \geq 0$, we have the following:

$$\|x_{k+1} - x^*\| \leq q_x^{k+1} \|x_0 - x^*\| + k q_\theta q^k \|\theta_0 - \theta^*\|,$$

where $q_x \triangleq (1 + \gamma_f^2 G_{f,x}^2 - 2\gamma_f \eta_f)^{1/2}$, $q_\theta \triangleq \gamma_f L_\theta$, $q_g \triangleq (1 + \gamma_g^2 G_g^2 - 2\gamma_g \eta_g)^{1/2}$, and $q \triangleq \max(q_x, g_g)$.

Constant steplength schemes for strongly convex problems II

Remark: Notably, learning leads to a degradation in the convergence rate from the standard **linear** rate to a **sub-linear** rate. Furthermore, it is easily seen that when we have access to the true θ^*, the original rate may be recovered.
‡

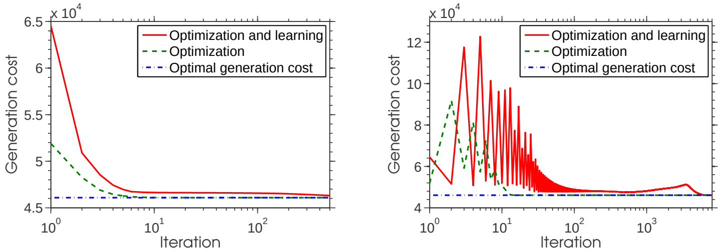


Figure 1 : Strongly convex problems and learning: Constant steplength (l) and Diminishing steplength (r)

Constant steplength schemes for strongly convex problems III

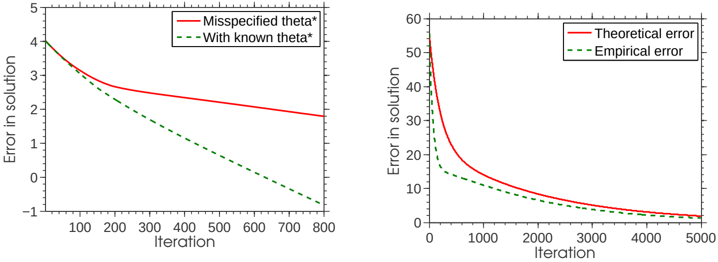


Figure 2 : Strongly convex optimization and learning: Impact on rate (l) and empirical vs. theor. rate (r)

‡ *We provide some numerics on a small production planning problem with 5 plants with capacity and ramping requirements. We assume that either cost is misspecified (Opt) or demand is misspecified (VIs).*

Misspecified convex optimization I

Assumption 6

The function f is convex in x with constant η_f for all $\theta \in \Theta$ and the function g is strongly convex with constant η_g.

Assumption 7

(a) *The sets X and Θ are compact and $\sup_{x \in X} \|x\| \leq C$, where C is a constant.*

(b) *The gradient map $\nabla_x f(x;\theta)$ is uniformly Lipschitz continuous in θ with constant $G_{f,\theta}$:*

$$\|\nabla_x f(x,\theta_1) - \nabla_x f(x,\theta_2)\| \leq G_{f,\theta}\|\theta_1 - \theta_2\|, \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$$

Assumption 8

There exists a constant $L_{f,\theta}$ such that
$|f(x,\theta_1) - f(x,\theta_2)| \leq L_{f,\theta}\|\theta_1 - \theta_2\|, \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$

Misspecified convex optimization II

Proposition 2 (Constant steplength scheme with averaging)

Let Assumptions 1, 2, 6, 7 and 8 hold and stepsizes $\gamma_{f,k}$ and $\gamma_{g,k}$ be fixed at constants γ_f and γ_g so that $0 < \gamma_g < 2/G_g$ and $0 < \gamma_f \leq 1/G_{f,x}$. Let the sequence $\{x_k, \theta_k\}$ be generated by Algorithm 1 and suppose $\bar{x}_k$ is defined as

$$\bar{x}_k \triangleq \frac{\sum_{i=0}^{k-1} x_{i+1}}{k}.$$

Then the following hold:

(i) *In addition, if $a_x = \frac{\|x_0 - x^*\|^2}{2\gamma_f}$, $a_\theta \triangleq \|\theta_0 - \theta^*\|$, and $b_\theta \triangleq \frac{CG_{f,\theta}}{1-q_g}$, then the following holds:*

$$|f(\bar{x}_K, \theta_K) - f(x^*, \theta^*)| \leq \frac{a_x}{K} + a_\theta \left(\frac{b_\theta}{K} + L_{f,\theta} q_g^K \right).$$

(ii) $\lim_{k \to \infty} f(\bar{x}_k, \theta_k) = f(x^*, \theta^*)$.

Misspecified convex optimization III

Remarks:

- Unlike in the case of strongly convex optimization, there is **no** degradation in the standard rate of convergence in function values which is $\mathcal{O}(1/K)$.
- Contribution from learning is given by

$$\|\theta_0 - \theta^*\| \left(L_{f,\theta} q_g^K + \frac{b_\theta}{K} \right).$$

- Some intuition:
  - The first term arises from the effort to learn the correct θ^*
  - The second term is an **interaction** term between x and θ through $L_{f,\theta}$ and is mitigated by averaging
  - Both terms are scaled by $\|\theta_0 - \theta^*\|$.
  - The overall rate does not degrade (but gets modified)

Misspecified convex optimization IV

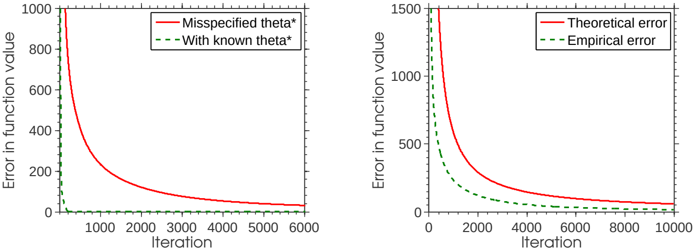


Figure 3 : Convex optimization and strongly convex learning: Impact on rate (l) and empirical vs. theor. (r)

Nonsmooth convex optimization I

Assumption 9

The function g is continuously differentiable in θ, strongly convex, and the gradient map $\nabla_\theta g(\theta)$ is Lipschitz continuous in θ with constant G_g.

Assumption 10 (**Subgradient boundedness)**

There exists an $M > 0$ such that $\|d_k\| \leq M$ for all $d_k \in \partial f(x_k, \theta_k)$ and for all $\theta_k \in \Theta$.

Assumption 11

There exists a constant $L_{f,\theta}$ such that
$|f(x, \theta_1) - f(x, \theta_2)| \leq L_{f,\theta}\|\theta_1 - \theta_2\| \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$

We consider the following subgradient-based analog of Algorithm 1:

Algorithm 2 (**Joint subgradient scheme)**

Given an $x_0 \in X$ and a $\theta_0 \in \Theta$ and sequences $\{\gamma_{f,k}, \gamma_{g,k}\}$, then

$$x_{k+1} := \Pi_X\left(x_k - \gamma_{f,k} d_k\right), \qquad \forall k \geq 0, \qquad (\text{nsOpt}(\theta_k))$$
$$\theta_{k+1} := \Pi_\Theta\left(\theta_k - \gamma_{g,k}\nabla_\theta g(\theta_k)\right), \qquad \forall k \geq 0, \qquad (\text{Learn})$$

where $d_k \in \partial f(x_k, \theta_k)$.

Proposition 3 (Rate analysis with averaging)

Let Assumptions 9 , 10, and 11 hold. Let $\gamma_{g,k}$ *be fixed at* γ_g *such that* $0 < \gamma_g < 2/G_g$*. Consider the sequence* $\{x_k, \theta_k\}$ *generated by Algorithm 2 and* $\bar{x}_k \triangleq \frac{\sum_{i=0}^{k} \gamma_{f,i} x_i}{\sum_{i=0}^{k} \gamma_{f,i}}$*. Then the following hold:*

(i) *If* $\gamma_{f,k}$ *is defined based on Assumption 3 with* $\gamma_{f,0} \leq 2\eta_f / G_{f,x}^2$ *and* $\gamma_g \leq 2/G_g$*, then*

$$\lim_{k \to \infty} |f(\bar{x}_k, \theta_k) - f(x^*, \theta^*)| = 0.$$

(ii) *Suppose Algorithm 2 is to be terminated after* K *iterations and* γ_f *(the optimal constant steplength) is defined as* $\gamma_{f,K} = \frac{\|x_0 - x^*\|}{M\sqrt{K+1}}$*, then*

$$|f(\bar{x}_K, \theta_K) - f(x^*, \theta^*)| \leq \frac{d_x}{\sqrt{K+1}} + d_\theta \left(L_{f,\theta} q_g^K + \frac{c_\theta}{(K+1)} \right),$$

where $d_x = M\|x_0 - x^*\|$, $d_\theta = \|\theta_0 - \theta^*\|$, *and* $c_\theta = 2L_{f,\theta}/(1 - q_g)$.

Nonsmooth convex optimization III

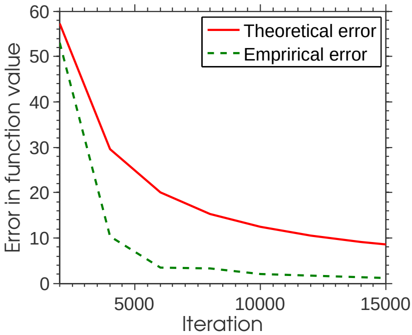
Remark: Standard subgradient methods for convex optimization display a convergence rate of $\mathcal{O}(1/\sqrt{K})$ in function value [BV04] using optimal constant steplength [SDR09]

- Joint scheme shows **no** degradation in the rate, not even in a constant factor sense.
- Modification in the rate is given by

$$\|\theta_0 - \theta^*\| \left(L_{f,\theta} q_g^K + \frac{b_\theta}{K} \right).$$

- Identical to the smooth case

Nonsmooth convex optimization IV



Misspecified variational inequality problems I

The misspecified optimization problem is now generalized to a variational inequality problem:

$$(y - x)^T F(x; \theta^*) \geq 0, \qquad \forall y \in X. \qquad (\mathcal{V}(\theta^*))$$

Assumption 12

(a) *The function g is differentiable, strongly convex with constant η_g, and Lipschitz continuous in gradient with constant G_g.*

(b) *The map F is monotone in x and uniformly Lipschitz continuous in x and θ with constants $L_{F,x}$ and $L_{F,\theta}$, respectively:*

$$\|F(x_1;\theta) - F(x_2;\theta)\| \leq L_{F,x}\|x_1 - x_2\| \quad \forall x_1, x_2 \in X, \quad \forall \theta \in \Theta,$$
$$\|F(x,\theta_1) - F(x,\theta_2)\| \leq L_{F,\theta}\|\theta_1 - \theta_2\| \quad \forall \theta_1, \theta_2 \in \Theta, \quad \forall x \in X.$$

Extragradient schemes I

Algorithm 3 (A joint extragradient scheme)

Given an $x_0 \in X$ and a $\theta_0 \in \Theta$ and a steplength τ,

$$z_{k+1} := \Pi_X(x_k - \tau F(x_k;\theta_k)) \qquad \forall k > 0, \qquad (\text{Extra}_x(\theta_k))$$

$$x_{k+1} := \Pi_X(x_k - \tau F(z_{k+1};\theta_k)) \qquad \forall k > 0, \qquad (\text{Extra}_z(\theta_k))$$

$$\theta_{k+1} := \Pi_\Theta(\theta_k - \gamma_g \nabla_\theta g(\theta_k)) \qquad \forall k > 0. \qquad (\text{Learn})$$

Theorem 1 (Convergence of extragradient scheme)

Let Assumption 12 holds and Θ is bounded. In addition, assume that stepsize $\gamma_{g,k}$ is fixed at γ_g, where $\gamma_g \leq \frac{2}{G_g}$. Let $\{x_k, \theta_k\}$ be the sequence generated by Algorithm 3 with

$$\tau^2 \leq \frac{1}{L_{F,x}^2 + L_{F,\theta}\|\theta_0 - \theta^*\|}.$$

Then, $\{x_k\}$ converges to a point in X^* and $\{\theta_k\}$ converges to $\theta^* \in \Theta$ as $k \to \infty$.

Extragradient schemes II

Remark:

- Standard extragradient methods require that $\tau \leq \frac{1}{L_{f,x}}$ (cf. [FP03b]).
- This variant requires that

$$\tau \leq \sqrt{\frac{1}{L_{f,x}^2 + L_{f,\theta}\|\theta_0 - \theta^*\|}}.$$

- When $\theta_0 = \theta^*$, we recover the original result.

Iteratively (Tikhonov) regularized schemes I

- Tikhonov regularization techniques [Tik63, TA76, FP03b] have proved useful in solving monotone variational inequality problems.
- Specifically, such techniques construct a sequence $\{x_k\}$ where
$$x_k = \Pi_X(x_k - \gamma_k(F(x_k) + \epsilon_k x_k)), \quad \forall k \geq 0$$
implying that $x_k \in \mathrm{SOL}(X, F + \epsilon_k \mathbf{I})$, where $\{\epsilon_k\} \to 0$ and $\{x_k\} \to x^* \in X^*$.
- Challenge: obtaining x_k requires solving a strongly monotone VI exactly (or with increasing accuracy) at every step
- An alternative lies in using *iterative* Tikhonov regularization where a projected gradient step is taken at every step [Pol87, KS10]
$$x_{k+1} := \Pi_X(x_k - \gamma_k(F(x_k) + \epsilon_k x_k)), \quad \forall k \geq 0.$$
Under suitable assumptions of $\{\gamma_k, \epsilon_k\}$, convergence can be recovered.
- We consider an extension of this scheme to the misspecified regime.

Algorithm 4 (A regularized projection scheme)

Given an $x_0 \in X$ and $\theta_0 \in \Theta$ and sequences $\{\gamma_{f,k}\}$ and $\{\epsilon_k\}$,

$$x_{k+1} := \Pi_X\left(x_k - \gamma_{f,k}(F(x_k, \theta_k) + \epsilon_k x_k)\right) \qquad \forall k > 0, \qquad (\mathrm{Var}(\theta_k, \epsilon_k))$$
$$\theta_{k+1} := \Pi_\Theta\left(\theta_k - \gamma_{g,k} \nabla_\theta g(\theta_k)\right) \qquad \forall k > 0. \qquad (\mathrm{Learn})$$

Iteratively (Tikhonov) regularized schemes II

In our analysis, we consider two auxiliary sequences $\{x_k^t\}$ and $\{z_k^t\}$, defined as follows:

$$x_k^t := \Pi_X(x_k^t - \gamma_{f,k}(F(x_k^t, \theta_k) + \epsilon_k x_k^t)) \quad \forall k > 0, \qquad (\text{Tik}(\theta_k))$$

$$z_k^t := \Pi_X(z_k^t - \gamma_{f,k}(F(z_k^t, \theta^*) + \epsilon_k z_k^t)) \quad \forall k > 0. \qquad (\text{Tik}(\theta^*))$$

- $\{z_k^t\}$ is the Tikhonov trajectory under perfect information (θ^* is known)
- $\{x_k^t\}$ is the Tikhonov trajectory under belief θ_k
- Proof of convergence shows that $\|x_k - x_k^t\| \to 0$ as $k \to \infty$ and $\|x_k^t - z_k^t\| \to 0$ as $k \to \infty$.
- Crucial Lemma:

Lemma 1

Let Assumptions 12, 13 and 14(d) hold. Suppose x_k^t and x_{k-1}^t are defined by Tik(θ_k) and Tik(θ_{k-1}) respectively. Then, we have that $\|x_k^t - x_{k-1}^t\|$ can be bounded as follows:

$$\|x_k^t - x_{k-1}^t\| \leq \frac{L_{F,\theta} q_g^{k-1} C_g}{\epsilon_k} + \frac{M}{\epsilon_k}|\epsilon_{k-1} - \epsilon_k|,$$

where $q_g \triangleq \sqrt{1 - 2\gamma_g \eta_g + \gamma_g^2 G_g^2}$, $C_g \triangleq \|\theta_0 - \theta^*\|(1 + q_g)$, and M is the constant defined in Assumption13.

Iteratively (Tikhonov) regularized schemes III

Assumption 13

The set X is compact and $\sup_{x \in X} \|x\| \leq M$, where M is a constant.

Assumption 14

The following hold:

(a) $0 < \gamma_{f,k} \leq \frac{\epsilon_k}{(L_{F,x}+\epsilon_k)^2} \leq \frac{\epsilon_0}{L_{F,x}^2}$ *for all k;*

(b) $\gamma_{f,k}\epsilon_k < 1$ *and* $\sum_{k=1}^{\infty} \gamma_{f,k}\epsilon_k = \infty$*;*

(c) $\lim_{k\to\infty} \frac{|\epsilon_{k-1}-\epsilon_k|}{\gamma_{f,k}\epsilon_k^2} = 0$*;*

(d) $\gamma_{g,k} \triangleq \gamma_g$ *such that* $\gamma_g \leq 2\eta_g/G_g^2$ *and* $\lim_{k\to\infty} \frac{q_g^{k-1}}{\gamma_{f,k}\epsilon_k^2} = 0$*, where* $q_g \triangleq \sqrt{1 - 2\gamma_{g,k}\eta_g + \gamma_{g,k}^2 G_g^2}$.

Theorem 2 (**Convergence of regularized scheme)**

Let Assumptions 12, 13 and 14 hold. Consider the sequence $\{x_k, \theta_k\}$ generated by Algorithm 4. Then, $\{x_k\}$ converges to x^* as $k \to \infty$, where x^* denotes the least-norm solution of X^* and $\{\theta_k\}$ converges to $\theta^* \in \Theta$.

Introduction of uncertainty I

- Computational problem: We consider the stochastic generalization of optimization/variational inequality problems.
- Specifically, such a problem requires an $x^* \in X$ such that

$$(x - x^*)^T \mathbb{E}[F(x^*; \theta^*, \xi(\omega))] \geq 0, \qquad \forall x \in X, \qquad (\mathrm{P}_x(\theta^*))$$

  where $\xi : \Omega \to \mathbb{R}^d$, $F : X \times \mathbb{R}^d \to \mathbb{R}^n$, $X \subseteq \mathbb{R}^n$, and $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the probability space
- Learning problem: The vector θ^* lies in the solution set of (P_θ):

$$\min_{\theta \in \Theta} g(\theta), \text{ where } g(\theta) \triangleq \mathbb{E}[g(\theta; \eta)]. \qquad (\mathrm{P}_\theta)$$

(P_x): Stochastic Optimization Problem

Algorithm 5 (**Coupled SA schemes for stochastic opt. problems)**

Step 0. Given $x_0 \in X, \theta_0 \in \Theta$ and sequences $\{\gamma_{k,x}, \gamma_{k,\theta}\}$, $k := 0$

Step 1.

$$x^{k+1} := \Pi_X \left(x^k - \gamma_{k,x}(\nabla_x f(x^k;\theta^k) + w^k) \right), \qquad k \geq 0 \qquad (\text{Opt}_k)$$

$$\theta^{k+1} := \Pi_\Theta \left(\theta^k - \gamma_{k,\theta}(\nabla_\theta g(\theta^k) + v^k) \right), \qquad k \geq 0 \qquad (\text{Learn}_k)$$

$w^k \triangleq \nabla_x f(x^k;\theta^k,\xi^k) - \nabla_x f(x^k;\theta^k)$ and $v^k \triangleq \nabla_\theta g(\theta^k;\eta^k) - \nabla_\theta g(\theta^k)$.

Step 2. If $k > K$, stop; else $k : k+1$, go to Step. 1.

Assumptions

Assumption 1 (Problem properties, A1-1)

Suppose the following hold:

(i) For every $\theta \in \Theta$, $f(x;\theta)$ is strongly convex (μ_x) and continuously differentiable with Lipschitz continuous gradients (L_x) in x.

(ii) For every $x \in X$, the gradient $\nabla_x f(x;\theta)$ is Lipschitz continuous in θ with constant L_θ.

(iii) The function $g(\theta)$ is strongly convex and continuously differentiable with Lipschitz continuous gradients in θ with convexity constant μ_θ and Lipschitz constant C_θ, respectively.

Assumption 2 (Steplength requirements, A2-1)

Let $\{\gamma_{k,x}\}$ and $\{\gamma_{k,\theta}\}$ be chosen such that $\sum_{k=0}^{\infty} \gamma_{k,x} = \infty$, $\sum_{k=0}^{\infty} \gamma_{k,x}^2 < \infty$ and $\gamma_{k,\theta} = \gamma_{k,x} L_\theta^2/(\mu_x \mu_\theta)$.

Assumption 3 (A3)

[§] Let the following hold: $\mathbb{E}[w^k \mid \mathcal{F}_k] = 0$ and $\mathbb{E}[v^k \mid \mathcal{F}_k] = 0$ *a.s.* for all k. Furthermore, $\mathbb{E}[\|w^k\|^2 \mid \mathcal{F}_k] \le \nu_x^2$ and $\mathbb{E}[\|v^k\|^2 \mid \mathcal{F}_k] \le \nu_\theta^2$ *a.s.* for all k.

[§] We define a new probability space $(Z, \mathcal{F}, \mathbb{P})$, where $Z \triangleq \Omega \times \Lambda$, $\mathcal{F} \triangleq \mathcal{F}_x \times \mathcal{F}_\theta$ and $\mathbb{P} \triangleq \mathbb{P}_x \times \mathbb{P}_\theta$. We use $\mathcal{F}_k$ to denote the sigma-field generated by the initial points (x^0, θ^0) and errors (w^l, v^l) for $l = 0, 1, \cdots, k-1$, i.e., $\mathcal{F}_0 = \left\{(x^0, \theta^0)\right\}$ and $\mathcal{F}_k = \left\{(x^0, \theta^0), \left((w^l, v^l), l = 0, 1, \cdots, k-1\right)\right\}$ for $k \ge 1$. We make the following assumptions on the filtration and errors.

Main results

Proposition 4 (Almost-sure convergence under strong convexity of f)

Suppose (A1-1), (A2-1) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, $x^k \to x^*$ and $\theta^k \to \theta^*$ a.s. as $k \to \infty$, where x^* denotes the unique solution to $(\mathrm{P}_x(\theta^*))$.

- Proof relies on super-martingale convergence theorem
- Surpising aspects:
  - The steplength sequences run on the same timescale; merely scaled variants
  - The overall variational problem in (x, θ) is not necessarily monotone but can be solved[¶]; what does this suggest regard the solution of more general complementarity/equilibrium/variational problems

[¶] No available schemes for solving non-monotone stochastic variational inequality problems

Weakening strong convexity of (P$_x$)

Assumption 4 (A1-2)

Suppose the following holds in addition to (A1-1 (ii)) and (A1-1 (iii)) For every $\theta \in \Theta$, $f(x;\theta)$ is convex and continuously differentiable with Lipschitz continuous gradients in x with Lipschitz constant L_x.

Furthermore, we make the following assumptions on the steplength sequences employed in the algorithm.

Assumption 5 (A2-2)

Let $\{\gamma_{k,x}\}$, $\{\gamma_{k,\theta}\}$ and some constant $\tau \in (0,1)$ be chosen such that $\sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty$ and $\sum_{k=0}^{\infty} \gamma_{k,\theta}^{2} < \infty$, $\sum_{k=0}^{\infty} \gamma_{k,x} = \infty$ and $\sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty$, $\beta_k = \frac{\gamma_{k,x}^{\tau}}{2\gamma_{k,\theta}\mu_\theta} \downarrow 0$ as $k \to \infty$.

Proceeding as in the previous result, we present a convergence result under these weakened conditions.

Theorem 2 (Almost-sure convergence under convexity of f)

Suppose (A1-2), (A2-2) and (A3) hold. Suppose X is bounded and the solution set X^* of $(\mathrm{P}_x(\theta^*))$ is nonempty. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, $\theta^k \to \theta^*$ a.s. as $k \to \infty$, and x^k converges to a random point in X^* a.s. as $k \to \infty$.

Notably, in merely convex regimes, $\gamma_{k,x}$ and $\gamma_{k,\theta}$ are run at differing timescales; specifically, $\gamma_{k,x} \to 0$ at a faster rate than $\gamma_{k,\theta} \to 0$.

Rate estimates I

Proposition 5 (Rate estimates for strongly convex f)

Suppose (A1-1) and (A3) hold.[a] Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, the following hold:

$$\mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{Q_\theta(\lambda_\theta)}{k} \text{ and } \mathbb{E}[\|x^k - x^*\|^2] \leq \frac{Q_x(\lambda_x)}{k},$$

$$\text{where } Q_\theta(\lambda_\theta) \triangleq \max\left\{\lambda_\theta^2 M_\theta^2 (2\mu_\theta\lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2]\right\},$$

$$Q_x(\lambda_x) \triangleq \max\left\{\lambda_x^2 \widetilde{M}^2 (\mu_x\lambda_x - 1)^{-1}, \mathbb{E}[\|x^1 - x^*\|^2]\right\},$$

$$\text{and } \widetilde{M} \triangleq \sqrt{M^2 + \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x\lambda_x}}.$$

[a] Suppose $\gamma_{x,k} = \lambda_x/k$ and $\gamma_{\theta,k} = \lambda_\theta/k$ with $\lambda_x > 1/\mu_x$ and $\lambda_\theta > 1/(2\mu_\theta)$. Let $\mathbb{E}[\|\nabla_x f(x^k;\theta^k) + w^k\|^2] \leq M^2$ and $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$ for all $x^k \in X$ and $\theta^k \in \Theta$.

- Under strong convexity, optimization and learning recovers **optimal rate** of SA
- Naturally, when $\theta_1 = \theta^*$, we recover the original optimization result

Rate estimates II

Theorem 3 (Rate estimates under convexity of f)

Suppose (A1-2) and (A3) hold.[a] Let $\{x^k, \theta^k\}$ be computed via Algorithm 5.[b]Then the following holds for $1 \leq i \leq k$:

$$\mathbb{E}[|f(\tilde{x}_{i,k}; \theta^k) - f(x^*; \theta^*)|] \leq \frac{\sqrt{Q_\theta(\lambda_\theta)}D_\theta + C_{i,k}\sqrt{B_k}}{\sqrt{k}},$$

where $C_{i,k} = \frac{k}{k-i+1}$ and $B_k = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k))(M^2 + M_x^2)$.

[a] Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$, $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$ and $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$ for all $x^k \in X$ and $\theta^k \in \Theta$.

[b] For $1 \leq i, t \leq k$, we define $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$, $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$ and $D_X \triangleq \max_{x \in X} \|x - x^1\|$. Suppose for $1 \leq t \leq k$ $\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1+\ln k)}{(M^2+M_x^2)k}}$, where $Q_\theta(\lambda_\theta) \triangleq \max\left\{\lambda_\theta^2 M_\theta^2 (2\mu_\theta\lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2]\right\}$, and $\gamma_{\theta,k} = \lambda_\theta / k$ with $\lambda_\theta > 1/(2\mu_\theta)$.

- Averaging in stochastic convex optimization leads to $O(1/\sqrt{k})$
- Averaging with learning leads to bound given loosely by $O\left(\sqrt{\ln(k)}/\sqrt{k}\right)$.
- Degradation in learning is $O\left(\sqrt{\ln(k)}\right)$.

Constant steplength error bounds

In many multiagent systems, constant steplengths (or gain sequences) are convenient; can one quantify these errors?

Proposition 6

Suppose (A3) holds. Suppose $\gamma_{\theta,k} = \gamma_{x,k} := \gamma$. Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ and $\mathbb{E}[\|\nabla_x f(x^k;\theta^k) + w^k\|^2] \leq M^2$ for all $x^k \in X$. Suppose $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$ and $a_k \triangleq \mathbb{E}[A_k]$. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5.
Suppose (A1-1) holds. Then, the following holds:

$$\limsup_{k\to\infty} a_k \leq \frac{1}{2\mu_x}\gamma M^2 + \frac{L_\theta^2}{2\mu_x^2}\frac{\gamma\nu_\theta^2}{(2\mu_\theta - \gamma C_\theta^2)}.$$

Suppose (A1-2) holds. Then, the following holds:

$$\limsup_{k\to\infty} |\mathbb{E}[f(x^k;\theta^k) - f(x^*;\theta^*)]| \leq \frac{1}{2}\gamma M^2 + \frac{1}{2}\gamma^{1-\tau}M_x^2 + \underbrace{\frac{\gamma^\tau \nu_\theta^2 L_\theta^2}{4\mu_\theta - 2\gamma C_\theta^2} + D_\theta\sqrt{\frac{\gamma\nu_\theta^2}{2\mu_\theta - \gamma C_\theta^2}}}_{\text{Degradation from learning}}$$

where $0 < \tau < 1$.

- Utility of this result; we've set $\gamma_x = \gamma_\theta$; But we may optimize this error bound in the choices of steplengths

Summary of rate statements

	Computation	Computation & Learning
Det. Strongly convex/diff.	Linear	Sublinear
Det. convex/diff.	$\mathcal{O}(1/K)$	$\mathcal{O}(1/K + q_g^K)$
Det. convex/nonsmooth.	$\mathcal{O}(1/\sqrt{K})$	$\mathcal{O}(1/\sqrt{K}) + \mathcal{O}(1/K + q_g^K)$
Stoch. Strongly convex	$O\left(\frac{1}{k}\right)$	$O\left(\frac{1}{k}\right)$
Stoch. Convex	$O\left(\frac{1}{\sqrt{k}}\right)$	$O\left(\frac{\sqrt{\ln(k)}}{\sqrt{k}}\right)$

(P_x): Stochastic variational inequality problem

Algorithm 6 (Coupled SA schemes for Stochastic variational probs.)

Step 0. Given $x_0 \in X, \theta_0 \in \Theta$ and sequences $\{\gamma_{k,x}, \gamma_{k,\theta}\}$, $k := 0$

Step 1.

$$x^{k+1} := \Pi_X \left(x^k - \gamma_{k,x}(F(x^k;\theta^k) + w^k) \right) \qquad (\text{Comp}_k)$$

$$\theta^{k+1} := \Pi_\Theta \left(\theta^k - \gamma_{k,\theta}(G(\theta^k) + v^k) \right), \qquad (\text{Learn}_k)$$

where $w^k \triangleq F(x^k;\theta^k,\xi^k) - F(x^k;\theta^k)$ and $v^k \triangleq G(\theta^k;\eta^k) - G(\theta^k)$.

Step 2. If $k > K$, stop; else $k := k+1$, go to Step. 1.

We begin by stating an assumption similar to (A1-1) on the mapping F.

Assumption 6 (A1-3)

(Identical to A1-1) with $\nabla f(x;\theta)$ replaced by $F(x;\theta)$

Main results I

Proposition 7 (Almost-sure convergence under strongly monotone F)
Suppose (A1-3), (A2-1) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 6. Then, $x^k \to x^*$ *a.s.* and $\theta^k \to \theta^*$ *a.s.* as $k \to \infty$, where x^* is the unique solution to VI$(X, F(\bullet; \theta^*))$.

- Result is similar to that for strongly convex problems

Main results II

Algorithm 7 (Coupled regularized SA schemes for stochastic VIs)

Step 0. Given $x_0 \in X, \theta_0 \in \Theta$ and sequences $\{\gamma_{k,x}, \gamma_{k,\theta}, \epsilon_k\}$, $k := 0$

Step 1.

$$x^{k+1} := \Pi_X \left(x^k - \gamma_{k,x}(F(x^k;\theta^k) + \underbrace{\epsilon_k x^k}_{\text{Tikhonov regular.}} + w^k) \right) \qquad (\text{Comp}_k)$$

$$\theta^{k+1} := \Pi_\Theta \left(\theta^k - \gamma_{k,\theta}(G(\theta^k) + v^k) \right), \qquad (\text{Learn}_k)$$

where $w^k \triangleq F(x^k;\theta^k,\xi^k) - F(x^k;\theta^k)$ and $v^k \triangleq G(\theta^k;\eta^k) - G(\theta^k)$.

Step 2. If $k > K$, stop; else $k : k + 1$, go to Step. 1.

- Unlike in optimization, we need to employ a Tikhonov regularizer, inspired by past work [KNS13]

Assumptions

The following assumptions will be made on both the decision variable and parameter.

Assumption 7 (A1-4)

(Similar to A1-3)

We also make the following assumptions on the steplength sequences employed in the algorithm.

Assumption 8 (A2-3)

Let $\{\gamma_{k,x}\}$, $\{\gamma_{k,\theta}\}$, $\{\epsilon_k\}$ and some constant $\tau \in (0,1)$ be chosen such that:

(i) $\sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty$ and $\sum_{k=0}^{\infty} \gamma_{k,\theta}^{2} < \infty$,

(ii) $\sum_{k=0}^{\infty} \gamma_{k,x}\epsilon^{k} = \infty$ and $\sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty$,

(iii) $\beta_k = \frac{\gamma_{k,x}^{\tau}}{2\gamma_{k,\theta}\mu_\theta} \downarrow 0$ as $k \to 0$.

(iv) $\sum_{k=0}^{\infty} \frac{(\epsilon_{k-1}-\epsilon_k)}{\epsilon_k} < \infty$.

Main results

Theorem 4
Suppose (A1-4) , (A2-3) and (A3) hold. Suppose X is bounded and the solution set X^* of VI($X, F(\bullet, \theta^*)$) is nonempty. Let $\{x^k, \theta^k\}$ be computed via Algorithm 7. Then, $\theta^k \to \theta^*$ *a.s.* as $k \to \infty$, and x^k converges to the least norm solution in X^* *a.s.* as $k \to \infty$.

- Again, $\gamma_{k,x}$ and $\gamma_{k,\theta}$ are decreased at different rates
- Unlike in the optimization setting, we recover the least-norm solution

Rate estimates I

- In the strongly monotone regime, we may recover the optimal rate of SA
- Without strong monotonicity, one avenue lies in averaging and working in a weak sharp regime; specifically, we assume that VI$(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$ possesses the MPS property, which is introduced in the following lemma.

Lemma 3

[Mar93] Let $H : X \to \mathbb{R}^n$ be a mapping that is monotone over the compact polyhedral set X. Let X^* be the solution set of VI$(X, H)^{\|}$ and there exists a positive number α s.t.

$$(x - x^*)^T H(x^*) \geq \alpha \, \mathrm{dist}(x, X^*), \quad \forall x \in X, \quad \forall x^* \in X^*,$$

where $\mathrm{dist}(x, X^*) \triangleq \min_{x^* \in X^*} \|x - x^*\|$.

Rate estimates II

Theorem 5 (**Rate estimates under monotonicity of F**)

Suppose (A1-4) and (A3) hold.[a] Let $\{x^k, \theta^k\}$ be computed via Algorithm 6. [b] Then there exists a positive number α such that for $1 \leq i \leq k$:

$$\mathbb{E}\left[\alpha \operatorname{dist}(\tilde{x}_{i,k}, X^*)\right] \leq C_{i,k}\sqrt{\frac{B_k}{k}},$$

where $C_{i,k} = \frac{k}{k-i+1}$ and $B_k = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1+\ln k))(M^2 + M_x^2)$.

---

[a] Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$, $\mathbb{E}[\|F(x^k;\theta^k) + w^k\|^2] \leq M^2$ and $\mathbb{E}[\|G(\theta^k) + v^k\|^2] \leq M_\theta^2$ for all $x^k \in X$ and $\theta^k \in \Theta$. Suppose X is a compact polyhedral set, the solution set X^* of VI($X, \mathbb{E}[F(\bullet;\theta^*,\xi)]$) is nonempty, and x^* is a point in X^*. Suppose VI($X, \mathbb{E}[F(\bullet;\theta^*,\xi)]$) possesses the MPS property.

[b] For $1 \leq i, t \leq k$, we define $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$, $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$ and $D_X \triangleq \max_{x\in X}\|x - x^1\|$. Suppose for $1 \leq t \leq k$ $\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1+\ln k)}{(M^2+M_x^2)k}}$, where $Q_\theta(\lambda_\theta) \triangleq \max\left\{\lambda_\theta^2 M_\theta^2(2\mu_\theta\lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2]\right\}$, and $\gamma_{\theta,k} = \lambda_\theta/k$ with $\lambda_\theta > 1/(2\mu_\theta)$.

- Akin to merely convex regimes, averaging allows for prescribing rates
- Degradation from learning is $O\left(\sqrt{\ln(k)}\right)$.

[‖] If the VI(X, H) possesses the minimum principle sufficiency (MPS) property

Constant steplength errors

Proposition 8

Suppose (A3) holds. Suppose $\gamma_{\theta,k} = \gamma_{x,k} := \gamma_x$. Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ and $\mathbb{E}[F(x^k;\theta^k) + w^k\|^2] \leq M^2$ for all $x^k \in X$. Suppose $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$ and $a_k \triangleq \mathbb{E}[A_k]$. Suppose X is a compact polyhedral set, the solution set X^* of VI$(X, F(\bullet, \theta^*))$ is nonempty, and x^* is a point in X^*. Suppose VI$(X, F(\bullet, \theta^*))$ possesses the MPS property. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5.

Suppose (A1-3) holds. Then, the following holds:

$$\limsup_{k\to\infty} a_k \leq \frac{1}{2\mu_x}\gamma M^2 + \frac{L_\theta^2}{2\mu_x^2}\frac{\gamma\nu_\theta^2}{2\mu_\theta - \gamma C_\theta^2};$$

Suppose (A1-4) holds. Then, there exists a positive number α such that:

$$\limsup_{k\to\infty} \mathbb{E}[\mathrm{dist}(x^k, X^*)] \leq \frac{1}{\alpha}\left[\frac{1}{2}\gamma M^2 + \frac{1}{2}\gamma^{1-\tau}M_x^2 + \frac{\gamma^\tau \nu_\theta^2 L_\theta^2}{4\mu_\theta - 2\gamma C_\theta^2}\right],$$

where $0 < \tau < 1$.

Diminishing steplength

Table 1 : Distributed scheme for learning x^* and θ^* in a stochastic regime: $\xi \sim U[-\theta^*/2, \theta^*/2]$

N	W	$\frac{\mathbb{E}[\|x^K - x^*\|]}{1+\|x^*\|}$	$\frac{\mathrm{ERR}}{1+\|x^*\|}$	$\frac{\|\mathbb{E}[\theta^K - \theta^*\|]}{1+\|\theta^*\|}$	$\frac{\mathrm{ERR}}{1+\|\theta^*\|}$
10	2	7.4×10^{-2}	1.2×10^{10}	4.7×10^{-2}	5.0×10^{4}
10	4	6.5×10^{-2}	2.3×10^{10}	3.7×10^{-2}	5.1×10^{4}
10	6	5.8×10^{-2}	3.8×10^{10}	2.9×10^{-2}	5.1×10^{4}
10	8	5.8×10^{-2}	6.9×10^{10}	2.2×10^{-2}	6.4×10^{4}
10	10	6.7×10^{-2}	1.1×10^{11}	1.9×10^{-2}	7.5×10^{4}

- $\gamma_{k,x} = 10/k$ and $\gamma_{k,\theta} = 10/k$.
- $K = 10000$.
- ERR : theoretical error in Proportion 5.

Averaging

Table 2 : Distributed scheme for learning x^* and θ^* in a stochastic regime: $\xi \sim U[-\theta^*/2, \theta^*/2]$

N	W	$\frac{\mathbb{E}[\lvert f(\tilde{x}_{1,K};\theta^K) - z^* \rvert]}{1+\lVert z^* \rVert}$	$\frac{\mathrm{ERR}}{1+\lVert x^* \rVert}$	γ_X
10	2	1.2×10^{-1}	1.7×10^5	68
10	4	1.9×10^{-1}	2.1×10^5	92
10	6	1.1×10^{-1}	1.2×10^5	127
10	8	1.2×10^{-1}	1.5×10^5	152
10	10	1.4×10^{-1}	2.4×10^5	161

- $\gamma_{K,\theta} = 10/K$, $z^* = f(x^*;\theta^*)$.
- $K = 10000$.
- ERR : theoretical error in Theorem 3.

Regret

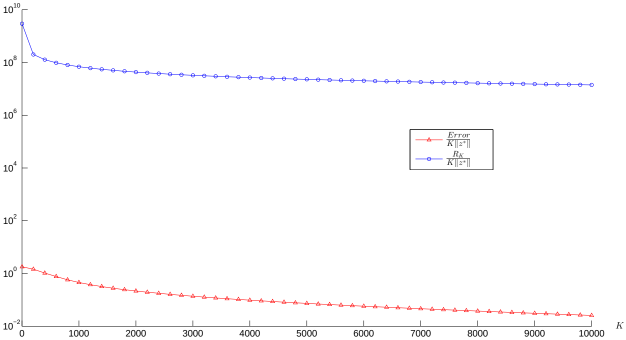


Figure 4 : Computing x^* and learning θ^* ($\xi \sim U[-\theta^*/2, \theta^*/2]$, $N = 5$, $W = 5$)

- $\gamma_{k,x} = k^{-0.8}$, $\gamma_{k,\theta} = 10/k$, $z^* = f(x^*; \theta^*)$.
- $K = 10000$.
- ERR : theoretical error in Theorem **??**.

Concluding remarks

A broad framework for resolving misspecified stochastic optimization/variational problems:

- Asymptotics for gradient/subgradient/extragradient/iterative regularization schemes for deterministic problems
- (a.s.) Asymptotics for stochastic approximation (and regularized counter-parts) for stochastic problems
- Rate statements for gradient/subgradient schemes with quantification of impact; Similar statements for mean-squared error for stochastic approximation schemes

Key findings:

- Natural extensions of gradient-type schemes are provably convergent
- Recover optimal rates upto constant factor modifications in some regimes; degradation in other regimes.
- **Seemingly non-monotone problems in full-space can be solved via first order schemes with modest rate degradation at worst**

Ongoing work:

- Misspecified Markov Decision Processes (as an alternative to Q-learning) where transition matrices need to be learnt
- Consensus (distributed optimization) under imperfect information

- H. Jiang, and U. V. Shanbhag, "*On the solution of stochastic optimization and variational problems in imperfect information regimes*", Under review.

- H. Jiang and U. V. Shanbhag, "*On the solution of stochastic optimization problems in imperfect information regimes*", to appear in Proceedings of the Winter Simulation Conference (2013).

K. J. Astrom and B. Wittenmark.
Adaptive Control.
Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 2nd edition, 1994.

J. R. Birge and F. Louveaux.
Introduction to Stochastic Programming: Springer Series in Operations Research.
Springer, 1997.

Dimitri P. Bertsekas, Angelia Nedić, and Asuman E. Ozdaglar.
Convex analysis and optimization.
Athena Scientific, Belmont, MA, 2003.

Stephen Boyd and Lieven Vandenberghe.
Convex Optimization.
Cambridge University Press, New York, NY, USA, 2004.

W. L. Cooper, T. Homem-de Mello, and A. J. Kleywegt.
Models of the spiral-down effect in revenue management.
Oper. Res., 54:968–987, September 2006.

W. L. Cooper, T. Homem-de Mello, and A. J. Kleywegt.
Learning and pricing with models that do not explicitly incorporate competition.
Working paper, 2012.

F. Facchinei and J. S. Pang.
Finite-dimensional variational inequalities and complementarity problems. Vol. I.
Springer Series in Operations Research. Springer-Verlag, New York, 2003.

Francisco Facchinei and Jong-Shi Pang.
Finite-dimensional variational inequalities and complementarity problems. Vol. I.
Springer Series in Operations Research. Springer-Verlag, New York, 2003.

J. C. Gittins.
Multi-armed bandit allocation indices.
Wiley-Interscience Series in Systems and Optimization, Chichester: John Wiley & Sons, Ltd., 1989.

J. M. Harrison, N. B.. Keskin, and A. Zeevi.
Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies.
submitted to Operations Research.

T. Hastie, R. Tibshirani, and J. H. Friedman.
The elements of statistical learning: data mining, inference, and prediction: with 200 full-color illustrations.
New York: Springer-Verlag, 2001.

J. Koshal, A. Nedic, and U. V. Shanbhag.
Regularized iterative stochastic approximation methods for stochastic variational inequality problems.
IEEE Trans. Automat. Contr., 58(3):594–609, 2013.

A. Kannan and U. V. Shanbhag.
Distributed iterative regularization algorithms for monotone Nash games.
Proceedings of the IEEE Conference on Decision and Control (CDC), pages 1963–1968, 2010.

K. L. Moore.
Iterative Learning Control for Deterministic Systems.
Springer-Verlag Series on Advances in Industrial Control. Springer-Verlag, London, 1993.

George Nemhauser and Laurence Wolsey.
Integer and combinatorial optimization.
Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1999.
Reprint of the 1988 original, A Wiley-Interscience Publication.

B. T. Polyak.
Introduction to optimization.
Optimization Software, Inc., New York, 1987.

A. Shapiro, D. Dentcheva, and A. Ruszczyński.
Lectures on stochastic programming, volume 9 of *MPS/SIAM Series on Optimization*.
SIAM, Philadelphia, PA, 2009.
Modeling and theory.

A. N. Tikhonov and V. Arsénine.
Méthodes de resolution de problèmes mal posés.
Éditions Mir, Moscow, 1976.
Traduit du russe par Vladimir Kotliar.

A. N. Tikhonov.
On the solution of incorrectly put problems and the regularisation method.
In *Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963)*, pages 261–265. Acad. Sci. USSR Siberian Branch, Moscow, 1963.

M. Zinkevich.
Online convex programming and generalized infinitesimal gradient ascent.
In Tom Fawcett and Nina Mishra, editors, *ICML*, pages 928–936. AAAI Press, 2003.